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MALLIAVIN CALCULUS FOR TWO-PARAMETER PROCESSES

D. NUALART and M. SANZ

Abstract. In this paper we apply the Malliavin Calculus to derive the existence of a density for the law of the solution of a stochastic differential equation with respect to a multidimensional two-parameter Wiener process.

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0. Introduction. In this paper we prove the existence of a density for the probability law on R^m induced by the solution of the stochastic integral equations.

$$X_z^i = x^i + \int_{[0,z]} [A_j^i(X_r)\, dW_r^j + B^i(X_r)\, dr] \ , \quad i = 1,\ldots,m, \qquad (0.1)$$

$z \in R_+^2$, where $W_z = (W_z^1,\ldots,W_z^d)$ is a d-dimensional two-parameter Wiener process, $x = (x^1,\ldots,x^m) \in R^m$, and assuming some conditions on the coefficients A_j^i and B^i. If these coefficients are smooth, it is known (cf. Cairoli [2], Hajek [4]) that (0.1) has a unique continuous solution, which has a particular Markov property. There exists a transition semigroup corresponding to these Markov processes, but it acts on continuous functions over sets of the form $\{(x,t),\ x \geq s\} \cup \{(s,y),\ y \geq t\}$ and we cannot expect that the probability law of X_z^i satisfy a second order partial differential equation.

In the case of an ordinary stochastic differential equation with respect to the Brownian motion, Malliavin has developped in [6] probabilistic techniques to show the existence and smoothness of density for the solution of these equations under Hörmander's conditions. Alternative approaches to Malliavin's theory were given by Shigekawa [8], Bismut [1] and Stroock [9]. It is not difficult to extend the Malliavin calculus to two-parameter Wiener functionals. However when we apply this calculus to the solutions of (0.1) some technical difficulties appear, in relation with the following facts:

(a) The inner products $\langle\langle X_z^i, X_z^k \rangle\rangle$ (in the notation of Stroock) are not solutions of a similar system of equations, because the stochastic differentiation rules with respect to the two-parameter Wiener process involve the presence of double integrals over the set $\{(z,z') \in R_+^2 \times R_+^2,$ $z = (x,y),\ z' = (x',y'),\ x \leqslant x',\ y \geqslant y'\}$ (cf. [10])

(b) The system (0.1) do not provide a flow of transformations of R^m. Moreover if we consider a linear system of equations, the solution is not invertible, in general.

Here we have followed Shigekawa's presentation of Malliavin Calculus, and using this theory we have proved the existence of a density for the two-parameter Wiener functional X_z assuming that the vector space spanned by the vector fields $A_1,\dots,A_d$, $A_i \nabla A_j$, $1 \leqslant i,j \leqslant d$, $A_i \nabla (A_j \nabla A_k)$, $1 \leqslant i,j,k \leqslant d,\dots$, at the point x (where $A_i \nabla A_j$ denotes the covariant derivative of A_j in the direction of A_i), is R^m. This property is strictly weaker than the restricted Hörmander's conditions, as we show in an example. Actually we have proved (see [7]) that it also implies the smoothness of the density.

1. <u>Some results on Malliavin Calculus</u>. The set of parameters will be $T = [0,1]^2$, with the partial ordering $(s_1,t_1) \leqslant (s_2,t_2)$ if and only if $s_1 \leqslant s_2$ and $t_1 \leqslant t_2$; $(s_1,t_1) < (s_2,t_2)$ means that $s_1 < s_2$ and $t_1 < t_2$. If $z_1 < z_2$, $(z_1,z_2]$ will denote the rectangle $\{z \in T, z_1 < z \leqslant z_2\}$. We put $R_z = [0,z]$, and $z_1 \boxtimes z_2 = (s_1,t_2)$ if $z_1 = (s_1,t_1)$ and $z_2 = (s_2,t_2)$. If $f: R_+^2 \longrightarrow R$, $f((z_1,z_2])$ means $f(z_1) - f(z_1 \boxtimes z_2) - f(z_2 \boxtimes z_1) + f(z_2)$. The Lebesgue measure of a Borel set $B \subset R_+^2$ is denoted by $|B|$.

Our probability space (Ω,F,P) is the canonical space associated to the d-dimensional two-parameter Wiener process. We also consider the filtration $\{F_z, z \in T\}$, where F_z is generated by the functions $\{\omega(s), \omega \in \Omega, s \leqslant z\}$ and the null sets of F. The family $\{F_z, z \in T\}$ satisfies the usual conditions of [3].

The following subset of Ω plays an important role:

$$H = \{\omega \in \Omega, \text{ there exists } \dot\omega^i \in L^2(T),\ i = 1,\dots,d, \text{ such that } \omega^i(z) = \int_{R_z} \dot\omega^i(r)\, dr , \text{ for any } z \in T \text{ and for any } i\} .$$

H is a Hilbert space with the inner product

$$< \omega_1, \omega_2 >_H = \int_T \sum_{i=1}^{d} \dot\omega_1^i(r) \; \dot\omega_2^i(r)\, dr.$$

A measurable function defined on (Ω,F,P) is called a Wiener functional. A Wiener functional $F: \Omega \longrightarrow R$ is smooth if there exists some $n \geqslant 1$ and a C^2-function f on R^n such that

(i) f and its derivatives up to the second order have at most polynomial growth order,

(ii) $F(\omega) = f(\omega(z_1),\dots,\omega(z_n))$ for some $z_1,\dots,z_n \in T$.

Every smooth functional is Fréchet-differentiable, and we have

$$DF(\omega_o)(\omega) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_j^i} (\omega_o(z_1),\ldots,\omega_o(z_n))\, \omega^j(z_i) .$$

We also need the operator L defined on smooth functionals as follows:

$$LF(\omega) = \sum_{j=1}^{d} \sum_{i,k=1}^{n} \frac{\partial^2 f}{\partial x_j^i\, \partial x_j^k} (\omega(z_1),\ldots,\omega(z_n))\, z_i \wedge z_k - DF(\omega)(\omega) ,$$

where $z_i \wedge z_k = (x_i \wedge x_k)(y_i \wedge y_k)$, if $z_i = (x_i,y_i)$, $i = 1,\ldots,n$.

For any $p \geqslant 1$, L_H^p will denote the space of Wiener functionals $F: \Omega \longrightarrow H$ such that $E(\|F\|_H^p) < \infty$. If we fix $\omega \in \Omega$ and a smooth functional F , $DF(\omega) : H \longrightarrow R$ is a continuous linear map, and, so, it may be considered as an element of H . In this sense we have $DF \in L_H^p$, for any $p \geqslant 1$.

Let $H(p_1,p_2;p_3)$, $p_1,p_2,p_3 \geqslant 1$, be the space of real valued Wiener functionals F such that there exists a sequence of smooth functionals $\{F_k, k \geqslant 1\}$ satisfying:

(a) $F_k \longrightarrow F$ in L^{p_1} ,

(b) $\{DF_k , k \geqslant 1\}$ is a Cauchy sequence in $L_H^{p_2}$, and

(c) $\{LF_k , k \geqslant 1\}$ is a Cauchy sequence in L^{p_3}.

For a Wiener functional $F \in H(p_1,p_2;p_3)$ we define $DF = \lim_k DF_k$ and $LF = \lim_k LF_k$. $H(p_1,p_2;p_3)$ is a Banach space with the norm $\|F\|_{p_1} + \|DF\|_{p_2} + \|LF\|_{p_3}$. We set $H_\infty = \bigcap_{p \geqslant 2} H(p,p;p)$. If $F \in H_\infty$, we will say that a sequence of smooth functionals $\{F_k, k \geqslant 1\}$ is an approximating sequence for F if $\lim_k (\|F - F_k\|_p + \|DF - DF_k\|_p + \|LF - LF_k\|_p) = 0$, for any p .

Let $F^i \in H_\infty$ for $i = 1,..,n$, and let $u: R^n \longrightarrow R$ be twice continuously differentiable function such that u and its first and second derivatives have at most polynomial growth order. If we set $F = (F^1,\dots,F^n)$, then $u \circ F \in H_\infty$, and the following differentiation rules hold:

$$D(u \circ F) = \left(\frac{\partial u}{\partial x_i} \circ F \right) DF^i ,$$

$$L(u \circ F) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \circ F \right) < DF^i, DF^j >_H + \left(\frac{\partial u}{\partial x_i} \circ F \right) LF^i .$$

The next result is the main theorem in Shigekawa's paper [8] adapted to the Wiener process with two parameters, and with a slight difference consisting in the use of the space $H(1,2;1)$. The proof given by Shigekawa can be extended to these new conditions without any change.

Theorem 1.1. Let $F = (F^1,\dots,F^m)$ be an R^m-valued two-parameter Wiener functional. We assume that F satisfies the following conditions:

(i) $F^i \in H(1,2;1)$, for $i = 1,\dots,m$.

(ii) $< DF^i, DF^j >_H \in H(1,2;1)$, for $i,j = 1,\dots,m$.

(iii) $\det (< DF^i, DF^j >_H) \neq 0$ a.s.

Then, the probability law of F is absolutely continuous with respect to the Lebesgue measure.

In the next section we will employ this result to show the existence of a density for the law induced by the solution of a stochastic differential system.

2. Application to stochastic differential equations. Consider mappings $A: R^m \times R^n \longrightarrow R^n \otimes R^d$ and $B: R^m \times R^n \longrightarrow R^n$ such that

(i) All their components have bounded and continuous first order derivatives.

(ii) $x \longrightarrow A(x,0)$ and $x \longrightarrow B(x,0)$ are slowly increasing functions, that is $|A(x,0)| \leqslant k(1+|x|^{\alpha})$ and $|B(x,0)| \leqslant k(1+|x|^{\beta})$ for some positive integers α and β and positive constant k.

<u>Lemma 2.1.</u> Let $\alpha = \{\alpha_z, z \in T\}$ be a continuous and adapted n-dimensional stochastic process such that for each $p \geqslant 2$, $E\{\sup_{z \in T} |\alpha_z|^p\} < \infty$. Consider a continuous and adapted m-dimensional process $X = \{X_z, z \in T\}$ such that $E\{\sup_{z \in T} |X_z|^p\} < \infty$, for every $p \geqslant 2$, and fix $v \in T$. Then, there is a unique continuous and adapted n-dimensional process $Y = \{Y_z, z \geqslant v\}$ satisfying the stochastic differential system

$$Y_z^i = \alpha_z^i + \int_{[v,z]} [A_j^i(X_r,Y_r)\, dW_r^j + B^i(X_r,Y_r)\, dr] \quad , \quad i=1,\dots,n, \quad (2.1)$$

and the following property holds:

$$E\{\sup_{z \geqslant v} |Y_z|^p\} < \infty, \text{ for any } p \geqslant 2 .$$

The solution of (2.1) (with an arbitrary initial point v) can be approximated by polygonal paths, as it is stated in the next lemma.

For any $k \geqslant 1$ we consider the set $\mathcal{S}^k$ of points $(i2^{-k}, j2^{-k})$, $i,j = 0,\dots,2^k$. If $z \in T$ put $\mathcal{S}_z^k = \{u \leqslant z\colon u \in \mathcal{S}^k$, or $u=v\boxtimes z$ with $v \in \mathcal{S}^k$, or $u=z\boxtimes v$ with $v \in \mathcal{S}^k$ or $u=z\}$. Define $\phi_k(z) = \sup\{u \in \mathcal{S}^k, u \leqslant z\}$ and $\psi_k(z) = \inf\{u \in \mathcal{S}^k, u \geqslant z\}$.

<u>Lemma 2.2.</u> Let $\alpha = \{\alpha(z), z \in T\}$ and $\alpha_k = \{\alpha_k(z), z \in T\}$, $k \geqslant 1$ be continuous and adapted n-dimensional processes satisfying

$$\lim_{k \to \infty} E\{\sup_{z \in T} |\alpha_k(z) - \alpha(z)|^p\} = 0$$

for each $p \geqslant 2$, and let X be as in the preceding lemma. Consider the process $Y_z = (Y_z^1,\dots,Y_z^n)$, $z \geqslant v$ defined by (2.1). Consider also the process $Y_z^{(k)} = (Y_z^{k,1},\dots,Y_z^{k,n})$, $z \geqslant v$ given by

$$Y_z^{k,i} = \alpha_k^i(z) + \int_{[\psi_k(v)\wedge z, z]} [A_j^i(X_{\phi_k(r)}^{(k)}, Y_{\phi_k(r)}^{(k)}) \, dW_r^j$$

$$+ B^i(X_{\phi_k(r)}^{(k)}, Y_{\phi_k(r)}^{(k)}) \, dr] ,$$

where $X^{(k)} = \{X_z^{(k)} , z \in T\}$, $k \geqslant 1$ is a sequence of continuous and adapted m-dimensional stochastic processes such that for each $p \geqslant 2$, $E\{\sup_{z \in T} | X_z^{(k)} - X_z |^p \} \xrightarrow[k\to\infty]{} 0$. Then

$$\lim_{k\to\infty} \sup_{v \in T} \{E [\sup_{z \geqslant v} | Y_z - Y_z^{(k)} |^p]\} = 0 , \forall p \geqslant 2 .$$

This result is a generalization to the two-parameter case of lemma V.2.1 from [5] .

<u>Lemma 2.3</u>. Let $X_z = (X_z^1,\dots,X_z^m)$, $z \in T$ be the process which satisfies

$$X_z^i = x^i + \int_{R_z} [A_j^i(X_r) \, dW_r^j + B^i(X_r) \, dr] , \quad i = 1,\dots,m .$$

where $A: R^m \longrightarrow R^m \otimes R^d$ and $B: R^m \longrightarrow R^m$ have bounded and continuous derivatives up to the second order. Then $X_z^i \in H_\infty$, for $i=1,\dots,m$, $z \in T$.

<u>Proof</u>. For any $n \geqslant 1$ consider the process $X_z^{(n)} = (X_z^{n,1},\dots,X_z^{n,m})$ defined by the recursive system

$$X_z^{n,i} = x^i + \int_{R_z} [A_j^i(X_{\phi_n(r)}^{(n)}) \, dW_r^j + B^i(X_{\phi_n(r)}^{(n)}) \, dr] , \quad i = 1,\dots,m .$$

The random variables $X_z^{n,i}$, $z \in T$ are smooth functionals. We are going to prove that $\{X_z^{n,i} , n \geqslant 1\}$ is an approximating sequence for X_z^i .

First, by lemma 2.2 we have $\lim_n E\{\sup_{z\in T} |X_z^{(n)} - X_z|^p\} = 0$, for all $p \geq 1$. If $z \in (0,1]^2$ and $u = \sup\{v \in S^n, v < z\}$, $X_z^{n,i}$ is given by

$$X_z^{n,i} = X_{z\boxtimes u}^{n,i} + X_{u\boxtimes z}^{n,i} - X_u^{n,i} + A_j^i(X_u^{(n)})\, W^j(\,(u,z]\,) + B^i(X_u^{(n)})|\,(u,z]\,| .$$

Fix an element $\omega \in \Omega$. Differentiating term by term we obtain

$$DX_z^{n,i}(\omega) = DX_{z\boxtimes u}^{n,i}(\omega) + DX_{u\boxtimes z}^{n,i}(\omega) - DX_u^{n,i}(\omega) + \frac{\partial A_j^i}{\partial x_k}(X_u^{(n)}(\omega))$$

$$.DX_u^{n,k}(\omega)\, \omega^j(\,(u,z\,]) + A_j^i(X_u^{(n)}(\omega)\,)\, e^j\, \varepsilon_{(u,z]} + \frac{\partial B^i}{\partial x_k}(X_u^{(n)}(\omega)\,)$$

$$.DX_u^{n,k}(\omega)\,|\,(u,z\,]|\,,$$

where $\varepsilon_{(u,z]} = \varepsilon_z - \varepsilon_{z\boxtimes u} - \varepsilon_{u\boxtimes z} + \varepsilon_u$, $\varepsilon_z = \int_T 1_{R_z}(u)\,du$, and e^j is the vector of R^d given by $(e^j)^i = \delta_i^j$. From now on we will omit the dependence on ω.

Denote by $U_z^{n,i}$ the derivative of $DX_z^{n,i}$ in the sense

$$< DX_z^{n,i}\,, h >_H = \int_{R_z} \sum_{j=1}^{d} U_z^{n,i,j}(r)\, \dot{h}^j(r)\, dr\,.$$

Then, $U_z^{n,i,j}(r)=0$ if $r \notin R_z$, and for $r \in R_z$ we have

$$U_z^{n,i,j}(r) = A_j^i(X_{\phi_n(r)}^{(n)}) + \int_{[\psi_n(r)\wedge z,z\,]} [\frac{\partial A_h^i}{\partial x_k}(X_{\phi_n(s)}^{(n)})\, U_{\phi_n(s)}^{n,k,j}(r)$$

$$.dW_s^h + \frac{\partial B^i}{\partial x_k}(X_{\phi_n(s)}^{(n)})\, U_{\phi_n(s)}^{n,k,j}(r)\, ds\,]\,.$$

For a fixed r let us consider the processes $\{U_z^{i,j}(r),\ z \geq r\}$ solution of

$$U_z^{i,j}(r) = A_j^i(X_r) + \int_{[r,z]} [\frac{\partial A_h^i}{\partial x_k}(X_s)\, U_s^{k,j}(r)\, dW_s^h + \frac{\partial B^i}{\partial x_k}(X_s)$$

$$.U_s^{k,j}(r)\, ds\,] \qquad (2.2)$$

$i = 1,\dots,m;\ j = 1,\dots,d$.

Applying lemma 2.2 to the processes $\{X_z^{n,i},\ U_z^{n,i,j}(r);\ i=1,\dots,m;\ j=1,\dots,d;\ z \geqslant r\}$ and $\{X_z^i, U_z^{i,j}(r);\ i=1,\dots,m;\ j=1,\dots,d; z \geqslant r\}$ we obtain

$$\sup_{r \in T} E\{\sup_{z \geqslant r} |U_z(r) - U_z^{(n)}(r)|^p\} \xrightarrow[n\to\infty]{} 0 .$$

In consequence, $\{DX_z^{n,i},\ n \geqslant 1\}$ is a Cauchy sequence in L_H^p for any $p \geqslant 1$, and the derivative $DX_z = (DX_z^1,\dots,DX_z^m)$ satisfies

$$< DX_z^i,\ h >_H = \int_{R_z} \sum_{j=1}^{d} U_z^{i,j}(r)\ \dot{h}^j(r)\ dr\ , \quad \text{a.s.}, \qquad (2.3)$$

for any $h \in H$.

It remains to prove that $\{LX_z^{n,i},\ n \geqslant 1\}$ is a Cauchy sequence in L^p for all $p \geqslant 1$, $i = 1,\dots,m$. Applying the differentiation rules we have

$$LX_z^{n,i} = \beta_n^i(z) + \int_{R_z} [\frac{\partial A_j^i}{\partial x_k}(X_{\phi_n(s)}^{(n)})\ LX_{\phi_n(s)}^{n,k}\ dW_s^j + \frac{\partial B^i}{\partial x_k}(X_{\phi_n(s)}^{(n)})$$

$$.LX_{\phi_n(s)}^{n,k}\ ds\ ] ,$$

where,

$$\beta_n^i(z) = \int_{R_z} \{ [\frac{\partial^2 A_j^i}{\partial x_k \partial x_h}(X_{\phi_n(s)}^{(n)}) < DX_{\phi_n(s)}^{n,k},\ DX_{\phi_n(s)}^{n,h} >_H$$

$$- A_j^i(X_{\phi_n(s)}^{(n)})\ ] dW_s^j + \frac{\partial^2 B^i}{\partial x_k \partial x_h}(X_{\phi_n(s)}^{(n)}) < DX_{\phi_n(s)}^{n,k},\ DX_{\phi_n(s)}^{n,h} >_H ds \}.$$

Consider now the stochastic differential system

$$LX_z^i = \beta^i(z) + \int_{R_z} [\frac{\partial A_j^i}{\partial x_k}(X_r) LX_r^k\ dW_r^j + \frac{\partial B^i}{\partial x_k}(X_r)\ LX_r^k\ dr\ ],$$

with

$$\beta^i(z) = \int_{R_z} \{ [\frac{\partial^2 A_j^i}{\partial x_k \partial x_h}(X_r) < DX_r^k,\ DX_r^h >_H - A_j^i(X_r)]\ dW_r^j$$

$$+ \frac{\partial^2 B^i}{\partial x_k \partial x_h}(X_r) < DX_r^k,\ DX_r^h >_H dr \}.$$

It can be checked that $E\{\sup_{z \in T} |\beta_n(z) - \beta(z)|^p\} \xrightarrow[n\to\infty]{} 0$ for any $p \geq 1$, and therefore, that $\{LX_z^{n,i}, n \geq 1\}$ is a Cauchy sequence in L^p, by lemma 2.2 applied to the processes $\{X_z^{n,i}, LX_z^{n,i}, i=1,\dots,m; z \in T\}$ and $\{X_z^i, LX_z^i, i = 1,\dots,m; z \in T\}$. □

<u>Theorem 2.4</u>. Let $X_z = (X_z^1,\dots,X_z^m)$ be the solution of the stochastic differential system (0.1), where A_j^i and B^i have bounded and continuous derivatives of any order. Assume further that the following property holds:

(P) The vector space spanned by the vector fields $A_1,\dots,A_d$, $A_i^\nabla A_j$, $1 \leq i,j \leq d$, $A_i^\nabla (A_j^\nabla A_k)$, $1 \leq i,j,k \leq d,\dots$, at the point x has full rank.

Then for any point (s,t) with $st \neq 0$, the law of the random vector X_{st} admits a density function.

<u>Proof</u>. We have to check conditions (i), (ii) and (iii) of theorem 1.1. The first condition follows from lemma 2.3. Let $U_z(r)$ be the process introduced in the proof of lemma 2.3, which satisfies (2.2), and call $S_{ij} = \langle DX_z^i , DX_z^j \rangle_H$. Then, from (2.3) we have

$$S_{ij} = \int_{R_z} \sum_{k=1}^{d} (\xi_1^i(r,z) A_k^1(X_r) \xi_{1'}^j(r,z) A_k^{1'}(X_r)) dr,$$

where , for any r, the process $\{ \xi_j^i(r,z), z \geq r\}$ is defined as the solution of the stochastic differential system

$$\xi_j^i(r,z) = \delta_j^i + \int_{[r,z]} [\frac{\partial A_h^i}{\partial x_k} (X_u) \xi_j^k(r,u) dW_u^h + \frac{\partial B^i}{\partial x_k} (X_u) \xi_j^k(r,u) du] .$$

By Burkholder and Hölder inequalities and Gronwall's lemma it is easy to obtain the following estimate:

For all $r,r' \leq z$ and $p > 2$, $E\{ |\xi(r,z) - \xi(r',z)|^p\} \leq C |r - r'|^{p/2}$ (2.4)

Therefore, by means of Kolmogorov's continuity criterium, a version of $\{ \xi(r,z), r \leq z \}$ can be choosen with almost surely continuous paths.

For each $n \geq 1$ let $\xi_j^{n,i}(r,z)$ be the process defined recursively by the equation

$$\xi_j^{n,i}(r,z) = \delta_j^i + \int_{[\psi_n(r)\wedge z,z]} [\frac{\partial A_h^i}{\partial x_k}(X^{(n)}_{\phi_n(s)})\ \xi_j^{n,k}(r,\ \phi_n(s))\ dW_s^h$$

$$+ \frac{\partial B^i}{\partial x_k}(X^{(n)}_{\phi_n(s)})\ \xi_j^{n,k}(r,\phi_n(s)\ ds]\ .$$

The random variables $\{\ \xi_j^{n,i}(r,z),\ n \geq 1\}$ are smooth functionals and by a slight modification of lemma 2.3 one can see that they form an approximating sequence for $\xi_j^i(r,z)$, and that this approximation is uniform in r. The same conclusion is true for the random variables $\{A_j^i(X^{(n)}_{\phi_n(r)}),\ n \geq 1\}$ and $A_j^i(X_r)$.

Therefore, after having noticed that $\{S_{ij}^n = \int_{R_z} \sum_{k=1}^{d} (\ \xi_1^{n,i}(r,z)\ A_k^1(X^{(n)}_{\phi_n(r)})\ \xi_{1'}^{n,j}(r,z)\ A_k^{1'}(X^{(n)}_{\phi_n(r)})\ )\ dr,\ n \geq 1\}$ is a sequence of smooth functionals we deduce that $\{S_{ij}^n,\ n \geq 1\ \}$ is an approximating sequence for S_{ij} and so, condition (ii) of theorem 1.1 holds.

Set $C_o = \{A_k,\ 1 \leq k \leq d\ \}$ and for $j \geq 1$, $C_j = \{A_k^{\nabla} A,\ 1 \leq k \leq d,\ A \in C_{j-1}\}$. By property (P) there exists a positive integer j_o such that the linear span of $\bigcup_{j=0}^{j_o} C_j$ at the point x has dimension m.

For each $\sigma \leq s,\ \omega \in \Omega$, denote by $K_\sigma(\omega)$ the linear span of $\{A_k(X_{x't}(\omega)\ ),\ x' \leq \sigma\ ,\ k=1,\dots,d\}$, and $K_{0^+}(\omega) = \bigcap_{\sigma \leq s} K_\sigma(\omega)$. We point out the following facts: (a) $K_\sigma(\omega)$ increases with σ. (b) By the Blumenthal zero-one law, $K_{0^+}(\omega)$ does'nt depend on ω a.s. (c) Let $\rho = \inf\ \{\sigma\ ,\ \dim\ K_\sigma > \dim K_{0^+}\}$. ρ is a strictly positive stopping time with respect to the filtration $\{F_{\sigma t},\ 0 \leq \sigma \leq s\}$, and $\forall \sigma < \rho(\omega)$, $K_\sigma(\omega) = K_{0^+}(\omega)$.

Assume that condition (iii) of theorem 1.1 is not satisfied, that means $P\{\ \omega\ ,\ \inf_{|\lambda|=1} \lambda^t S(\omega)\lambda = 0\} > 0$, where $S = (S_{ij})_{i,j=1,\dots,m}$, and consequently $P\{\exists\lambda,\ |\lambda|=1\ ,\ \int_{R_z}\sum_{k=1}^{d}(\lambda_i\xi_1^i(r,z)\ A_k^1(X_r))^2\ dr = 0\} > 0.$

Then, using (2.4) we have

$$P\{\exists\lambda,\ |\lambda|=1,\ \sum_{k=1}^{d}(\lambda_i\ A_k^i(X_{\sigma t}))^2 = 0, \forall\sigma \leq s\} > 0.$$

By (b) and (c) this implies that there exists λ, $|\lambda| = 1$ such that

$$P\{\ \omega\ ,\ \lambda \text{ ortogonal to } K_\sigma(\omega)\ ,\forall\sigma < \rho(\omega)\} > 0\ ,$$

in consequence $\lambda_i A_k^i(x) = 0$ for all $k=1,\dots,d$.

Applying Itô's formula in the first coordinate (see [10]), we have

$$A_k^i(X_{\sigma t}) = A_k^i(x) + \int_{R_{\sigma t}}\{\frac{\partial A_k^i}{\partial x_1}(X_{\upsilon t})\ A_h^1(X_{\upsilon\tau})\ dW_{\upsilon\tau}^h + [\frac{\partial A_k^i}{\partial x_1}(X_{\upsilon t})\ B^1(X_{\upsilon\tau})$$
$$+\frac{1}{2}\frac{\partial^2 A_k^i}{\partial x_1\partial x_r}(X_{\upsilon t})\sum_{j=1}^{d}A_j^1(X_{\upsilon\tau})\ A_j^r(X_{\upsilon\tau})]\ d\upsilon\ d\tau\}\ .$$

For any $k=1,\dots,d$, $\{\lambda_i A_k^i(X_{\sigma\wedge\rho,t})\ ,\ \sigma\leq s\}$ is a continuous semimartingale, which is equal to zero on a set of positive probability, in consequence on this set

$$\int_{R_{\sigma t}}(\lambda_i\frac{\partial A_k^i}{\partial x_1}(X_{\upsilon t})\ A_h^1(X_{\upsilon\tau}))^2\ d\upsilon\ d\tau = 0,$$

$\forall h,k = 1,\dots,d, \forall\sigma \leq \rho.$

In particular, $\lambda_i\frac{\partial A_k^i}{\partial x_1}(X_{0t})\ A_h^1(X_{0\tau}) = \lambda_i(A_h^\nabla A_k)^i(x) = 0, \forall h,k = 1,\dots,d.$

Repeating the same argument as before to the continuous semimartingale $\{\lambda_i\frac{\partial A_k^i}{\partial x_1}(X_{\upsilon\wedge\rho,t})\ A_h^1(X_{\upsilon\wedge\rho,t}),\ \upsilon\leq s\}$, we show that $\lambda_i A^i(x) = 0$ for all $A\in C_2$, and recursively, $\lambda_i A^i(x) = 0$, for all $A \in C_j$, $j \geq 2$, which is contradictory with the hypothesis (P). □

In the one parameter case, the existence of a density for the solution of a stochastic differential equation can be proved under Hörmander's conditions. Suppose that the vector fields $B, A_1,\dots,A_d$ have bounded derivatives of any order greater than or equal to one. Then the following assumption suffices for the existence of a density:

(H) The vector space spanned by $A_1,\dots,A_d$, $[A_i,A_j]$, $1 \leqslant i,j \leqslant d$, $[A_i,[A_j,A_k]]$, $1 \leqslant i,j,k \leqslant d,\dots$, at the point x is R^m.

Actually a more general condition, using the Lie brackets formed with the vector field B as generators would be sufficient.

Condition (P) is weaker than (H) and, in fact, theorem 2.4 can be applied to a family of situations that did not appear in the one parameter case. Consider, for instance, the following example. Assume that m=2, d=1 $A_1^1 = 1$, $A_1^2 = x^1$, and $x = (0,0)$. Then condition (H) does not hold, and the one parameter solution to $X_t^1 = W_t^1$, $X_t^2 = \int_0^t W_s^1\, dW_s^1 = \frac{1}{2}[(W_t^1)^2 - t]$ satisfies $2X_t^2 = (X_t^1)^2 - t$. However, in the two-parameter case, theorem 2.4 can be used, and, for st $\neq 0$, the joint distribution of the random variables $X_{st}^1 = W_{st}^1$, $X_{st}^2 = \int_{R_{st}} W_z^1\, dW_z^1$ has a density on R^2. Remark that here the stochastic differentiation rules (cf. [10]) claim that $X_{st}^2 = -\int_{R_{st}\times R_{st}} 1_D(z,z')\, dW_z\, dW_{z'} + \frac{1}{2}[(W_{st}^1)^2 - st]$, and X_{st}^2 is not a function of X_{st}^1.

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