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EDWARD JAMES MCSHANE

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EXISTENCE THEOREMS FOR ORDINARY PROBLEMS OF THE CALCULUS OF VARIATIONS

(PART I)

by EDWARD JAMES MCSHANE (Göttingen).

The search for existence theorems for the parametric problem of the calculus of variations has resulted in the finding of theorems of a highly satisfactory generality, for which TONELLI and HAHN are chiefly to be thanked. However, the analogous problem for the ordinary, or non-parametric problem has not been advanced to a comparable state of completeness. TONELLI (1) has obtained existence theorems for problems

$$\int F(x, y, y')dx = \min.$$

under the assumption that $F(x, y, y') \geqq M|y'|^{\alpha} - N$, where $M > 0$ and $\alpha > 1$; and these theorems have been extended to space by GRAVES (2). But these theorems fail to apply to many of the most interesting non-parametric problems, for example, to the rather frequently occuring problem

$$\int \varphi(y)\sqrt{1+y'^2}dx = \min.$$

Hence for space problems there is need of a considerable extension of known results. For the plane problem there are existence theorems (3) for integrals not satisfying the condition $F(x, y, y') \geqq M|y'|^{\alpha} - N$. It seems that these theorems are not all in reach of the method here developed. On the other hand, we here obtain theorems not previously known, applying for example to the problem of the brachistochrone.

In the present paper the properties of semi-continuity of ordinary integrals and the existence theorems for ordinary problems are found by a detour through the more complete theory of the parametric problem. Given an ordinary integral

$$I[y] = \int F(x, y, y')dx,$$

(1) TONELLI: *Fondamenti di Calcolo delle Variazioni*, Vol. II, pp. 281-307.

(2) L. M. GRAVES: *On the Existence of the Absolute Minimum*, etc., Annals of Mathematics, Vol. 28 (1927), pp. 153-170.

(3) TONELLI, loc. cit., Vol. II, p. 370. — TONELLI, Rendiconti della R. Acc. dei Lincei, 1° sem. 1932. — M. NAGUMO, Japanese Journal of Math., Vol. VI (1929), p. 173.

we can find a parametric integral

$$J[C]=\int G(x, y, x', y')ds$$

which for all curves $y=y(x)$, say with continuous derivatives, agrees with $I[y]$. This integral $J[C]$ is not a parametric integral of the usual type, for G is not defined when $x'<0$. Nevertheless, the integral $J[C]$ can be handled by suitable modifications of the methods used for parametric integrals. We study the integral $J[C]$, not as a functional on the class K of curves $y=y(x)$ with absolutely continuous $y(x)$, but on the larger class $\overline{K}$ of rectifiable curves $x(s)$, $y(s)$ with $x'(s)\geqq 0$. On this class we establish the semi-continuity (under suitable hypotheses) of $J[C]$, and we find in $\overline{K}$ a minimizing curve for $J[C]$. The problem then reduces to that of finding hypotheses under which this minimizing curve is actually a curve of K, with a representation $y=y(x)$ in which $y(x)$ is absolutely continuous. Stated in this terminology, the method by which TONELLI obtains most of his theorems (that of Vol. II, p. 370, being excepted) requires proving that all the curves C of $\overline{K}$ for which $J[C]<M$ (M=constant) are also curves $y=y(x)$ of K, and are in fact equi-continuous. The present method requires only that the minimizing curve in $\overline{K}$ belong to K, and for this purpose weaker hypotheses suffice.

In the first part of the paper we develop the properties of the integral $J[C]$ and its relation to the ordinary integral $I[y]$. Then a method is developed by which the integrand $G(x, y, x', y')$ can be approximated by functions possessing the properties of the parametric integrands usually studied. This leads to theorems on the semi-continuity of the functional $J[C]$; these theorems imply corollaries concerning the semi-continuity of $I[y]$ which (for problems in space) are stronger than any in the literature. In the second part of the paper (to be published later) these theorems on the semi-continuity of $J[C]$ are utilized in establishing the existence of a minimizing curve of $J[C]$ in the extended class $\overline{K}$. An immediate consequence of this is a theorem on the existence of a solution of the problem $I[y]=\min.$ which is more general than the theorem of TONELLI cited in footnote 1, and also includes NAGUMO's theorem (footnote 3). A theorem is proved which permits the extension of the results to the case in which the field A containing the class K of curves is unbounded; the hypotheses here are likewise weaker than in corresponding theorems in the literature.

We then proceed to consider more restricted classe of curves K; K is no longer a « complete class » in the sense of TONELLI, but is the class of all absolutely continuous curves $y=y(x)$ joining two given points or two given closed point sets. Also the field A in which the curves lie is restricted to be a « cylindrical set »; for example, a rectangle in the plane or a parallelopiped in space. On the other hand, the previous hypothesis $G(x, y, x', y')\rightarrow+\infty$ as $x'\rightarrow 0$ is replaced by the weaker hypothesis $\frac{\partial}{\partial x'}G(x, y, x', y')\rightarrow-\infty$ as $x'\rightarrow 0$. Proceeding further in this direction, an existence theorem is found which applies, in particular, to the problem

$$\int \varphi(y) \sqrt{1+y'^2}\,dx = \text{min.}$$

for continuous positive $\varphi(y)$, and also to the brachistochrone problem.

It is to be hoped that the study of the integral $J[C]$ presented in § 2 will be useful to other students of this problem, as it has already been of use to me in related questions (4).

§ 1. - Functions and Curves.

Throughout this paper we shall be concerned with the study of a functional

$$I[y] = \int F(x, y^1, \ldots, y^q, y^{1'}, \ldots, y^{q'})\,dx.$$

All of our theorems and methods are equally valid in space of any number of dimensions; but for simplicity of terminology we shall discuss only the case in which there are two dependent variables ($q=2$). The extension to the general case is quite obvious.

The symbols y and y' shall be used to denote the pairs (y^1, y^2) and $(y^{1'}, y^{2'})$ respectively. Correspondingly, we shall often use the words « the function $y(x)$ » to denote « the pair of functions $y^1(x)$, $y^2(x)$ ». We shall use a modification of the tensor summation convention; the repetition of a *Greek letter* suffix in any term shall denote the summation over all values of that suffix. Thus $y^\alpha_n \eta^\alpha_n = y^1_n \eta^1_n + y^2_n \eta^2_n$, the summation extending over all values of α but not over n.

Since we shall attack the ordinary problem through the parametric problem, it is useful to have an alternative notation in which no one axis is singled out. Consequently the coordinate axes will be given two notations; first (x, y^1, y^2), as already mentioned, and second (z^0, z^1, z^2). Thus $x=z^0$, $y^1=z^1$, $y^2=z^2$. We shall feel free to substitute the symbol z or (z^0, z^1, z^2) for the symbol (x, y) or (x, y^1, y^2) wherever it is convenient.

As a result of this convention some suffixes will have the ranges (1, 2) and others the range (0, 1, 2). In each case the letter bearing the suffix will indicate the range. But as a safeguard against confusion we shall use the letters α, β, i when the range is (1, 2) and λ, μ, j when the range is (0, 1, 2).

In dealing with parametric problems unit vectors z' occur so frequently as to deserve a special symbol. We shall connote that a vector is a unit vector by attaching the subscript u; thus the symbol $\bar{z}'_u$ denotes a vector such that $\bar{z}^{\lambda'}_u \bar{z}^{\lambda'}_u = 1$.

Being given two sets of continuous functions $y_1(x)$, $a_1 \leqq x \leqq b_1$, and $y_2(x)$,

(4) *The Du Bois-Reymond Relation in the Calculus of Variations*, to be published in Math. Annalen; *Über die Unlösbarkeit eines einfachen Problems der Variationsrechnung*, to be published in Nachr. Ges. Wiss. Göttingen.

$a_2 \leqq x \leqq b_2$, we define the distance dist (y_1, y_2) in the following way: First we define $y_1(x) = y_1(a_1)$ for $x < a_1$ and $y_1(x) = y_1(b_1)$ for $x > b_1$, and likewise we define $y_2(x) = y_2(a_2)$ for $x < a_2$ and $y_2(x) = y_2(b_2)$ for $x > b_2$. The distance dist (y_1, y_2) is then defined to be the greatest of the three numbers:

1°) $|a_1 - a_2|$;

2°) $|b_1 - b_2|$;

3°) the maximum distance between the points $(x, y_1(x))$ and $(x, y_2(x))$ for $-\infty < x < +\infty$.

If the functions $y_0(x)$, $y_1(x)$,.... are continuous, we define $\lim_{n\to\infty} y_n(x) = y_0(x)$ to mean $\lim_{n\to\infty}$ dist $(y_0, y_n) = 0$.

For curves in parametric form we use the following definition of distance: Let the curves C_1 and C_2 be defined by the respective equations C_1: $z = z_1(t)$, $a_1 \leqq t \leqq b_1$ and C_2: $z = z_2(\tau)$, $a_2 \leqq \tau \leqq b_2$, where $z_1(t)$, $z_2(\tau)$ are triples of continuous functions. The distance dist (C_1, C_2) is defined to be the number d with the following properties:

1°) for every $\varepsilon > 0$ there exists a topological mapping of (a_1, b_1) on (a_2, b_2) with preservation of sense (that is, a continuous monotonic function $\tau(t)$ such that $\tau(a_1) = a_2$, $\tau(b_1) = b_2$) for which all corresponding points $z_1(t)$ and $z_2(\tau(t))$ have distance less than $d + \varepsilon$;

2°) there exists no such correspondence for which all corresponding points have distance less than d. Correspondingly, we say that a sequence of curves $\{C_n\}$ approaches a curve C_0 as limit if $\lim_{n\to\infty}$ dist $(C_0, C_n) = 0$.

We readily find that

$$\begin{aligned} &\text{dist}\,(y_1, y_2) = \text{dist}\,(y_2, y_1), \\ &\text{dist}\,(y_1, y_2) + \text{dist}\,(y_2, y_3) \geqq \text{dist}\,(y_1, y_3), \\ &\text{dist}\,(C_1, C_2) = \text{dist}\,(C_2, C_1), \\ &\text{dist}\,(C_1, C_2) + \text{dist}\,(C_2, C_3) \geqq \text{dist}\,(C_1, C_3). \end{aligned}$$

We also recognize that dist $(y_1, y_2) = 0$ if and only if y_1 is identically equal to y_2. The equation dist $(C_1, C_2) = 0$ we accept as the *definition* of the identity of the curves C_1 and C_2.

The following interrelation of these definitions will be needed later: If the continuous functions $y_n(x)$, $a_n \leqq x \leqq b_n$, tend to the continuous function $y_0(x)$, $a_0 \leqq x \leqq b_0$, then the curves C_n: $x = t$, $y = y_n(t)$, $a_n \leqq t \leqq b_n$ tend to the curve C_0: $x = \tau$, $y = y_0(\tau)$, $a_0 \leqq \tau \leqq b_0$. For the curves C_n, C_0 we use the alternative notation $z = z_n(t)$, $z = z_0(\tau)$ respectively. Let us extend the range of definition of the functions $z_0(\tau)$ by defining [5] $z_0(\tau) = (\tau, z_0^1(a_0), z_0^2(a_0))$ for $\tau < a_0$ and $z_0(\tau) = (\tau, z_0^1(b_0), z_0^2(b_0))$ for $\tau > b_0$. Then the functions $z_0(\tau)$ are uniformly continuous for all τ; that is, the inequality $|\tau_1 - \tau_2| < \delta$ implies dist $(z_0(\tau_1), z_0(\tau_2)) < \omega(\delta)$,

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[5] That is, $z_0^0(\tau) = x = \tau$, $z_0^i(\tau) = z_0^i(a_0) = y_0^i(a_0)$, $i = 1, 2$.

where $\omega(\delta)$ approaches 0 with δ. Let $\tau=\tau(t)$ be a linear mapping of the interval (a_n, b_n) on (a_0, b_0); it is readily seen that $|t-\tau(t)|\leqq|a_n-a_0|+|b_n-b_0|$. Then

$$\text{dist}\,(C_0, C_n)\leqq\max \text{dist}\,(z_0(\tau(t), z_0(t))+\max \text{dist}\,(z_0(t), z_n(t))$$
$$\leqq\omega(|a_n-a_0|+|b_n-b_0|)+\max\{[y_0^1(t)-y_n^1(t)]^2+[y_0^2(t)-y_n^2(t)]^2\}^{\frac{1}{2}}.$$

As $n\to\infty$ the right member of this inequality approaches 0, proving our statement.

§ 2. - The Integrand and the Associated Parametric Functional.

We shall assume throughout that the integrand satisfies the following conditions:

$$F(x, y, y') \tag{2.1}$$

is continuous, together with all its partial derivatives of first and second order, for all points (x, y) *of a closed set* A *in* $(q+1)$*-dimensional space and for all finite values of* $y^1,\ldots, y^q$.

The frequently occurring partial derivatives with respect to $y^{1'}$, $y^{2'}$ will be denoted by F_1, F_2, respectively; likewise the second partial derivative of F with respect to $y^{i'}$ and $y^{h'}$ will be denoted by F_{ih} $(i, h=1, 2)$.

In accordance with the usual terminology we shall say that *the functional*

$$I[y]=\int F(x, y, y')dx$$

is positive quasi-regular on A *if the Weierstrass E function* ([6])

$$E(x, y, y', \overline{y}')\equiv F(x, y, \overline{y}')-F(x, y, y')-(\overline{y}^{\alpha'}-y^{\alpha'})F_\alpha(x, y, y')$$

is non-negative for all (x, y) *on* A *and all* y' *and* $\overline{y}'$.

It is well known that $I[y]$ is quasi-regular on A if and only if the inequality $y^{\alpha'}y^{\beta'}F_{\alpha\beta}(x, y, \overline{y}')\geqq 0$ holds for all (x, y) on A and all y' and $\overline{y}'$.

We shall say that $I[y]$ is *positive quasi-regular semi-normal on* A if it is positive quasi-regular on A, and if moreover to each point (x, y) of A there corresponds a vector y_0' such that $E(x, y, y_0', \overline{y}')>0$ for all vectors $\overline{y}'\neq y_0'$. In the plane case $(q=1)$, this is equivalent to TONELLI's definition.

Our method of procedure will be to study the integral $I[y]$ indirectly, by means of the *associated parametric* functional

$$J[C]=\int_C G(x, y, x', y')dt=\int_C G(z, z')dt.$$

In this functional the integrand G (the *associated parametric integrand*) is defined for $x'>0$ by the equation

$$G(x, y, x', y')=x'F\left(x, y, \frac{y'}{x'}\right). \tag{2.2}$$

([6]). α here ranges over the numbers 1, 2.

It is evident that for $x'>0$ the function G is positively homogeneous of degree 1 in (x', y'), and that G is continuous together with all its first and second partial derivatives for (x, y) on A and all (x', y'). On introducing the symbols

$$G_0(z, z') = G_{z^{0'}}(z, z') = G_{x'}(x, y, x', y'),$$
$$G_i = G_{z^{i'}}(z, z') = G_{y^{i'}}(x, y, x', y') \qquad (i=1, 2),$$

we find immediately

$$G_i(z, z') = G_i(x, y, x', y') = F_i\left(x, y, \frac{y'}{x'}\right) \qquad (i=1, 2), \tag{2.3}$$

$$G_0(z, z') = G_0(x, y, x', y') = F\left(x, y, \frac{y'}{x'}\right) - \frac{1}{x'} y^{\alpha'} F_\alpha\left(x, y, \frac{y'}{x'}\right). \tag{2.4}$$

Also if we define as usual (7)

$$\mathcal{E}(z, z', \bar{z}') = G(z, \bar{z}') - \bar{z}^{\lambda'} G_\lambda(z, z')$$

we find with little calculation that

$$\mathcal{E}(z, z', \bar{z}') = \bar{x}' E\left(x, y, \frac{y'}{x'}, \frac{\bar{y}'}{\bar{x}'}\right).$$

Thus if $I[y]$ is quasi-regular, $\mathcal{E}(z, z', \bar{z}')$ is non-negative for all $z=(x, y)$ on A, all y' and $\bar{y}'$, and all positive values of x' and $\bar{x}'$. We prefer not to speak of G as quasi-regular; we shall instead say as usual that an integral

$$\int G^*(z, z')\,dt$$

is positive quasi-regular if G^* satisfies the usual conditions of continuity and homogeneity for *all* z' and if further its $\mathcal{E}$-function, $\mathcal{E}^*(z, z', \bar{z}')$, is non-negative for all z in A, *all* $\bar{z}'$, and *all* $z' \neq (0, 0, 0)$.

In case $I[y]$ is quasi-regular, we can extend the definition of $G(z, z')$ to include $z^{0'}=x'=0$. For then

$$G_{00}(z, z') = G_{x'x'}(x, y, x', y') = \frac{1}{x'^3}\left(y^{\alpha'} y^{\beta'} F_{\alpha,\beta}\left(x, y, \frac{y'}{x'}\right)\right) \geqq 0 \tag{2.5}$$

for $x'>0$. Hence the function $G(x, y, x', y')$, regarded as a function of x' alone for fixed x, y, y', is convex, and as x' approaches zero $G(x, y, x', y')$ must approach a finite limit or $+\infty$. This limit, finite or infinite, we accept as the definition of $G(x, y, 0, y')$.

The function $G(z, z')$ thus defined may fail to be continuous for $z^{0'}=0$; but we can prove

Lemma 2.1. - *If $I[y]$ is positive quasi-regular on A, the associated parametric integrand $G(z, z')$, defined for all z on A and all z' with $z^{0'} \geqq 0$, is a lower semi-continuous function of its six arguments.*

For arguments (z, z') with $z^{0'}>0$ the function $G(z, z')$ is continuous, and a fortiori lower semi-continuous. It remains only to show that if $z_0^{0'}=0$, for every

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(7) λ here has the range 0, 1, 2.

sequence (z_n, z'_n) of arguments tending to (z_0, z_0') and having z_n in A and $z^{0'} \geqq 0$ the inequality

$$\liminf G(z_n, z'_n) \geqq G(z_0, z_0') \tag{2.6}$$

holds. We give the proof for the case $G(z_0, z_0')$ finite; the proof for $G(z_0, z_0') = +\infty$ requires only trivial modifications.

Since (z_n, z'_n) tends to (z_0, z_0'), there exists an M such that $|z_n^{j'}| \leqq M$ and $|z_n^j| \leqq M$ for all n. For the arguments

$$|z^j| \leqq M, \qquad z^{0'} = M, \qquad |z^{i'}| \leqq M \qquad (i=1, 2)$$

the function G_0 is continuous, hence bounded, say $\leqq N$. By (2.5), G_0 is a monotonic increasing function of $z^{0'}$, hence the inequality

$$G_0(z, z') \leqq N \tag{2.7}$$

holds for all arguments such that $|z^j| \leqq M$, $|z^{i'}| \leqq M$, $0 < z^{0'} \leqq M$.

Let now ε be any positive number. By the definition of $G(z_0, z_0')$ we can find a positive ν less than $\varepsilon/2N$ such that

$$G(z_0, \nu, z_0^{1'}, z_0^{2'}) \geqq G(z_0, z_0') - \frac{\varepsilon}{4}.$$

For all $(z, z^{1'}, z^{2'})$ in a neighborhood of $(z_0, z_0^{1'}, z_0^{2'})$ the inequality

$$G(z, \nu, z^{1'}, z^{2'}) > G(z_0, z_0') - \frac{\varepsilon}{2}$$

holds, for since ν is positive G is continuous. Hence by (2.7) we have

$$G(z, z') \geqq G(z_0, z_0') - \frac{\varepsilon}{2} - N\nu > G(z_0, z_0') - \varepsilon,$$

valid for all z, $z^{1'}$, $z^{2'}$ in a neighborhood of z_0, $z_0^{1'}$, $z_0^{2'}$ and all $z^{0'} \leqq \nu$. These conditions being satisfied for all but a finite number of the (z_n, z'_n), we have

$$\liminf G(z_n, z'_n) \geqq G(z_0, z_0') - \varepsilon\,;$$

and this holding for every $\varepsilon > 0$, inequality (2.6) is established.

By an exactly similar proof we can establish

Lemma 2.2. - *If $I[y]$ is positive quasi-regular on A and $G(z, z')$ is its associated parametric integrand, then $G_0(z, z')$ is an upper semi-continuous function of its six arguments.*

In the proof, inequality (2.7) is replaced by (2.5).

As a corollary of lemma 2.1, we have

Lemma 2.3. - *If A is bounded and closed and $I[y]$ is positive quasi-regular on A, then $G(z, z'_u)$ is bounded below for all z on A and all unit vectors z'_u with $z_u^{0'} \geqq 0$.*

For a lower semi-continuous function assumes its least value on a bounded closed set, and the values of G are all finite or $+\infty$.

In order to have the right to pass back and forth between the functional $I[y]$

and the associated parametric functional $J[C]$, we need to be able to recognize when a curve $z=z(t)$, $z^{0'}(t)\geqq 0$, can be represented in ordinary form by equations $y=y(x)$, where the $y(x)$ are absolutely continuous; and we also need to know that for curves C represented by equations $y=y(x)$, where the $y(x)$ are absolutely continuous, the two functionals are identical: $J[C]=I[y]$. For this purpose we establish a sequence of lemmas.

Lemma 2.4. - *Let C be a rectifiable curve $z=z(s)$, $z^{0'}(s)\geqq 0$, $0\leqq s\leqq L$, where s is the length of arc. In order that C can be represented in the form $y=\bar{y}(x)$, with absolutely continuous functions $\bar{y}(x)$, it is necessary and sufficient that $x'(s)=z^{0'}(s)>0$ except at most on a set of s of measure 0.*

For if C has such a representation, then

$$s(x)=\int_{x(0)}^{x}[1+\bar{y}^{\alpha'}\bar{y}^{\alpha'}]^{\frac{1}{2}}dx$$

is absolutely continuous. But the absolutely continuous monotic function $x=x(s)$ has an absolutely continuous inverse if and only if $x'(s)>0$ almost everywhere ([8]).

Conversely, if $x'(s)>0$ almost everywhere, $s(x)$ is absolutely continuous, by the theorem just cited. But each $y^i(s)$ is Lipschitzian; hence ([9]) $\bar{y}^i(x)\equiv y^i(s(x))$ is absolutely continuous.

Lemma 2.5. - *If*

a) the functions $y(x)$, $a\leqq x\leqq b$, are absolutely continuous and lie in A;

b) the equations $z=\zeta(s)$, $0\leqq s\leqq L$, are the parametric representation of the curve C: $y=y(x)$, $a\leqq x\leqq b$, with the length of arc s as parameter; then if either one of the integrals ([10])

$$I[y]=\int_a^b F(x, y, y')dx,$$

$$J[C]=\int_0^L G(\zeta, \zeta')ds$$

exists so does the other, and the two have the same value.

([8]) Carathéodory: *Vorlesungen über Reelle Funktionen*, p. 584. This book will henceforth be referred to as « Carathéodory ».

([9]) Carathéodory, p. 555.

([10]) It will be noticed that the integrand may fail to be defined for a set of measure 0. Here and henceforward we adopt the convention: If a function $f(x)$ is defined at all points of a set E except those belonging to a set N of measure 0, we define

$$\int_E f(x)dx\equiv\int_{E-N} f(x)dx,$$

provided that the latter integral exists.

Let us define

$$F^*(x) = \int_a^x F(x, y(x), y'(x))dx = \int_a^x G(x, y(x), 1, y'(x))dx.$$

This function is absolutely continuous, and $x(s)$ is absolutely continuous and monotic increasing; hence ([11]) $F^*(x(s))$ is an absolutely continuous function of x. This implies ([12]) that $G(\zeta(s), 1, y'(\zeta^0(s))) \cdot \zeta^{0\prime}(s)$ is summable, and

$$\int_0^L G(\zeta(s), 1, y'(\zeta^0(s))) \cdot \zeta^{0\prime}(s)ds = \int_a^b F(x, y, y')dx = I[y].$$

But by the absolute continuity of $y^i(x)$, the derivatives $y'(x)$ exist except at a set of x of measure 0, which corresponds to a set of s of measure 0. Also, for almost all s the derivative of $\zeta^0(s)$ exists and is positive. Hence almost everywhere

$$y^{i\prime}(\zeta^0(s)) \cdot \zeta^{0\prime}(s) = \frac{dy^i}{ds} = \zeta^{i\prime}(s) \qquad (i=1, 2).$$

Therefore by the homogeneity of G we have

$$I[y] = \int_0^L G(\zeta(s), \zeta'(s))ds.$$

Conversely, if $J[C]$ exists, we define

$$G^*(s) = \int_0^s G(\zeta, \zeta')ds = \int_0^s G(\zeta(s), 1, y'(\zeta^0(s))) \cdot \zeta^{0\prime}(s)ds.$$

Since $s(x)$ is monotonic and absolutely continuous, $G^*(s(x))$ is absolutely continuous, and by the theorem cited we conclude that the integral

$$\int_a^b G(x, y(x), 1, y'(x)) \cdot x'(s(x)) \cdot s'(x)dx = \int_a^b F(x, y, y')dx$$

exists and has the value $J[C]$.

It is worthy of notice that we have not needed to suppose that $J[C]$ is quasi-regular, since $\zeta^{0\prime}(s) > 0$ almost everywhere, and for the remaining values of s we can for example set $G=0$.

Lemma 2.6. - *If $I[y]$ is positive quasi-regular on A, for every set of absolutely continuous functions $y(x)$, $a \leqq x \leqq b$, of A either $I[y]$ exists finitely*

([11]) Carathéodory, p. 556.

([12]) Carathéodory, p. 563.

or $I[y]=+\infty$. *Likewise for all rectifiable curves* C: $z=z(t)$, $z_0'\geqq 0$, *in* A *either* $J[C]$ *exists finitely or* $J[C]=+\infty$.

The function $F(x, y, y')$ is continuous for all (x, y) on the curves $y=y(x)$ and all finite y'; hence [13] $F(x, y(x), y'(x))$ is measurable. If we write

$$F(x, y, y')=E(x, y, 0, y')+F(x, y, 0)+y^{\alpha'}F_\alpha(x, y, 0)$$
$$\geqq F(x, y, 0)-|y^{\alpha'}F_\alpha(x, y, 0)|,$$

we notice that the expression on the right is summable, so that the integral from a to b of F is finite or $+\infty$.

Likewise for the arguments [z on C; z' such that $z^{0'}\geqq 0$] the function $G(z, z')$ is lower semi-continuous, hence [14] $G(z(t), z'(t))$ is measurable. Since G is bounded below for unit vectors z', say $G\geqq -M$, the integrand $G(z(t), z'(t))$ is at least $-M[z^{\lambda'}z^{\lambda'}]^{\frac{1}{2}}$, and its integral is finite or $+\infty$.

Lemma 2.7. - *If*

a) $I[y]$ *is positive quasi-regular on* A;

b) the functions $y(x)$, $a\geqq x\geqq b$, *are absolutely continuous and lie in* A;

c) the equations $z=\bar{z}(t)$, $t_0\leqq t\leqq t_1$, *form a representation of the curve* C: $y=y(x)$, $a\leqq x\leqq b$, *and the functions* $\bar{z}(t)$ *are absolutely continuous*; *then the integrals*

$$I[y]=\int_a^b F(x, y(x), y'(x))dx \tag{2.8}$$

and

$$J[C]=\int_{t_0}^{t_1} G(z(t), z'(t))dt \tag{2.9}$$

have the same value, finite or infinite.

We compare $I[y]$ and $J[C]$ with the integral

$$\int_0^L G(\zeta(s), \zeta'(s))ds, \tag{2.10}$$

where $z=\zeta(s)$ is the representation of C with length of arc as parameter. We first suppose that (2.10) is finite, and define

$$F^{**}(s)=\int_0^s G(\zeta(s), \zeta'(s))ds;$$

to the integrand we assign the value 0 on the set of measure 0 on which it is

(13) Carathéodory, p. 377.

(14) Carathéodory, p. 377.

undefined. Since the functions $z(t)$ are absolutely continuous, the length of arc s is a monotonic absolutely continuous function of t,

$$s=\int_{t_0}^{t}(z^{\lambda'}z^{\lambda'})^{\frac{1}{2}}dt.$$

Hence $F^{**}(s(t))$ is absolutely continuous ([15]).

Consequently ([16]), $G(\zeta(s(t)), \zeta'(s(t)))\frac{ds}{dt}$ is summable, and

$$\int_{t_0}^{t_1} G(\zeta(s(t)), \zeta'(s(t)))\frac{ds}{dt}\,dt=\int_{0}^{L} G(\zeta(s), \zeta'(s))ds.$$

Since the $\zeta^{j'}(s)$ are bounded we have

$$\zeta^{j'}(s(t))\cdot\frac{ds}{dt}=\frac{d\zeta^{j}(s(t))}{dt}=\bar{z}^{j'}(t)$$

for almost all t; hence, making use of the homogeneity of G,

$$\int_{t_0}^{t_1} G(\bar{z}(t), \bar{z}'(t))dt=\int_{0}^{L} G(\zeta(s), \zeta'(s))ds. \tag{2.11}$$

By lemma 2.5, this implies

$$I[y]=J[C]. \tag{2.12}$$

We now digress from our proof to make a remark which is not without interest. As yet we have made no use of the hypothesis of quasi-regularity. If for almost all t the inequality $\frac{ds}{dt}>0$ holds, so that the function $s=s(t)$ has an absolutely continuous inverse $t=t(s)$, we can repeat the above argument with interchange of z and ζ to prove that whenever $J[C]$ exists, so does the integral (2.10), and the two are equal. Hence using lemma 2.5, we see that if $\frac{ds}{dt}>0$ for almost all t, then if either one of the integrals $I[y]$, $J[C]$ exists, so does other, and the two are equal. This is true whether or not the integral is quasi-regular.

We return to the proof of the lemma. By lemma 2.6, the only case remaining to be considered is that in which the integral (2.10) has the value $+\infty$. Suppose then that this is so.

For all unit vectors z_n with $z_u^{0'}\geqq 0$ we define

$$G_N(z, z'_u)=(\text{smaller of } G(z, z'_u) \text{ and } N),$$

([15]) CARATHÉODORY, p. 556.

([16]) CARATHÉODORY, p. 563.

and we extend the definition of G_N to all vectors z' with $z^{0'} \geqq 0$ by homogeneity. The previous proof then applies ([17]) to show that

$$\int_{t_0}^{t_1} G_N(\bar{z}(t), \bar{z}'(t))dt = \int_0^L G_N(\zeta, \zeta')ds.$$

Letting $N \to \infty$, the integral on the right tends to ∞, hence so does the integral on the left. But for every N this last integral is less than $J[C]$, hence $J[C] = +\infty$. Since by lemmas 2.5 and 2.6 we have $I[y] = +\infty$, equation (2.12) holds in this case also, and the proof of the lemma is complete.

As a final remark, whenever we speak of a rectifiable curve C: $z = z(t)$, $a \leqq t \leqq b$, it will always be understood without further mention that the functions $z(t)$ are absolutely continuous. Under this assumption the statements « $z^0(t)$ is monotonic non-decreasing » and « $z^{0'}(t) \geqq 0$ wherever it is defined » are equivalent. The latter statement we shall abbreviate to « $z^{0'}(t) \geqq 0$ ».

§ 3. - The Figuratrix.

Let us suppose that $G(z, z')$ is a parametric integrand, continuous together with its derivatives G_j for all z on A and all $z' \neq (0, 0, 0)$ and positively homogeneous of degree 1 in z', and that for a fixed point z the inequality $G(z, z') > 0$ holds for all $z' \neq (0, 0, 0)$. The equation $w = G(z, z')$ then represents in four dimensional space a conical surface with vertex at the origin, while the inequality $w \geqq G(z, z')$ represents the interior of this cone. We shall be particularly interested in the intersection of the cone with the hyperplane $w = 1$. This is easily obtained in terms of the function G. For let z'_u be any unit vector, and let $w_u = G(z, z'_u)$. Since by hypothesis $w_u > 0$, from the homogeneity of G it follows that

$$1 = G\left(z, \frac{z'_v}{w_u}\right).$$

Therefore if we proceed in the direction of the unit vector z'_u we find that at a distance $\frac{1}{w_u}$ from the origin we reach a point $\frac{z'_u}{w_u}$ for which cone and plane intersect.

Thus the intersection of cone and plane can be specified in polar coordinates (r, z'_u) by saying that in the direction of the unit vector z'_u the radius vector is

$$r(z'_u; z) = \frac{1}{G(z, z'_u)}.$$

The figure defined by the equation we shall call the *figuratrix* ([18]) of $G(z, z')$ at the place z.

([17]) We recall that $G(z, z'_u)$ is already known to be bounded below (Lemma 2.3).

([18]) This is not the same as the figuratrix of TONELLI, which is the cone itself.

The utility of the figuratrix lies in the fact that *the figuratrix is a convex surface if and only if $G(z, z')$ is quasi-regular at the point z.* For suppose that G is quasi-regular at z. At an arbitrary point z' the tangent hyperplane to the cone $w = G(z, z')$ is given by the equation

$$w = \overline{w}(\overline{z}') = G(z, z') + (\overline{z}^{\lambda'} - z^{\lambda'}) G_\lambda(z, z') = \overline{z}^{\lambda'} G_\lambda(z, z'), \tag{3.1}$$

where the $\overline{z}'$ are the running coordinates. Then at any point $\overline{z}'$ we have

$$G(x, y, \overline{x}', \overline{y}') - \overline{w}(\overline{z}') = \mathcal{E}(z, z', \overline{z}') \geqq 0,$$

so that the cone $w \geqq G(z, \overline{z}')$ lies entirely within the half-space $w \geqq \overline{w}(\overline{z}')$ above the tangent hyperplane. Considering now the intersection of these figures with the hyperplane $w = 1$, we find that the intersection of the cone with the hyperplane $w = 1$ — that is, the figuratrix — lies entirely to one side of the intersection of the tangent hyperplane with the hyperplane $w = 1$ — that is, the plane tangent to the figuratrix. But this tangent plane space can be chosen to be any arbitrary tangent plane. Hence the figuratrix lies entirely to one side of any one of its tangent planes, and is therefore convex.

Conversely, suppose that the figuratrix is convex. Choosing an arbitrary tangent hyperplane to the cone $w = G(x, y, x', y')$, say $w = \overline{z}^{\lambda'} G_\lambda(z, z')$, the intersection of this hyperplane with $w = 1$ is a plane tangent to the figuratrix. The figuratrix lies entirely to one side of this tangent space, hence the cone lies entirely to one side of the tangent plane, and

$$G(z, \overline{z}') - \overline{z}^{\lambda'} G_\lambda(z, z')$$

never changes sign. Since this sign is sometimes positive (namely, where the $\overline{z}'$ are such that the sum of the last terms is 0), it is non-negative, and $\mathcal{E}(z', z', \overline{z}') \geqq 0$ for all $z' \neq (0, 0, 0)$ and all $\overline{z}'$.

With only trivial modifications the above discussion applies to integrands $G(z, z')$, $z^{0'} \geqq 0$, which are associated with integrands $F(x, y, y')$ in ordinary form. We adopt the convention that $1 : G(z, z') = 0$ if $G(z, z') = +\infty$. As before, we find that the figuratrix is convex if and only if $E(x, y, y', \overline{y}') \geqq 0$ for all y' and $\overline{y}'$.

§ 4. - Semi-continuity of Parametric Integrals.

These preliminaries being set forth, we proceed to the proof of the theorem on semi-continuity of parametric integrals from which we shall later deduce the corresponding theorem for integrands in ordinary form. The theorem is based on two lemmas.

Lemma 4.1. - *Let $G(z, z')$ be continuous with its partial derivatives G_j for all z on a closed set A and all $z' \neq (0, 0, 0)$. Let the curves C_p: $z = z_p(t)$, $a \leqq t \leqq b$, $p = 0, 1, 2, \ldots$, lie in A and satisfy the conditions* *a)* $\lim z_n(t) = z_0(t)$ *uniformly for* $a \leqq t \leqq b$;

b) all the function $z_0(t)$, *satisfy a Lipschitz condition with the same constant* M.

Let $Z_0'(t)$ *be a triple of measurable functions, nowhere equal to* (0, 0, 0), *which is equal to* $z_0'(t)$ *wherever the derivatives* $z_0'(t)$ *are all defined and not all* [19] 0. *Then*

$$\lim_{n\to\infty}\left\{J[C_n]-J[C_0]-\int_a^b \mathcal{E}(z_0, Z_0', z'_n)dt\right\}=0.$$

We can assume without loss of generality that A is bounded as well as closed; for by *a)* all the curves C_0, C_n lie within a sufficiently large sphere about the origin, and we can restrict our attention to the portion of A lying in that sphere. The equation

$$z_0^{\lambda'} G_\lambda(z_0, Z_0') = G(z_0, z_0')$$

holds everywhere except on the set of measure 0 on which the $z_0'(t)$ are not all defined. For either the $z_0'(t)$ are not all 0, in which case $Z_0' = z_0'$ and the equation is a consequence of the homogeneity of G; or else the z_0' are all 0, and both sides of the equation reduce to 0. Hence we have

$$\int_a^b \{ G(z_n, z'_n) - G(z_0, z_0') - \mathcal{E}(z_0, Z_0', z'_n) \} dt$$
$$= \int_a^b \{ G(z_n, z'_n) - G(z_0, z'_n) \} dt + \int_a^b (z_n^{\lambda'} - z_0^{\lambda'}) G_\lambda(z_0, Z_0') dt.$$

On the set of points $[z \text{ in } A;\ |z^{j'}| \leqq M]$ the function G is uniformly continuous, hence the integrand in the first term on the right tends uniformly to 0. The function $G_1(z_0, Z_0')$ is measurable and bounded, for it is homogeneous of degree 0 in Z_0', and for every interval (h, k) in (a, b) we have

$$\int_h^k (z_n^{1'} - z_0^{1'}) dt = [z_n(k) - z_0(k)] - [z_n(h) - z_0(h)] \to 0.$$

Hence [20]

$$\lim_{n\to\infty} \int_a^b (z_n^{1'} - z_0^{1'}) G_1(z_0, Z_0') dt = 0.$$

A similar argument applies to the other terms on the right, and the lemma is established.

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(19) For example, we can set $Z_0'(t) = (1, 0, 0)$ wherever $z_0'(t)$ is undefined or is (0, 0, 0).

(20) Hobson: *Theory of Functions of a Real Variable*, Vol. II, § 279.

Lemma 4.2. - *Let* $\{C_n\}$ *be a sequence of curves all of length* $\leqq M$ *and approaching a curve* C_0 *as limit. Then for* C_0 *and for a subsequence of* $\{C_n\}$ *there exist representations satisfying the hypotheses of lemma* 4.1.

For the curve C_n, of length L_n, we choose as parameter $t=\frac{s}{L_n}$, where s is the length of arc. We thus have a representation $z=z_n(t)$, $0\leqq t\leqq 1$, in which the functions $z_n(t)$ satisfy a Lipschitz condition of constant $L_n\leqq M$. By ASCOLI's theorem it is possible to choose from the sequence $\{C_n\}$ a subsequence — which we continue to call $\{C_n\}$ — for which the functions $z_n^0(t)$ converge uniformly to a limit function $z_0^0(t)$; and this limit function satisfies a Lipschitz condition of constant M. From this subsequence we choose again a subsequence — which we again call $\{C_n\}$ — for which the functions $z_n^1(t)$ converge uniformly to a limit function $z_0^1(t)$. The relation $z_n^0(x)\rightrightarrows z_0^0(t)$ continues of course to hold. Repeating the process, we arrive at a subsequence — which we still call $\{C_n\}$ — such that the function $z_n^j(t)$ tend uniformly to the respective limits $z_0^j(t)$ $(j=0, 1, 2)$. Hence the curve $z=z_0(t)$ is a limit curve of the sequence $\{C_n\}$. But $\{C_n\}$ has the unique limit C_0. Hence $z=z_0(t)$ forms a representation of C_0, and the lemma is proved.

We are now able to prove

THEOREM 4.1. - *If* $J[C]$ *is positive quasi-regular on a closed set* A, *then for every* $M>0$ *it is lower semi-continuous on the class of all curves lying in* A *and having length less than* M.

Suppose the contrary; there then exists a sequence of curves $\{C_n\}$, all of length less than M, tending to a limit curve C_0, and

$$\liminf J[C_n]<J[C_0].$$

From these we select a subsequence of curves, which we again call $\{C_n\}$, such that

$$\lim J[C_n]<J[C_0]. \tag{4.1}$$

According to lemma 4.2, for C_0 and for a subsequence of $\{C_n\}$ — which we again call $\{C_n\}$ — there exist a representation satisfying the hypothesis of lemma 4.1. Then by lemma 4.1

$$\lim_{n\to\infty}\left\{J[C_n]-J[C_0]-\int_a^b \mathcal{E}(z_0, Z_0', z'_n)dt\right\}=0$$

where $Z_0'=z_0'$ if z_0' is defined and not $(0,0,0)$ and otherwise $Z_0'=(1,0,0)$. It follows that

$$\liminf J[C_n]=J[C_0]+\liminf\int_a^b \mathcal{E}(z_0, Z_0', z'_n)dt\geqq J[C_0].$$

This contradicts inequality (4.1) and establishes the theorem.

§ 5. - Proof of a Basic Lemma.

A natural analogue to theorem 4.1 would be a proof that under proper hypotheses, the functional $I[y]$ is lower semi-continuous on the class of all absolutely continuous functions $y(x)$ lying in A. But this class of functions is not closed under the limiting processes which we shall later apply, and we have need of a somewhat more general theorem. Let us denote by K_a the class of all curves $y=y(x)$, $a \leqq x \leqq b$, lying in A, for which the functions $y(x)$ are absolutely continuous. This is included in the class $\overline{K_a}$ of all rectifiable curves $x=x(t)$, $y=y(t)$, $t_0 \leqq t \leqq t_1$, lying in A and such that $x'(t) \geqq 0$. By use of theorem 4.1 we shall prove that under suitable hypotheses on the integrand $F(x, y, y')$ the associated parametric functional $J[C]$ is lower semi-continuous on $\overline{K_a}$, which implies as a corollary that $I[y]$ is lower semi-continuous on K_a. But the associated parametric integrand does not possess the properties needed for the direct application of Theorem 4.1, and we must therefore first prove:

Lemma 5.1. - *Hypotheses*:

1°) *$F(x, y, y')$ satisfies the conditions of continuity* (2.1) *on a bounded closed set A*;

2°) *$I[y]=\int F(x, y, y')dx$ is positive quasi-regular on A*;

3°) *k is a positive number.*

Conclusion:

For every $h>0$ it is possible to define a function $G^{(h)}(z, z')$ with the following properties:

1°) *$G^{(h)}$ is positively homogeneous of degree* 1 *in z' and is continuous together with its partial derivatives $G_j^{(h)}$ $(j=0, 1, 2)$ for all z on A and all $z' \neq (0, 0, 0)$*;

2°) *for all (z, z') such that $z^{0'} \geqq 0$ the inequality*

$$G^{(h)}(z, z') \leqq G(z, z') + k[z^{\lambda'} z^{\lambda'}]^{\frac{1}{2}}$$

holds, $G(z, z')$ being the parametric integrand associated with $F(x, y, y')$;

3°) *there is a constant b such that the inequality $G^{(h)}(z, z'_u) \geqq b$ holds for all h, all z on A and all unit vectors z'_u*;

4°) *the equation $\lim\limits_{h \to 0} G^{(h)}(z, z') = G(z, z') + k[z^{\lambda'} z^{\lambda'}]^{\frac{1}{2}}$ holds for all (z, z') such that $z^{0'} \geqq 0$*;

5°) *the integral $J^{(h)}[C] = \int_C G^{(h)}(z, z')ds$ is quasi-regular.*

Remark. - Added in Proof: It is worth noticing that neither here nor in Theorem 6.1 is any essential use made of the derivatives F_i. It is in fact enough to assume that $F(x, y, y')$ is continuous and that it is a convex function of y' for each (x, y) in A.

Let us first define

$$(5.1)\qquad \begin{aligned}\bar{G}(z, z') &\equiv G(z, z') - z^{\lambda'} G_{\lambda}(z, 1, 0, 0) + k[z^{\lambda'} z^{\lambda'}]^{\frac{1}{2}} \\ &= \mathcal{E}(z, 1, 0, 0, z') + k[z^{\lambda'} z^{\lambda'}]^{\frac{1}{2}} \geqq k[z^{\lambda'} z^{\lambda'}]^{\frac{1}{2}}.\end{aligned}$$

The last inequality implies that for every unit vector z'_u the inequality $\bar{G}(z, z'_u) \geqq k$ holds. It is thus possible to form the figuratrix $r = r(z'_u; z) = \dfrac{1}{\bar{G}(z, z'_u)}$, $z^{0'}_u \geqq 0$, of $\bar{G}$ for every point z of A, and the inequality $r \leqq \frac{1}{k}$ holds for all z and all z'_u for which $z^{0'}_u \geqq 0$. By the continuity of $\bar{G}$ there is a number m such that $\frac{m}{2} > \bar{G}(z, 1, 0, 0) \geqq k$ for all z in A. Consequently the point with coordinates $z^{0'} = \frac{2}{m}$, $z^{1'} = z^{2'} = 0$, is interior to the figuratrix $r = r(z'_u; z)$, no matter which point z of A we may choose.

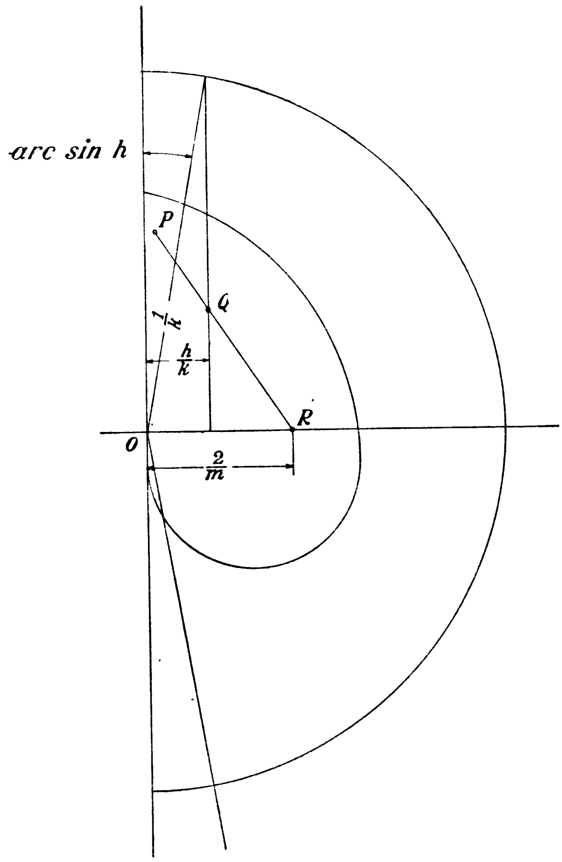

Fig. 1.

Henceforward we shall assume that we are dealing with values of $h \leqq \frac{k}{m}$, and shall define $G^{(h)}$ for such values. We can then define $G^{(h)} \equiv G^{\left(\frac{k}{m}\right)}$ for all values of h greater than $\frac{k}{m}$; such values are however of no particular interest, since we shall use $G^{(h)}$ in a limiting process in which $h \to 0$.

Our next step is to cut out those points in or on the figuratrix for which $z^{0'} = 0$, and at which the continuity properties of $r(z'_u; z)$ as a function of z are undetermined. We do this by restricting our attention to the convex body C_z defined by the inequalities $0 \leqq r \leqq r(z'_u; z)$, $z^{0'}_u \geqq h$. The body C_z is actually convex, being the intersection of the figuratrix and its interior ([21]) with the cone $z^{0'}_u \geqq h$.

([21]) When we speak of the interior of the figuratrix we consider that it is closed by adding to it the necessary portion of the plane $z^{0'} = 0$. Analytically, the interior of the figuratrix is the set of points (r, z'_u) such that $z^{0'}_u > 0$ and $0 < r < r(z'_u; z)$.

Interpreting the figuratrix $r=r(z'_u;\,z)$ as a surface in three-dimensional space, we state that for every point P interior to or on this figuratrix there is a point Q of the body C_z whose distance from D is less than $\frac{2mh}{k^2}$. If P is itself in C_z, this is trivial. If not, we consider the plane determined by the origin, the point P and the point R with rectangular coordinates $\left(\frac{2}{m}, 0, 0\right)$ $\left(\text{or polar coordinates } z'_u=(1, 0, 0),\ r=\frac{2}{m}\right)$. We join P to R by a line segment, and choose Q to be the point of this line for which $z^{0'}=\frac{h}{k}$ (cf. Fig. 1). The point Q belongs to C_z. For first the whole line segment PR is interior (except perhaps for P) to the figuratrix, by convexity, and a fortiori Q is interior to the figuratrix. Second, Q has the direction cosine $z_u^{0'}=\frac{z^{0'}}{\overline{OQ}}\geqq\frac{h}{k}:\left(\frac{1}{k}\right)=h$. The distance PQ satisfies the inequality

$$\overline{PQ}:\overline{PR}\leqq\frac{h}{k}:\frac{1}{m},$$

and since $\overline{PR}<\frac{2}{k}$, this yields

$$\overline{PQ}<\frac{2mh}{k^2},$$

as was to be proved.

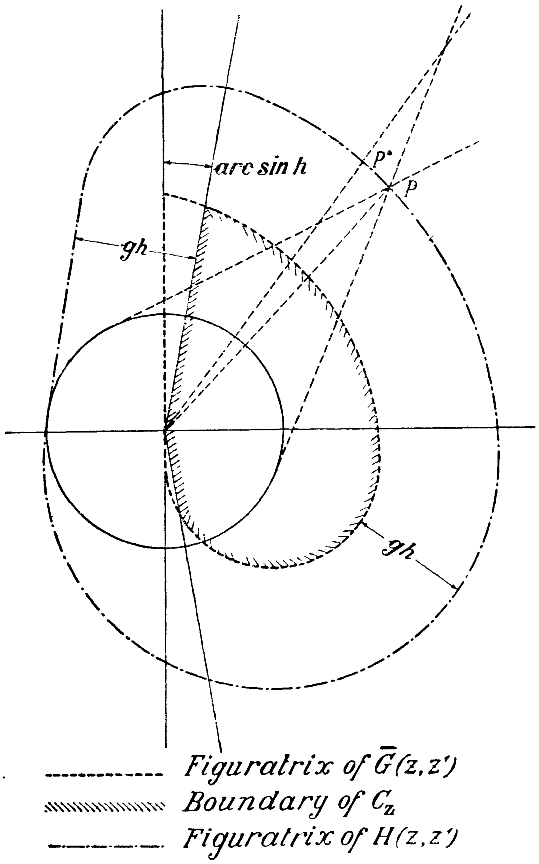

Fig. 2.

The number $\frac{2m}{k^2}$ will occur so frequently that we denote it by g,

$$g=\frac{2m}{k^2}. \tag{5.2}$$

From the body C_z we now proceed to form another convex body Γ_z consisting of all points whose distance from the body C_z is equal to or less than gh. (A cross section of the resulting figure is shown in Fig. 2). That Γ_z is convex is easily seen. Moreover, from the preceding paragraph we know that Γ_z contains the entire figuratrix $r=r(z'_u;\,z)$ in its interior. Also, since C_z contained the origin, Γ_z contains the entire sphere $z^{\lambda'}z^{\lambda'}\leqq(gh)^2$. The boundary $\bar{\Gamma}_z$ of the body Γ_z can thus be represented in polar coordinates by an equation $r=\varrho(z'_u;\,z)$, where z'_u ranges over the class of *all* unit vectors, and not merely those for which $z^{0'}\geqq 0$.

From the fact that Γ_z contains the figuratrix we see that for all (z, z'_u) such that $z_u^{0'} \geqq 0$ the inequality

$$\varrho(z'_u;\, z) \geqq r(z'_u;\, z) \tag{5.3}$$

holds.

We now proceed to use the convex surface $\overline{\Gamma}_z$ to define the function $H(x, y, x', y')$ of which it is the figuratrix. To do this we have only to set

$$H(z, z'_u) = \frac{1}{\varrho(z'_u;\, z)}$$

and extend the definition of H to all vectors z' by the usual condition of positive homogeneity. From the convexity of $\overline{\Gamma}_z$ we know that

(5.4) the $\mathcal{E}$-function of H is everywhere $\geqq 0$;

and from the inequality (5.3) we see that

$$H(z, z') \leqq \overline{G}(z, z') \tag{5.5}$$

for all z and for all z' such that $z^{0'} \geqq 0$.

Moreover, for all points z of A and all z' such that $z^{0'} \geqq 0$ the equation

$$\lim_{h \to 0} H(z, z') = \overline{G}(z, z') \tag{5.6}$$

holds. For let $\overline{P}$ be a point with polar coordinates $\bar{z}'_u$, $\bar{r}$, where $\bar{r} > r(\bar{z}'_u;\, z)$. The point $\overline{P}$ lies outside of the body $r \leqq r(z'_u;\, z)$, hence it has a positive distance δ from that body. A fortiori, it has a distance $\geqq \delta$ from the body C_z, no matter what value h has. If now h be small enough so that $gh < \delta$, then P also lies outside of Γ_z, and $\bar{r} > \varrho(z'_u;\, z)$, so that

$$\bar{r} \geqq \limsup_{h \to 0} \varrho(\bar{z}'_u;\, z).$$

But $\bar{r}$ can be chosen as close as we wish to $r(\bar{z}'_u;\, z)$. Hence

$$r(\bar{z}'_u;\, z) \geqq \limsup_{h \to 0} \varrho(\bar{z}'_u;\, z);$$

and taking the reciprocals

$$\overline{G}(z, \bar{z}'_u) \leqq \liminf_{h \to 0} H(z, \bar{z}'_u).$$

Comparing this with (5.5) we see that

$$\lim_{h \to 0} H(z, \bar{z}'_u) = \overline{G}(z, \bar{z}'_u).$$

Since this equation holds for all unit vector $\bar{z}'_u$ with $\bar{z}_u^{0'} \geqq 0$, it extends by the homogeneity of H and $\overline{G}$ to all vectors z' with $z^{0'} \geqq 0$, and equation (5.6) is established.

We now take up the somewhat tedious proof that H and its first partial derivatives H_j are continuous. To begin with we already know that the sphere $z^{\lambda'} z^{\lambda'} \leqq (gh)^2$ is contained in Γ_z for all z. This implies that $\varrho(z'_u; z)$ satisfies a Lipschitz condition of constant $\frac{2}{mh}$. (The value of this constant is unimportant; it is only important that it is independent of z). For let P, P^* be any two points of the boundary $\overline{\Gamma}_z$, and let $\Delta\vartheta \leqq \arcsin \frac{mh}{k}$ be the angle between them. The points P, P^* and the origin determine an ordinary plane, and in this plane we have the situation shown in Fig. 2 (which also contains details irrelevant to the present purpose). The figure will suggest the further steps needed to prove our statement.

Next, for each fixed z'_u the function $\varrho(z'_u; z)$ varies continuously with z. For let us consider the point $\overline{P}$ with polar coordinates $\bar{z}'_u$, $\varepsilon + \varrho(\bar{z}'_u; z)$. This point lies outside of Γ_z, hence the nearest point of C_z lies at a greater distance than gh from $\overline{P}$ — say at a distance $gh + \delta$. But for $z_u^{0'} \geqq h$ the function $r(z'_u; z)$ is uniformly continuous in z'_u and z, since for these values we have [22]

$$r(z'_u; z) = 1 : G(z, z'_u) = 1 : x'_u F\left(x, y, \frac{y'_u}{x'_u}\right)$$

and $y'_u : x'_u$ is bounded. Hence for all points z^* at a sufficiently small distance from z we have $r(z'_u; z^*) < r(z'_u; z) + \delta$, valid for all z'_u with $z_u^{0'} \geqq h$. Therefore the distance from $\overline{P}$ to the point with polar coordinates $(z'_u, r(z'_u; z^*))$ is greater than $gh + \delta - \delta$, and $\overline{P}$ also lies outside of the body Γ_{z^*}; that is,

$$\varrho(\bar{z}'_u; z^*) < \varrho(\bar{z}'_u; z) + \varepsilon.$$

A similar proof shows that

$$\varrho(\bar{z}'_u; z^*) > \varrho(\bar{z}'_u; z) - \varepsilon$$

for all points z^* near enough to z. Hence for fixed z'_u the function $\varrho(z'_u; z)$ is continuous in z.

From the two properties, that ϱ is continuous in z for fixed z'_u and that for all z it satisfies a Lipschitz condition of constant $\frac{2}{mh}$ with respect to z'_u, it follows that ϱ is a continuous function of all the variables z, z'_u. And since $\varrho \geqq gh > 0$, its reciprocal $H(z, z'_u)$ is also a continuous function of all variables; from which, by means of the homogeneity of H, we see that $H(z, z')$ is continuous for all z on A and all z'.

In order to prove that the partial derivatives H_j are continuous we first notice that for each point P on the boundary $\overline{\Gamma}_z$ of Γ_z there is exactly one nearest point $N(P)$ on the boundary $\overline{C}_z$ of C_z, as follows immediately from the convexity of C_z; and the distance between P and $N(P)$ is gh. Let now $\{z_n\}$ be

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[22] The notation is clear; $(x, y) = z$ and $(x'_u, y_u^{1'}, y_u^{2'}) = (z_u^{0}, z_u^{1'}, z_u^{2'})$.

any sequence of points of A tending to a limit point z and let $\{z'_{u,n}\}$ be any sequence of unit vectors tending to a unit vector z'_u; we must show that

$$\lim_{n\to\infty} H_j(z_n, z'_{u,n}) = H_j(z, z'_u), \qquad (j=0,1,2).$$

We define P_n to be the point with polar coordinates $(z'_{u,n}, \varrho(z'_{u,n}; z_n))$, and likewise define P to be the point with coordinates $(z'_u, \varrho(z'_u; z))$. By the recently established continuity of ϱ we know that P_n approaches P as $n\to\infty$. The behavior of the points $N(P_n)$ needs to be studied. Each point $N(P_n)$ lies on the boundary $\overline{C}_z$ and thus has coordinates z'_u, r satisfying one of the two conditions $z^{0'}_u > h$, $r = r(z'_u; z_n)$ or $z^0_u = h$, $r \leqq r(z'_u; z_n)$. By the uniform continuity of $r(z'_u; z)$ for such arguments we see that every limit point of the $N(P_n)$ lies on the boundary $\overline{C}_z$. Moreover, every limit point of the $N(P_n)$ is at distance gh from P. But there is only one point satisfying the two requirements, and that is $N(P)$. Hence as $n\to\infty$ the two relationships $P_n \to P$ and $N(P_n)\to N(P)$ both hold, so that the vector joining $N(P)$ to P varies continuously with z and z'_u.

The usefulness of this lies in the fact that the vector joining P to $N(P)$ is the normal vector to the boundary $\overline{\Gamma}_z$, as is seen without difficulty. We have thus proved that the direction cosines of the normal to the surface $r = \varrho(z'_u; z)$ are continuous functions of z'_u and z. It follows that if we write in normal form the equation of the plane tangent to that surface at the point $\bar{z}'_u$, the coefficients will be continuous functions of z and $\bar{z}'_u$. Let the equation of this tangent plane be $a^\lambda(z, \bar{z}'_u) z^{\lambda'} + p(z, \bar{z}'_u) = 0$.

Recollecting now that the figuratrix $\overline{\Gamma}_z$ is the intersection of the cone $w = H(z, z')$ with the hyperplane $w=1$, we write the equation of the hyperplane through the origin

$$a^\lambda(z, \bar{z}'_u) z^{\lambda'} + p(z, \bar{z}'_u) w = 0$$

whose intersection with $w=1$ is the above found plane tangent to the figuratrix. This hyperplane is tangent to the cone $w = H(x, y, x', y')$ along the element $z^{j'} = a\bar{z}^{j'}_u$, $a \geqq 0$. But for this tangent hyperplane we have the equation

$$w = H(z, \bar{z}'_u) + (z^{\lambda'} - \bar{z}^{\lambda'}_u) H_\lambda(z, z'_u)$$

or

$$z^{\lambda'} H_\lambda(z, \bar{z}'_u) - w = 0.$$

Comparing the coefficients, we find

$$H_j = -\frac{a^j}{p},$$

so that $H_j(z, z'_u)$ is a continuous function of z and z'_u. From this, by the homogeneity of H_j, it follows that $H_j(z, z')$ is continuous for all z on A and all $z' \neq (0,0,0)$.

All that remains is to define $G^{(h)}(z, z')$ by the equation

$$G^{(h)}(z, z') = H(z, z') + z^{\lambda'} G_\lambda(z, 1, 0, 0) \tag{5.7}$$

and to collect the above results. The homogeneity and continuity of $G^{(h)}$ and the continuity of the derivatives $G_j^{(h)}$ follow from (5.7) and the corresponding properties of H and H_j as just established. From (5.1), (5.5) and (5.7) we have the inequality

$$G^{(h)}(z, z') \leqq G(z, z') + k[z^{\lambda'} z^{\lambda'}]^{\frac{1}{2}}$$

for all z and all z' such that $z^{0'} \geqq 0$. Since $H \geqq 0$ and the derivatives $G_j(z, 1, 0, 0)$ are bounded below uniformly for all z on A, it follows from (5.7) that there exists a constant b such that

$$G^{(h)}(z, z'_u) \geqq b$$

for all h, all z on A and all unit vectors z'_u. From equations (5.6), (5.1) and (5.7) we establish the equation

$$\lim_{h \to 0} G^{(h)}(z, z') = G(z, z') + k[z^{\lambda'} z^{\lambda'}]^{\frac{1}{2}}$$

for all z' such that $z^{0'} \geqq 0$. And finally, from statement (5.4) and the observation that $G^{(h)}$ and H differ only by a linear function and thus have the same $\mathcal{E}$-function, we find that $\int G^{(h)}(z, z')ds$ is quasi-regular. The lemma is thus established.

§ 6. - Semi-continuity of Integrals in Ordinary Form.

The somewhat boresome proof of lemma 5.1 now bears fruit in the form of several theorems on the semi-continuity of integrals in ordinary form. The first of these theorems follows with almost no additional effort.

THEOREM 6.1. - *Let the integral* $I[y] = \int F(x, y, y')dx$ *satisfy the continuity conditions* (2.1) *and also be positive quasi-regular on a closed set* A. *Then on the class of all rectifiable curves* C: $x = x(t)$, $y = y(t)$, $x'(t) \geqq 0$, $a \leqq t \leqq b$, *lying in* A *and having lengths* $\leqq M$ (M a constant) *the parametric functional*

$$J[C] = \int_C G(z, z')ds$$

associated with $I[y]$ *is lower semi-continuous.*

Suppose the contrary; there then exists a curve C_0 in A and a sequence of curves $\{C_n\}$ in A tending to C_0, all having lengths $\leqq M$, and satisfying the inequality

$$\liminf J[C_n] < J[C_0]. \tag{6.1}$$

We first take up the case $J[C_0] < \infty$, in which case

$$\liminf J[C_n] = J[C_0] - \delta < J[C_0]. \tag{6.2}$$

From the sequence $\{C_n\}$ we can select a subsequence (which we continue to call $\{C_n\}$) such that

$$\lim J[C_n]=J[C_0]-\delta=J[C_0]-3kM, \tag{6.3}$$

where we have set $k=\frac{\delta}{3M}$. For this value of k we form the integrands $G^{(h)}(z, z')$ of lemma 5.1; the assumption there made that A is bounded causes no trouble, since we can here restrict our attention to the part of A lying in a (closed) sphere large enough to contain C_0 and all the C_n. By conclusion 2°) of that lemma we have for all h and all n the inequality

$$\begin{aligned} \int_{C_n} G^{(h)}(z_n, z'_n)ds &\leqq \int_{C_n} \{ G(z_n, z'_n)+k[z^{\lambda} z^{\lambda'}]^{\frac{1}{2}} \} ds \\ &\leqq \int_{C_n} G(z_n, z'_n)ds+kM=J[C_n]+\frac{\delta}{3}, \end{aligned} \tag{6.4}$$

since the integral of the last term is the length of C_n, which does not exceed M.

On the other hand, if the parameter s be the length of arc on C_0 the derivatives $z_0'(s)$ form a unit vector for almost all s, and by conclusions 2°), 3°) and 4°) of lemma 5.1 we have

$$\lim_{h\to 0} \int_{C_0} G^{(h)}(z_0, z_0')ds=\int_{C_0} G(z_0, z_0')ds+k\int_{C_0} [z^{\lambda'}z^{\lambda'}]^{\frac{1}{2}}ds \geqq J[C_0]. \tag{6.5}$$

Let us choose an h small enough so that

$$\int_{C_0} G^{(h)}(z_0, z_0')ds>J[C_0]-\frac{\delta}{3}. \tag{6.6}$$

Comparing (6.3), (6.4) and (6.6), we have

$$\liminf_{n\to\infty} \int_{C_n} G^{(h)}(z_n, z'_n)ds<\int_{C_0} G^{(h)}(z_0, z_0')ds-\frac{\delta}{3}, \tag{6.7}$$

so that the functional

$$\int_{C} G^{(h)}(z, z')ds$$

fails to be lower semi-continuous at C_0 on the class of all curves of length $\leqq M$. But this functional satisfies all the hypotheses of theorem 4.1, and hence must be lower semi-continuous on that class. This contradiction establishes our theorem for the case $J[C_0]<\infty$.

For the case $J[C_0]=\infty$ the same proof holds with only minor modifications. Equation (6.2) is replaced by

$$\liminf J[C_n]=\delta<J[C_0], \tag{6.8}$$

equation (6.3) by

$$\lim J[C_n] = \delta = kM, \tag{6.9}$$

with $k = \frac{\delta}{M}$; inequality (6.4) by

$$\int_{\dot{C}_n} G^{(h)}(z_n, z'_n) ds \leqq J[C_n] + kM = J[C_n] + \delta; \tag{6.10}$$

inequality (6.5) by

$$\lim_{h \to 0} \int_{\dot{C}_0} G^{(h)}(z_0, z_0') ds = \infty. \tag{6.11}$$

We choose h small enough so that

$$\int_{\dot{C}_0} G^{(h)}(z_0, z_0') ds \geqq 2\delta + 1. \tag{6.12}$$

On comparing (6.9), (6.10) and (6.12) we have

$$\liminf \int_{\dot{C}_n} G^{(h)}(z_n, z'_n) ds \leqq 2\delta \leqq \int_{\dot{C}_0} G^{(h)}(z_0, z_0') ds - 1, \tag{6.13}$$

and this inequality expresses the contradiction which establishes our theorem.

A direct corollary of theorem 6.1 is

Theorem 6.2 - *If $I[y]$ is positive quasi-regular on a closed set A, it is lower semi-continuous on the class of all absolutely continuous functions $y(x)$, $a \leqq x \leqq b$, lying in A and having total variation less than M* (M a constant).

We need only to compare Theorem 6.1 and lemma 2.4. Theorem 6.1 is adequate for the later needs of this paper, for in each case the sets of curves on which semi-continuity is needed will turn out to have uniformly bounded lengths. Nevertheless, the semi-continuity of integrals is in itself a matter of sufficient interest to justify the proving of a theorem in which the restriction as to lengths is removed. In preparation for this theorem we establish

Lemma 6.1. - *If $I[y] = \int F(x, y, y') dx$ is positive quasi-regular semi-normal* [23] *on A, to each point $\bar{z} = (\bar{x}, \bar{y})$ of A there correspond constants v^0, v^1, v^2 such that for the parametric integrand associated with F the inequality*

$$G(z, z'_u) + v^\lambda z_u^{\lambda'} \geqq k > 0$$

holds for all unit vectors z'_u with $z_u^{0'} \geqq 0$ and for all points z of A lying in a neighborhood of $\bar{z}$.

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[23] Defined in § 2.

We first observe that the effect of adding a linear sum $a^\alpha y^{\alpha'}+b$ to F is to add a linear sum $b^\lambda z^{\lambda'}$ to G, and vice versa. By the definition of semi-normality, there exists a pair $\bar{y}'$ such that

$$\Phi(x,y,y')\equiv F(\bar{x},\bar{y},y)-F(\bar{x},\bar{y},\bar{y}')-(y^{\alpha'}-\bar{y}^{\alpha'})F_\alpha(\bar{x},\bar{y},\bar{y}')=E(\bar{x},\bar{y},\bar{y}',y')>0$$

for all $y'\neq\bar{y}'$.

Now let Q be any number greater than $\bar{y}^{\alpha'}\bar{y}^{\alpha'}$ and also greater than 1. For all vectors Y' such that $Y^{\alpha'}Y^{\alpha'}=Q$ we have $\Phi(\bar{x},\bar{y},Y')>0$; hence by the continuity properties of Φ we find that $\Phi(\bar{x},\bar{y},Y')\geqq 3\mu>0$. Again using the continuity of Φ, there exists a neighborhood U of $(\bar{x},\bar{y})$ such that for every (x,y) of U the inequalities

$$|\Phi(x,y,\bar{y}')|\leqq\mu,$$
$$\Phi(x,y,Y')\geqq 2\mu$$

hold. Letting y' be any pair such that $y^{\alpha'}y^{\alpha'}>2Q$, we consider the arguments $\bar{y}'+t(y'-\bar{y}')$. The function $\varphi(t)=\Phi(x,y,\bar{y}'+t(y'-\bar{y}'))$ has for its second derivative the expression

$$\Phi_{\alpha\beta}(y^{\alpha'}-\bar{y}^{\alpha'})(y^{\beta'}-\bar{y}^{\beta'}).$$

This can not be negative, for Φ and F have the same E-function and $I[y]$ is positive quasi-regular. Therefore $\varphi(t)$ is a convex function. For a value t_0 between 0 and 1 the argument $\bar{y}'+t(y'-\bar{y}')$ has absolute value Q and is one of the Y', hence for this value we have $\varphi(t_0)\geqq 2\mu$. Also $\varphi(0)=\Phi(x,y,\bar{y}')\leqq\mu$. Hence

$$\Phi(x,y,y')=\varphi(1)\geqq\frac{1-t_0}{t_0}\mu+2\mu=\frac{1+t_0}{t_0}\mu\geqq\frac{\mu}{t_0}. \tag{6.14}$$

To estimate t_0, a figure will be convenient. We denote the origin $(y^{1'}=y^{2'}=0)$ by O, and the points $\bar{y}'+t(y'-\bar{y}')$ for $t=0$, t_0, 1 by $P(0)$, $P(t_0)$, $P(1)$ respectively. Then

$$t_0=\overline{P(0)P(t_0)}:\overline{P(0)P(1)}\leqq 2Q:(\overline{OP(1)}-Q)=2Q:([y^{\alpha'}y^{\alpha'}]^{\frac{1}{2}}-Q).$$

With this estimate for t_0, we find from (6.14) that

$$\Phi(x,y,y')\geqq\frac{\mu}{2Q}([y^{\alpha'}y^{\alpha'}]^{\frac{1}{2}})-\frac{\mu}{2}. \tag{6.15}$$

Now we form the parametric integrand $\Gamma(z,z')=\Gamma(x,y,x',y')$ associated with Φ. For all unit vectors x'_u, y'_u, $x'_u>0$, such that

$$[y_u^{\alpha'}y_u^{\alpha'}]^{\frac{1}{2}}:x'_u\geqq 2Q \tag{6.16}$$

the inequality (6.15) yields

$$\Gamma(x,y,x'_u,y'_u)=x'_u\Phi\left(x,y,\frac{y'_u}{x'_u}\right)\geqq x'_u\frac{\mu}{2Q}\frac{[y^{\alpha'}y^{\alpha'}]^{\frac{1}{2}}}{x'_u}-x'_u\frac{\mu}{2}. \tag{6.17}$$

From (6.16) follows by a simple calculation that $x'_u < \frac{1}{2Q}$ and

$$[y_u^{a'} y_u^{a'}]^{\frac{1}{2}} > \frac{\sqrt{3}}{2},$$

so that (6.17) implies that

$$\Gamma(x, y, x'_u, y'_u) \geqq \frac{\mu\sqrt{3}}{4Q} - \frac{\mu}{4Q} > \frac{\mu}{8Q} \tag{6.18}$$

for all unit vectors (x'_u, y'_u), $x'_u > 0$, satisfying (6.16). Letting $x'_u \to 0$, (6.18) continues to hold if $x'_u = 0$. Finally, for the remaining unit vectors (x'_u, y'_u), $x'_u > 0$, which do not satisfy (6.16), we have

$$\frac{y_u^{a'} y_u^{a'} + x'^2_u}{x'^2_u} < 4Q^2 + 1,$$

so that

$$x'_u > \frac{1}{(4Q^2+1)^{\frac{1}{2}}} > \frac{1}{4Q}. \tag{6.19}$$

For all unit vectors satisfying (6.19) and for all (x, y) on U the function $\Gamma(x, y, x'_u, y'_u)$ is bounded below, so that for such vectors there exists a constant $c > 0$ such that

$$\Gamma(x, y, x'_u, y'_u) + cx'_u > \frac{\mu}{8Q} \equiv k > 0. \tag{6.20}$$

By (6.18) this inequality holds also for all vectors which satisfy (6.16); in other words, (6.20) holds for all (x, y) on U and all unit vectors (x'_u, y'_u) with $x'_u \geqq 0$. But $\Gamma + cx'_u$ differs from $G(x, y, x'_u, y'_u)$ only by a linear sum $ax'_u + b^a y_u^{a'}$, where a, b^1, b^2 are constants, and the lemma is thus proved.

We here digress to indicate a contrast between the parametric and the non-parametric problems. For the parametric problem it has already been shown ([24]) that if at a point z we have $\mathcal{E}(z, z', \bar{z}') \geqq 0$ for all z' and $\bar{z}'$, and $\mathcal{E}(z, z', \bar{z}')$ is not identically 0, then there exists a set of constants v^j such that $G(z, z') + v^\lambda z^{\lambda'} \geqq k > 0$ for all z' (G being the integrand for which $\mathcal{E}$ is formed). From this it further follows that there exists a z_0' such that $\mathcal{E}(z, z_0', \bar{z}') > 0$ for all $\bar{z}'$ not of the form $(cz_0^{0'}, cz_0^{1'}, cz_0^{2'})$, $c \geqq 0$. We need only to choose z_0' as that unit vector for which $G(z, z'_u)$ is a minimum. The conclusion can then be reached as in the note cited, or even more simply as follows. At z_0' the function $r = r(z'_u; z)$ defining the figuratrix has its maximum value, say ϱ. The figuratrix then lies in the sphere $r = \varrho$. The tangent plane at $(z'_u, r(z'_u; z))$ is also tangent to the sphere, hence can have no other point in common with the figuratrix. This at once implies that $\mathcal{E}(z, z_0', \bar{z}') > 0$ for $\bar{z}' \neq cz_0'$ $(c \geqq 0)$.

([24]) E. J. McShane: *A Remark concerning Mr. Graves' Paper*, etc., Monatshefte für Math. und Physik, Vol. 39, p. 105.

On the other hand, in proving lemma 6.1 we have had to assume that there exists a y_0' such that $E(x, y, y_0', \overline{y}') > 0$ for all $\overline{y}' \neq y_0'$. In contrast with the parametric problem, it is *not* sufficient to assume that $E(x, y, y', \overline{y}') \geqq 0$ for all y', $\overline{y}'$ and is not identically zero, as is shown by the example $F(x, y, z, y', z') = [1 + y'^2]^{\frac{1}{2}}$. For this we have

$$E(x, y, z, y', z', \overline{y}', \overline{z}') = [1 + \overline{y}'^2]^{\frac{1}{2}} - [1 + y'^2]^{\frac{1}{2}} - (\overline{y}' - y')y'[1 + y'^2]^{-\frac{1}{2}}$$
$$= [1 + y'^2]^{-\frac{1}{2}} \{ [1 + \overline{y}'^2]^{\frac{1}{2}} [1 + y'^2]^{\frac{1}{2}} - 1 - y'\overline{y}' \}.$$

By the inequality of Schwarz, this is non-negative, and is zero only if $\overline{y}' = y'$. The associated parametric integrand is $G(x, y, z, x', y', z') = [x'^2 + y'^2]^{\frac{1}{2}}$. If to this we add any linear combination $c_1 x' + c_2 y' + c_3 z'$ and then consider the two sets of arguments $x' = y' = 0$, $z' = 1$ and $x' = y' = 0$, $z' = -1$, we see that it is impossible that any expression of the form $G + c_1 x' + c_2 y' + c_3 z'$ can be positive for all x', y', z' with $x' \geqq 0$.

We now take up the proof of

THEOREM 6.3. - *If $I[y]$ is positive quasi-regular semi-normal on A, the associated parametric functional $J[C] = \int G(z, z') dt$ is lower semi-continuous on the class of all rectifiable curves $z = z(t)$, $z^{0'}(t) \geqq 0$, lying in A.*

Suppose the contrary; there then exists a curve C_0: $z = z_0(t)$ of A and a sequence of curves C_n: $z = z_n(t)$ of A such that

$$\text{(6.21)} \qquad \liminf J[C_n] < J[C_0].$$

By lemma 6.1, to each point ζ on the curve C_0 there corresponds a set of coefficients $v^j(\zeta)$ and a neighborhood $U(\zeta)$ such that the inequality

$$\text{(6.22)} \qquad G(z, z'_u) + v^\lambda(\zeta) z_u^{\lambda'} \geqq k(\zeta) > 0$$

holds for all z in $U(\zeta)$ and all unit vectors z'_u with $z_u^{0'} \geqq 0$. A finite number of the neighborhoods $U(\zeta)$ cover the curve C_0. Hence we can subdivide C_0 into a finite number of arcs C_0^r $(r = 1, 2, \ldots, m)$, each lying in a neighborhood U^r which is one of the set $U(\zeta)$. Corresponding to this subdivision of C_0, we can subdivide each curve C_n into arcs C_n^r such that

$$\text{(6.23)} \qquad \lim_{n \to \infty} C_n^r = C_0^r.$$

But

$$\sum_{r=1}^{m} J[C_p^r] = J[C_p], \qquad (p = 0, 1, 2, \ldots),$$

and

$$\liminf J[C_n] \geqq \sum_{r=1}^{m} \liminf J[C_n^r];$$

hence inequality (6.21) implies that for at least one value of r we have

$$\liminf J[C_n^r] < J[C_0^r]. \tag{6.24}$$

We now show that inequality (6.24) can not be valid.

From the curves C_n^r we select a subsequence (we continue to call it $\{C_n^r\}$) for which $\lim J[C_n^r]$ exists, and

$$\lim J[C_n^r] < J[C_0^r]. \tag{6.25}$$

The curves C_n^r can not have uniformly bounded lengths, for then (6.25) would stand in contradiction with Theorem 6.1. Hence we can select a subsequence (which we again call $\{C_n^r\}$) such that

$$\lim L[C_n^r] = \infty, \tag{6.26}$$

where $L[C]$ denotes the length of C.

Without loss of generality we suppose that the curves C_p^r ($p=0, 1, 2,....$) have representations C_p^r: $z=z_p(t)$, $0 \leqq t \leqq 1$. By inequality (6.22) the inequality

$$G(z, z') + v^\lambda z^{\lambda'} \geqq k[z^{\lambda'} z^{\lambda'}]^{\frac{1}{2}} \tag{6.27}$$

holds for all z in U^r and all z' such that $z^{0'} \geqq 0$; the v^j are here the constants $v^j(\zeta)$ belonging to the neighborhood U^r, and $k>0$ is the corresponding $k(\zeta)$. The curve C_0^r lies in U^r, and by (6.23) so do all the C_n^r from a certain n on. Hence for all sufficiently large n we have

$$G(z_n, z'_n) \geqq k[z_n^{\lambda'} z_n^{\lambda'}]^{\frac{1}{2}} - v^\lambda z_n^{\lambda'}. \tag{6.28}$$

Integrating from 0 to 1, this yields

$$J[C_n^r] \geqq kL[C_n] - v^\lambda[z_n^{\lambda}(1) - z_n^{\lambda'}(0)]. \tag{6.29}$$

The last term on the right tends, by (6.23), to the limit $v^\lambda[z_0^\lambda(1) - z_0^\lambda(0)]$. Consequently by (6.26) we have

$$\lim J[C_n^r] = \infty,$$

which contradicts (6.24) and establishes our theorem.

A direct corollary of Theorem 6.3 is

THEOREM 6.4. - *If the integral $I[y]$ is positive quasi-regular semi-normal on A, it is lower semi-continuous on the class of all absolutely continuous functions $y=y(x)$ lying in A.*

We need only to compare Theorem 6.3 with lemma 2.4.

For integrals in the plane TONELLI has shown [25] that lower semi-continuity

[25] L. TONELLI, op. cit. [1], Vol. I, p. 397.

is a consequence of quasi-regularity. This theorem does not hold for integrals in three-space, as is shown by the example

$$I=\int_a^b [zy'-yz'+(x^2+y^2+z^2)\sqrt{1+y'^2+z'^2}]dx,$$

where we use the symbols (x, y, z) instead of (x, y^1, y^2). This integrand is quasi-regular, and except at the origin is semi-normal, and in fact regular; yet it is not lower semi-continuous. For corresponding to the functions $y=0$, $z=0$, $0 \leqq x \leqq 1$, we find

$$I=I_0=\int_0^1 x^2 dx=\frac{1}{3}.$$

Now construct the sequence of absolutely continuous functions

$$\begin{aligned}
y_n(x) &= \frac{1}{n}\cos 2n^3\pi x && \text{for} \quad 0 \leqq x \leqq \frac{1}{n};\\
y_n(x) &= \frac{1}{n} && \text{for} \quad \frac{1}{n} < x \leqq 1;\\
z_n(x) &= \frac{1}{n}\sin 2n^3\pi x && \text{for} \quad 0 \leqq x \leqq \frac{1}{n};\\
z_n(x) &= 0 && \text{for} \quad \frac{1}{n} < x \leqq 1.
\end{aligned}$$

Splitting the integral I into the sum of two integrals, we have

$$\int_0^1 (z_n y'_n - y_n z'_n)dx = -\int_0^{\frac{1}{n}} 2\pi n dx = -2\pi$$

$$\int_0^1 (x_n^2+y_n^2+z_n^2)\sqrt{1+y'^2_n+z'^2_n}dx \leqq \int_0^{\frac{1}{n}} \frac{3}{n^2}\sqrt{1+4\pi^2 n^4}dx$$

$$+\int_{\frac{1}{n}}^1 \left(x^2+\frac{1}{n^2}\right)\sqrt{1}dx \leqq \int_0^{\frac{1}{n}} \frac{3}{n^2}\cdot 4\pi n^2 dx + \frac{1}{3}+\frac{1}{n^2} \leqq \frac{12\pi}{n}+\frac{1}{3}+\frac{1}{n^2}.$$

Hence for the functions $y_n(x)$, $z_n(x)$ the integral has a value I_n such that

$$I_n \leqq -2\pi+\frac{1}{3}+\frac{12\pi}{n}+\frac{1}{n^2},$$

and so

$$\liminf I_n \leqq -2\pi+\frac{1}{3} < \frac{1}{3} = I_0,$$

proving that the integral is not lower semi-continuous.