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A CLASS OF GAP THEOREMS

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The present paper is devoted to the study of the following:

THEOREM I [1]. - *Let the Maclaurin series* $f(z)=\sum_{0}^{\infty} a_n z^{\lambda_n}$ *have the unit circle for its circle of convergence, and let there exist a function* $\Phi(\theta)$ *belonging to* L_2 *such that over the arc* (α, β),

$$\lim_{r\to 1-0}\int_{\alpha}^{\beta} |f(re^{i\theta})-\Phi(\theta)|^2 d\theta=0.$$

Let the λ_n*'s be arranged in increasing order, and let* $\beta>\alpha$. *Let*

$$\lambda_{n+1}-\lambda_n \geqslant L=\frac{16\pi}{\beta-\alpha}.$$

Then there exists a function $\Psi(\theta)$ *belonging to* L_2 *over* $(0, 2\pi)$ *and coinciding with* $\Phi(\theta)$ *over* (α, β) *such that*

$$\lim_{r\to 1-0}\int_{0}^{2\pi} |f(re^{i\theta})-\Psi(\theta)|^2 d\theta=0,$$

and

$$\int_{0}^{2\pi} |\Psi(\theta)|^2 d\theta \leqslant \frac{16\pi}{\beta-\alpha}\int_{\alpha}^{\beta} |\Psi(\theta)|^2 d\theta.$$

To prove this let us consider the polynomial $P(\theta)=\sum_{1}^{N} a_n e^{i\lambda_n\theta}$. We have for the FOURIER transform of $\dfrac{e^{i\lambda_n\theta}\sin\frac{L\theta}{2}}{\theta}$ the functions

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{iu\theta}e^{i\lambda_n\theta}\frac{\sin\frac{L\theta}{2}}{\theta}\, d\theta=\begin{cases}\sqrt{\frac{\pi}{2}} & \text{if} \quad |\lambda_n+u|<\frac{L}{2};\\ 0 & \text{if} \quad |\lambda_n+u|>\frac{L}{2}.\end{cases}$$

[1] Theorem I follows the argument of Theorem XL in the forthcoming book of PALEY and WIENER: *Fourier Transforms in the Complex Domain.*

These differ from zero over non-overlapping ranges and are orthogonal. By PLANCHEREL's theorem,

$$\int_{-\infty}^{\infty} |P(\theta)|^2 \frac{\sin^2 \frac{L\theta}{2}}{\theta^2}\, d\theta = \sum_1^N |a_n|^2 \frac{\pi L}{2}. \tag{1}$$

Hence

$$\int_{-\frac{\pi}{L}}^{\frac{\pi}{L}} |P(\theta)|^2 d\theta \leqslant \frac{\pi^3}{2L} \sum_1^N |a_n|^2$$

and in general

$$\int_{-\frac{\pi}{L}+y}^{\frac{\pi}{L}+y} |P(\theta)|^2 d\theta \leqslant \frac{\pi^3}{2L} \sum_1^N |a_n|^2. \tag{2}$$

Again, by (2), if $A > \frac{\pi}{L}$,

$$\int_{-A}^{A} |P(\theta)|^2 d\theta \leqslant \pi^2 A \sum_1^N |a_n|^2.$$

Thus

$$\left[\int_A^\infty + \int_{-\infty}^{-A}\right] \frac{|P(\theta)|^2}{\theta^2}\, d\theta = \int_A^\infty \frac{1}{\theta^2}\, d \int_{-\theta}^{\theta} |P(\theta)|^2 d\theta =$$

$$= \left| -\frac{1}{A^2} \int_{-A}^{A} |P(\theta)|^2 d\theta + 2 \int_A^\infty \frac{d\theta}{\theta^2} \frac{1}{\theta} \int_{-\theta}^{\theta} |P(x)|^2 dx \right| \leqslant$$

$$\leqslant 2\pi^2 \sum_1^N |a_n|^2 \int_A^\infty \frac{dx}{x^2} = \frac{2\pi^2}{A} \sum_1^N |a_n|^2.$$

Thus again by (1)

$$\int_{-A}^{A} |P(\theta)|^2 \frac{\sin^2 \frac{L\theta}{2}}{\theta^2}\, d\theta \geqslant \left[\frac{\pi L}{2} - \frac{2\pi^2}{A}\right] \sum_1^N |a_n|^2.$$

Now let $A = \frac{4\pi^2}{\pi L - 2C}$ where C is some sufficiently small positive constant. Then

$$\int_{-\frac{4\pi^2}{\pi L - 2C}}^{\frac{4\pi^2}{\pi L - 2C}} |P(\theta)|^2 d\theta \geqslant \frac{4}{L^2} \int_{-\frac{4\pi^2}{\pi L - 2C}}^{\frac{4\pi^2}{\pi L - 2C}} |P(\theta)|^2 \frac{\sin^2 \frac{L\theta}{2}}{\theta^2}\, d\theta \geqslant \frac{4C}{L^2} \sum_1^N |a_n|^2.$$

In particular, if $C=\frac{\pi L}{2}$,

$$\int_{-\frac{8\pi}{L}}^{\frac{8\pi}{L}} |P(\theta)|^2 d\theta \geqslant \frac{2\pi}{L}\sum_1^N |a_n|^2.$$

If $\beta-\alpha=\frac{16\pi}{L}$ it follows that

$$\int_\alpha^\beta |P(\theta)|^2 d\theta \geqslant \frac{\beta-\alpha}{8}\sum_1^N |a_n|^2 = \frac{\beta-\alpha}{16\pi}\int_0^{2\pi} |P(\theta)|^2 d\theta.$$

Now let us consider $f(re^{i\theta})$ where $r<1$. We have over $(0, 2\pi)$

$$f(re^{i\theta}) = \underset{N\to\infty}{\text{limit in mean}} \sum_1 a_n r^{\lambda_n} e^{i\lambda_n\theta} = \underset{N\to\infty}{\text{limit in mean}}\, f_N(re^{i\theta}).$$

We have just shown that

$$\int_0^{2\pi} |f_N(re^{i\theta})|^2 d\theta \leqslant \frac{16\pi}{\beta-\alpha}\int_\alpha^\beta |f_N(re^{i\theta})|^2 d\theta,$$

and it follows at once that

$$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leqslant \frac{16\pi}{\beta-\alpha}\int_\alpha^\beta |f(re^{i\theta})|^2 d\theta.$$

Thus

$$\overline{\lim_{r\to 1}} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \leqslant \frac{16\pi}{\beta-\alpha}\int_\alpha^\beta |\varphi(\theta)|^2 d\theta.$$

That is

$$\overline{\lim_{r\to 1}} \sum_1^\infty r^{2\lambda_n} |a_n|^2 \leqslant \frac{8}{\beta-\alpha}\int_\alpha^\beta |\varphi(\theta)|^2 d\theta,$$

whence we find that

$$\sum_1^\infty |a_n|^2 \leqslant \frac{8}{\beta-\alpha}\int_\alpha^\beta |\varphi(\theta)|^2 d\theta,$$

and if we put

$$\psi(\theta) \sim \sum a_n e^{i\lambda_n\theta},$$

then

$$\int_0^{2\pi} |\psi(\theta)|^2 d\theta \leqslant \frac{16\pi}{\beta-\alpha}\int_\alpha^\beta |\psi(\theta)|^2 d\theta$$

and using BESSEL equality for FOURIER Series we have

$$\lim_{r\to 1-0} \int_0^{2\pi} |f(re^{i\theta}) - \psi(\theta)|^2 d\theta = 0.$$

An intermediate corollary of Theorem I is:

THEOREM II. - *On the hypothesis of Theorem I, if $\psi(\theta)$ is the n-times repeated integral of $\psi^{(n)}(\theta)$ over (α, β), it is also the n-times repeated integral $\psi^{(n)}(\theta)$ over $(0, 2\pi)$, $\psi^{n}(\theta)$ is of period 2π and*

$$\int_0^{2\pi} |\psi^{(n)}(\theta)|^2 d\theta \leqslant \frac{16\pi}{\beta - \alpha} \int_\alpha^\beta |\psi^{(n)}(\theta)|^2 d\theta.$$

Let it be noted that the constant in this formula is independent of n. The theorem is obtained by applying Theorem I to $\sum (i\lambda_k)^n a_k z^{\lambda_k}$.

A further corollary is:

THEOREM III. - *On the hypothesis of Theorem I, if $\psi(\theta)$ is infinitely often differentiable, and*

$$\text{(a)} \qquad \int_\alpha^\beta |\psi^{(n)}(\theta)|^2 d\theta < A_n^2 B^n$$

where A_n is given and B is some positive quantity, then

$$\text{(b)} \qquad \int_0^{2\pi} |\psi^{(n)}(\theta)|^2 d\theta \leqslant A_n^2 C^n,$$

where C is some positive quantity. If over (α, β), (a) *is true, which is always the case if*

$$\text{(c)} \qquad |\psi^{(n)}(\theta)| < A_n B_1^n$$

for some B_1, then over the entire circle,

$$\text{(d)} \qquad |\psi^{(n-1)}\theta| < A_n C_1^n$$

for some C_1. In particular, if ψ is analytic over (α, β) it is analytic over the entire circle, and 1 *is not the radius of convergence of the circle, which contradicts our assumptions. Thus the circle $|r| = 1$ is a natural boundary.*

The only parts of Theorem III which need discussion are the transition from (b) to (d), and the analytic case. As to the transition from (b) to (d), let it be noted that as ψ is periodic, the real and the imaginary parts of $\psi^{(n)}(\theta)$ must vanish at places θ_1 and θ_2 (not necessarily identical) between 0 and 2π. Thus by the SCHWARZ inequality

$$\begin{aligned} |\psi^{(n-1)}(\theta)| &\leqslant \left| \int_{\theta_1}^{\theta} R\psi^{(n)}(w) dw \right| + \left| \int_{\theta_2}^{\theta} I\psi^{(n)}(w) dw \right| \leqslant \\ &\leqslant 2 \int_0^{2\pi} |\psi^{(n)}(w)| dw \leqslant 2 \sqrt{\int_0^{2\pi} dw \int_0^{2\pi} |\psi^{(n)}(w)|^2 dw} \leqslant \\ &\leqslant 2\sqrt{2\pi} A_n C^{\frac{n}{2}} \leqslant A_n C_1^n \end{aligned}$$

for some C_1.

As to the analytic case, let us notice that $\psi(\theta)$ is analytic over (α, β) when and only when there exists a C such that over (α, β)

$$\psi^{(n)}(\theta) \leqslant n!\, C^n.$$

Then over $(0, 2\pi)$ we shall have a B such that

$$\psi^{(n)}(\theta) \leqslant (n+1)!\, C_1^n \leqslant n!\, n C_1^n \leqslant n!\, C_2^n,$$

Thus $\psi(\theta)$ is analytic for all θ, and about each point of the circle of convergence, we may describe a small circle within which $f(z)$ may be continued analytically. By the HEINE-BOREL theorem, we may cover the entire unit circle with a finite number of overlapping circles of this sort, which together with the unit circle will contain a large circle concentric with the unit circle. The circle of convergence is hence not the unit circle.

The class of all functions satisfying (c) or (d) is quasi-analytic over its respective interval if

$$\int_0^\infty \log\left(\sum_{\nu=0}^\infty \frac{x^{2\nu}}{A_\nu^2}\right)\frac{dx}{1+x^2}$$

diverges. This statement is precisely equivalent to that of the divergence

$$\int_0^\infty \log\left(\sum_{\nu=0}^\infty \frac{x^{2\nu+2}}{A_\nu^2}\right)\frac{dx}{1+x^2}.$$

Thus we obtain

THEOREM IV. - *On the hypothesis of Theorem I, if $f(re^{i\theta})$ belongs to a quasi-analytic class along the sector (α, β) of the circle of convergence, it belongs to a related quasi-analytic class over the entire circle of convergence.*

The culminating theorem of this paper is

THEOREM V. - *Let $f(z)=\sum_0^\infty a_n z^{\lambda_n}$ have the unit circle as its circle of convergence, and let the λ_n's be a set of integers arranged in increasing order. Let $\lambda_{n+1}-\lambda_n \to \infty$. Then the unit circle is a natural boundary. Further, let $f(z)$ tend over some sector to continuous boundary values as $|z| \to 1$. Then if these boundary values are quasi-analytic over any sector, as in* (c), *with $A_n \geqslant n!$ they exist and belong to a related quasi-analytic class over the entire unit circle.*

Exactly similar results hold for the general DIRICHLET series $\sum a_n e^{\lambda_n z}$ in the abscissa of convergence, where λ_n are real but not necessarily integers, and are proved in exactly the same way.

To prove Theorem V let us notice that if we subtract an appropriate polynomial from $f(z)$, we can make the gaps between the remaining successive λ_n's all as large as we please. We therefore may apply Theorems III and IV to the remainder. We have already so defined the quasi-analytic class of (c) that the functions contained in it remain in it after the addition of an analytic function.

It will be observed that our gap-theorems include the classical HADAMARD gap-theorem, but not the more powerful gap-theorem of FABRY. It is still an open question whether the latter theorem can be established by methods of this type, which essentially go back to the fact that a lacunary FOURIER series represents a function with a certain type of generalized periodicity of arbitrary small period.