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INVERSE PROCESSES IN INVARIANTS, WITH APPLICATIONS TO THREE PROBLEMS IN MECHANICS

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(Jubilee memoir. Fortieth anniversary of the author's Philosophical Doctorate).

After a mathematical subject has been developed from various points of view, by different writers, there may have arisen a problem in criticism. One may be able to work synthetically with these viewpoints, to assert the predominance of one of them, perhaps, and to advance the subject, consequently, in a new direction. Criticism then becomes a phase of origination and more than a mere review of shortcomings in published work.

In a century and one third, invariant theory has been developed from many points of view. Without reciting its history we may say that, within the last quarter of this time, the bases of the subject have been materially broadened, by the introduction of novel groups of transformations, by the use of invariantive concepts in the field of analysis and by the introduction of the method of invariants into mechanics. Among the leading algorithms of invariant theory is that which makes the subject a branch of the theory of linear, partial differential equations and by extending this algorithm farther than others have done, we have unified some diverse viewpoints and originated a theory of inverse processes. An inverse problem, in general, is one in which the ground-quantics and the transformations are the unknowns, to be determined when the invariant is given.

I. - Introduction.

If a function of two sets,

$$f(x_1, x_2; \ a_0, \dots, a_e),$$

is homogeneous in each set, that is, if

$$\left[x \frac{\partial}{\partial x}\right] f = mf, \ \left[a \frac{\partial}{\partial a}\right] f = f; \quad \left[x \frac{\partial}{\partial x}\right] = x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2},$$

m a positive integer, and if a second function of the sets of degree g in the a_i,

$$D(x_1, x_2; \ a_0, \dots, a_e),$$

is invariant when the x_i are transformed by a scheme Q, f can be derived from D by a linear operator under certain conditions. Assuming that the ξ_i in,

$$\Delta = \xi_0 \frac{\partial}{\partial a_0} + \xi_1 \frac{\partial}{\partial a_1} + \dots + \xi_e \frac{\partial}{\partial a_e} \equiv \left[\xi \frac{\partial}{\partial a}\right],$$

are cogredient to the a_i, under the transformations S which Q induces upon the a_i, $\Delta^\sigma D$, $(\sigma = 0, 1, \dots, g)$, is invariant if different from zero. End for end symmetry in the variables x_1, x_2 is postulated for Δ. With $\sigma = g-1$, $\Delta^\sigma D$, being linear in the a_i must be the ground-form f of the invariant problem, but not uniquely, because the cogredient set ξ_i is not unique in general and D is only one of many invariants, here one of a fundamental system [1].

Assuming Q linear,

$$(1) \qquad Q: \quad x_i = \lambda_i x_1' + \mu_i x_2', \qquad (i=1, 2)$$

we may state the problem more inclusively by allowing f and therefore D to involve the parameters of the transformation, assumed at first to be arbitrary variables.

The essential fact about $\Delta^{g-1}D$ is that it is invariant and linear in $a_0, \dots, a_e$. With m left arbitrary, $\Delta^{g-1}D$ (also $\Delta^g D \neq 0$) will contain as factors powers of universal covariants of Q. There may be involved in these factors a case of the covariant [2],

$$\lambda_2 x_1^2 + (\mu_2 - \lambda_1) x_1 x_2 - \mu_1 x_2^2,$$

whose roots are the poles of Q and from which Q can be determined, and there may be among the factors, covariants properly universal. These will involve essentially only the variables; will be covariantive for all admitted values of the parameters λ_i, μ_i, whatever group or set Q represents. The latter universally covariant factors serve to determine Q.

This will seem to be, at first, mostly a binary theory. It is true that an invariant operator like Δ can be constructed on the basis of the determinant (invariant under linear transformations Q') [3],

$$u_x = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = (yz)_1 x_1 + (yz)_2 x_2 + (yz)_3 x_3 = u_1 x_1 + u_2 x_2 + u_3 x_3 .$$

This operator will be of the form,

$$\Delta' = u_1^m \frac{\partial}{\partial a_{m00}} + \binom{m}{1} u_1^{m-1} u_2 \frac{\partial}{\partial a_{m-110}} + \dots,$$

where a_{ijk} are the actual coefficients of a ternary quantic a_x^m or any set of parameters which could be considered as such coefficients. However, repeated operations Δ' upon any mixed form, such as,

$$(abu)^2 a_x^{m-2} b_x^{m-2}, \qquad (b \sim a),$$

often leads to a vanishing result before it leads to a form of the first degree in a_{ijk}. We shall show that the problem in three variables is, however, often solvable in important special forms.

II. - The transformation as unknown.

Thus, in $\Delta^{g-1}D$, we look to the factor linear in $a_0, \ldots, a_e$ for a form of the ground-function, and to the other factors for a means of determining Q. Consider the expression D_1, assumed invariant,

$$D_1 = (a_0 + a_1 + a_2 + a_3)(a_0 - a_1 + a_2 - a_3)(a_0^2 - a_0 a_2 + a_1 a_3 - a_3^2)^2.$$

If the four a_i in the order of their subscripts are regarded as cogredient to the coefficients of x_1^3, $x_1^2 x_2$, $x_1 x_2^2$, x_2^3, in the covariant,

$$(xy)^3 = (x_1 y_2 - x_2 y_1)^3,$$

an invariant operator is,

$$\Delta_1 = y_2^3 \frac{\partial}{\partial a_0} - 3 y_2^2 y_1 \frac{\partial}{\partial a_1} + 3 y_2 y_1^2 \frac{\partial}{\partial a_2} - y_1^3 \frac{\partial}{\partial a_3}.$$

We find,

$$\Delta_1^5 D_1 = 720\,(y_2^2 - y_1^2)^6 (a_0 y_2^3 + a_1 y_2^2 y_1 + a_2 y_2 y_1^2 + a_3 y_1^3).$$

Hence the ground-form f_1 is the arbitrary cubic, and the transformation is the cyclic Q_1 (or Q_2),

$$Q_1: \quad y_1 = \mu y_1' + \nu y_2', \quad y_2 = \nu y_1' + \mu y_2',$$
$$Q_2: \quad y_1 = \mu y_1' + \nu y_2', \quad y_2 = -\nu y_1' - \mu y_2'.$$

The expression $s = a_0^2 - a_0 a_2 + a_1 a_3 - a_3^2$ is also a cyclic invariant of f_1 but the ground-form corresponding is

$$\Delta_1 s = (2a_0 - a_2) y_2^3 - 3 a_3 y_2^2 y_1 - 3 a_0 y_2 y_1^2 + (-a_1 + 2a_3) y_1^3.$$

Since the group is cyclic, y_1, y_2 can be interchanged.

It is usually implied, when (y_1, y_2) is cogredient to (x_1, x_2), that y_1, y_2 are replaceable by x_1, x_2, respectively, but for cogredience, it is not necessary that y_1, y_2 should be linear in x_1, x_2.

Suppose that we wish to pass from y_i to x_i by the substitutions,

$$y_1 = x_1^2 + x_2^2, \quad y_2 = x_1^2 - x_2^2,$$

and that $e = 2$. If the function assumed invariant is,

$$D^1 = \lambda a_1^2 + a_0 a_2, \qquad (\lambda \text{ constant}),$$

Δ^1 being the operator,

$$\Delta^1 = (x_1^2 - x_2^2)^2 \frac{\partial}{\partial a_0} - 2(x_1^4 - x_2^4) \frac{\partial}{\partial a_1} + (x_1^2 + x_2^2)^2 \frac{\partial}{\partial a_2},$$

we get,

$$\Delta^1 D^1 = (a_0 - 4\lambda a_1 + a_2)\, x_1^4 + 2(a_0 - a_2)\, x_1^2 x_2^2 + (a_0 + 4\lambda a_1 + a_2)\, x_2^4 = f^1, \tag{2}$$
$$(\Delta^1)^2 D^1 = (8\lambda + 2)(x_1^4 - x_2^4)^2.$$

The latter form, as a universal covariant, determines the transformation,

$$Q^1: \quad x_1 = i\mu x_2', \quad x_2 = \mu x_1', \qquad (i = \sqrt{-1}).$$

The problem fails when $\lambda = -1/4$.

The choice of the functions $y_1(x)$, $y_2(x)$, cogredient to x_1, x_2, is in fact arbitrary, but unless these (i. e. ξ_i) and the expression D, assumed invariant, are fortunately chosen within the bounds of some essential invariant problem, the final function $\Delta^g D$ $(\neq 0)$ will prove to be invariant only with respect to the identical transformation.

Formal n-ary groups in less than n^2 parameters.

Under the ternary transformation with arbitrary ([1]) coefficients,

$$S_1: \quad x_i = \lambda_i x_1' + \mu_i x_2' + \nu_i x_3', \qquad (i = 1, 2, 3)$$

which we write more simply in the form,

$$S_1: \quad (x) = \begin{pmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{pmatrix} (x'),$$

the determinant,

$$d = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_2 & y_3 & y_1 \\ 1 & 1 & 1 \end{vmatrix},$$

may be assumed invariant, the relation of cogredience,

$$(y_1, y_2, y_3) \quad \text{Co.}\,(x_1, x_2, x_3),$$

being also assumed, but a result of these hypotheses is a particularization of S_1 to a four-parameter sub-group of the general, algebraic, homogeneous group.

This is proved as follows. Since

$$d = (y_3 - y_1)\, x_1 + (y_1 - y_2)\, x_2 + (y_2 - y_3)\, x_3,$$

([1]) An invariant is *algebraic* if the group is Q, or S_1. A *ground-quantic* is any covariant of degree unity in $a_0, \ldots, a_e$. An algebraic *process* is an analytic process which can be carried out rationally.

the differences (y_3-y_1), (y_1-y_2), (y_2-y_3) are contragredient to (x_1, x_2, x_3), under S_1. Hence we have the identities,

$$\begin{aligned}
y_3-y_1&=\lambda_1(y_3'-y_1')+\lambda_2(y_1'-y_2')+\lambda_3(y_2'-y_3')=(\lambda_3-\lambda_1)\,y_1'+(\mu_3-\mu_1)\,y_2'+(\nu_3-\nu_1)\,y_3',\\
y_1-y_2&=\mu_1(y_3'-y_1')+\mu_2(y_1'-y_2')+\mu_3(y_2'-y_3')=(\lambda_1-\lambda_2)\,y_1'+(\mu_1-\mu_2)\,y_2'+(\nu_1-\nu_2)\,y_3',\\
y_2-y_3&=\nu_1(y_3'-y_1')+\nu_2(y_1'-y_2')+\nu_3(y_2'-y_3')=(\lambda_2-\lambda_3)\,y_1'+(\mu_2-\mu_3)\,y_2'+(\nu_2-\nu_3)\,y_3'.
\end{aligned}$$

These lead to the relations,

$$\begin{array}{lll}
\lambda_2-\lambda_1=\lambda_3-\lambda_1, & \lambda_3-\lambda_2=\mu_3-\mu_1, & \lambda_1-\lambda_3=\nu_3-\nu_1,\\
\mu_2-\mu_1=\lambda_1-\lambda_2, & \mu_3-\mu_2=\mu_1-\mu_2, & \mu_1-\mu_3=\nu_1-\nu_2,\\
\nu_2-\nu_1=\lambda_2-\lambda_3, & \nu_3-\nu_2=\mu_2-\mu_3, & \nu_1-\nu_3=\nu_2-\nu_3,
\end{array}$$

whence S_1 takes the form,

$$S: \quad (x)=\begin{pmatrix} \lambda_2+\mu_2-\mu_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_1 \\ \lambda_2 & \mu_1 & \nu_1+\mu_2-\mu_1 \end{pmatrix}(x').$$

THEOREM. - *The transformations of the type S form a group, represented by* (S).

In proof, assume that the determinant δ of S is not zero,

$$\delta=(\lambda_2+\mu_2+\nu_1)(\mu_1-\mu_2)^2 \neq 0.$$

Then S is non-singular and its inverse is easily shown to be of the same type as S. Let T be an independent transformation of this type,

$$T: \quad (x)=\begin{pmatrix} L_2+M_2-M_1 & M_1 & N_1 \\ L_2 & M_2 & N_1 \\ L_2 & M_1 & N_1+M_2-M_1 \end{pmatrix}(x'),$$

then, if we combine S and T, we find,

$$S'=ST: \quad (x)=\begin{pmatrix} q+t-r & r & s \\ q & t & s \\ q & r & s+t-r \end{pmatrix}(x''),$$

in which,

$$\begin{aligned}
q&=L_2(\lambda_2+\mu_2-\mu_1)+(M_2+N_1)\,\lambda_2, & r&=(L_2+M_2-M_1)\,\mu_1+M_1\,\mu_2+N_1\,\mu_1,\\
s&=(L_2+M_2)\,\nu_1+N_1(\nu_1+\mu_2-\mu_1), & t&=L_2\,\mu_1+M_2\,\mu_2+N_1\,\mu_1.
\end{aligned}$$

Hence the product of two transformations of the set (S) is within the set (S); also the identical transformation, $(q=r=s=0,\ t=1)$, is within (S), q. e. d.

If we substitute $y_i=x_i$ in d, the resulting covariant d_x,

$$d_x=-x_1^2-x_2^2-x_3^2+x_1\,x_2+x_1\,x_3+x_2\,x_3,$$

is linearly factorable,

$$d_x = d_1 d_2 = (x_1 - wx_2 + w^2 x_3)(x_1 + w^2 x_2 - wx_3),$$

w being an imaginary cube root of -1. The forms d_1, d_2 are left invariant individually by the typical transformation of (S), (taken as S'). The invariant relations are,

$$d_1 = (t-r)\, d_1', \qquad d_2 = (t-r)\, d_2'.$$

There is a third (real) covariant of order one, viz.,

$$d_3 = qx_1 + rx_2 + sx_3\ ;\ d_3 = (q+s+t)\, d_3'.$$

It is a covariant of a subset of (S), (vary t). The vertices of the covariant triangle $d_1 d_2 d_3$ are the poles of S', one being real.

If we substitute from the inverse of T',

$$T':\quad (d) = \begin{pmatrix} 1 & -w & w^2 \\ 1 & w^2 & -w \\ q & r & s \end{pmatrix} (x),$$

for x_1, x_2, x_3, in the arbitrary ternary quantic,

$$f_m = a_x^m = \sum a_{ijk} x_1^i x_2^j x_3^k,$$

the result is an expansion of f_m in the arguments d_1, d_2, d_3,

$$f_m' = A_d^m = \sum A_{ijk} d_1^i d_2^j d_3^k.$$

The $\frac{1}{2}(m+1)(m+2)$ coefficients A_{ijk} are invariants of degree one in the a_{ijk} and appertain to a domain Ω which includes w, q, r, s. They are thus invariants of the subset of (S) and, with d_1, d_2, d_3, are called invariant elements. In Ω the invariant elements are a fundamental system of invariants and covariants. This is evident since the inverse of T' and the inverse of

$$A_{ijk} = A_{ijk}(a_{m00}, a_{m-110}, \ldots, a_{00m}),$$

serve to express any concomitant of f_m under the subset, as a rational, integral function, in Ω, of d_1, d_2, d_3, A_{ijk}. The A_{ijk} $(=0)$ are invariant hyperplanes expressed in the a_{ijk} as variables.

Concomitants of f_m under (S) and therefore of a domain Ω_1 free from w, q, r, s, are polynomials in the invariant elements of structure such that an algebraic simplification from Ω to Ω_1 is always possible. Such a system of polynomials is a HILBERT system within which there exists a set which forms a finite basis. The basis theory will be considered further in a later section.

A fundamental system for $f_m = a_x^m$ under the ternary algebraic group of nine parameters, *e.g.* the system of a cubic (GORDAN) and that of a quartic (E.

NOETHER), is involved with advanced formalism, but in any mixed form or contravariant of such a system, u_1, u_2, u_3 can be replaced by x_3-x_1, x_1-x_2, x_2-x_3 respectively. The result will be a covariant of f_m under (S). The method furnishes, as far as proved, only parts of complete systems under (S) but it gives new life to the algebra and geometry by reduction of the formal complication and an increase in the number of covariant configurations.

Formulae for the conic-form,

$$f_2 = ax_1^2 + bx_2^2 + cx_3^2 + 2hx_1x_2 + 2gx_1x_3 + 2fx_2x_3,$$

are added. The invariant elements are d_1, d_2, d_3, and the following expressions, in which $\zeta = (w^2+w)(q+r+s)$.

$$\begin{aligned}
\zeta^2 A &= (w^2r^2 - ws^2 - 2rs)a + (w^2q^2 + 2wqs + s^2)b + (-wq^2 - 2w^2qr + r^2)\, c \\
&\quad + 2(qs - w^2qr - w^2s^2 - wrs)h + 2(qr + w^2rs + wqs + wr^2)\, g \\
&\quad + 2(-rs + w^2qs - wqr - q^2)f, \\
\zeta^2 B &= (w^2s^2 - wr^2 - 2rs)a + (-2w^2qs - wq^2 + s^2)b + (w^2q^2 + 2wqr + r^2)\, c \\
&\quad + 2(qs + w^2rs + wqr + ws^2)\, h + 2(qr - w^2qs - w^2r^2 - wrs)\, g \\
&\quad + 2(w^2qr - wqs - rs - q^2)\, f, \\
\zeta^2\, C &= -3(a+b+c+2h+2g+2f), \\
\zeta^2 H &= (-r^2 - rs - s^2)a + (-q^2 - qs - s^2)b + (-q^2 - qr - r^2)\, c \\
&\quad + (2qr + qs + rs - s^2)h + (2qs + qr + rs - r^2)g + (2rs + qr + qs - q^2)\, f, \\
\zeta^2 G &= ((w-2)r - (w+1)s)a + (-(2w-1)s - (w-2)q)\, b \\
&\quad + ((w+1)q + (2w-1)r)c + (-(w-2)q + (w-2)\, r - 3ws)\, h \\
&\quad + ((w+1)q + 3(w-1)r - (w+1)s)g + (3q + (2w-1)(r-s))\, f, \\
\zeta^2 F &= (-(w+1)r + (w-2)s)a + ((w+1)q + (2w-1)s)\, b \\
&\quad + (-(w-2)q - (2w-1)r)c + (3(w-1)s + (w+1)(q-r))\, h \\
&\quad + (-3wr + (w-2)(s-q))g + (3q + (2w-1)(s-r))\, f.
\end{aligned}$$

The line-equation contravariant $(abu)^2$ now becomes a covariant conic-form expressible as,

$$\Gamma = \begin{vmatrix} a & h & g & x_3-x_1 \\ h & b & f & x_1-x_2 \\ g & f & c & x_2-x_3 \\ x_3-x_1 & x_1-x_2 & x_2-x_3 & 0 \end{vmatrix}.$$

The coefficients of d_x are cogredient to those of f_2, hence there exists an invariant operator O,

$$O = -2\frac{\partial}{\partial a} - 2\frac{\partial}{\partial b} - 2\frac{\partial}{\partial c} + \frac{\partial}{\partial h} + \frac{\partial}{\partial g} + \frac{\partial}{\partial f}. \tag{3}$$

If D is the discriminant of f_2, the following concomitants exist, (Cf. $\zeta^2 C$ above);

$$D, \quad OD, \quad O^2D=6(q+r+s)^2 C, \quad f_2, \quad \Gamma, \quad O\Gamma=2(q+r+s)^2 Cd_x, \quad O^2\Gamma=0.$$

In the twenty known concomitants of the simultaneous system of f_2 and a second arbitrary conic-form g_2 under S_1, both ground-quantics have the standing of covariants, but if we make g_2 the same as d_x, g_2 loses this property unless, at the same time, we particularize the group to (S). When this is done, the twenty quantics, (i. e. those of them that do not vanish), become concomitants of f_2 under (S), wherein also u_1, u_2, u_3, become, respectively, x_3-x_1, x_1-x_2 x_2-x_3.

* * *

If we proceed from the bi-quaternary form,

$$k=(y_4-y_1)x_1+(y_1-y_2)x_2+(y_2-y_3)x_3+(y_3-y_4)x_4,$$

and follow the method we have used in the case of d, there is obtained a six-parameter quaternary group (U), where,

$$U: \quad (x)=\begin{pmatrix} u+w-v & v & x & z \\ u & w & y & z+x-y \\ u+x-y & v & y+w-v & z \\ u & v+x-y & y & z+w-v \end{pmatrix}(x').$$

The transformation U leaves invariant the conicoid,

$$k_x=-x_1^2-x_2^2-x_3^2-x_4^2+x_1x_2+x_3x_4+x_1x_4+x_2x_3.$$

There is, probably, a corresponding group for any number, n, of variables x_i but the difficulties increase with n.

We can now readily write Δ-operators and algorithms to correspond with § I for quantics a_x^m (ternary) or b_x^m (quaternary) under the respective groups (S) and (U).

III. – Instances of determination of ground-quantics.

Consider the class of expansions obtained when the arbitrary quantic f_M of order,

$$M=(m+1)n-1, \qquad (n>0),$$

$$f_M=a_x^M=a_0x_1^M+a_1x_1^{M-1}x_2+\ldots+a_Mx_2^M,$$

is developed in the homogeneous form,

$$f_M=\varphi_{0n-1}f_1^m+\binom{m}{1}\varphi_{1n-1}f_1^{m-1}f_2+\ldots+\binom{m}{m}\varphi_{mn-1}f_2^m,$$

f_1, f_2 being arbitrary forms of order n,

$$f_1 = \beta_0 x_1^n + \beta_1 x_1^{n-1} x_2 + \ldots + \beta_n x_2^n = f_1(x),$$
$$f_2 = \gamma_0 x_1^n + \gamma_1 x_1^{n-1} x_2 + \ldots + \gamma_n x_2^n = f_2(x).$$

The expansion has a symbolic basis, by which is meant that it can be expressed also in the form,

$$f_M = \Phi f_1^m + \frac{\nabla\Phi}{\underline{|1}} f_1^{m-1} f_2 + \frac{\nabla^2\Phi}{\underline{|2}} f_1^{m-2} f_2^2 + \ldots + \frac{\nabla^m\Phi}{\underline{|m}} f_2^m,$$

wherein,

$$\Phi = \varphi_{0n-1}, \quad -\nabla = \left(\beta_0 \frac{\partial}{\partial\gamma_0} + \beta_1 \frac{\partial}{\partial\gamma_1} + \ldots + \beta_n \frac{\partial}{\partial\gamma_n}\right) = \left[\beta \frac{\partial}{\partial\gamma}\right],$$

and therefore in the form of a symbolic power,

$$f_M = (\Xi_1 f_1 + \Xi_2 f_2)^m = \Xi_f^m = \Xi_f'^m = \ldots .$$

The point is that the symbolism makes sense, for we have not only,

$$\Xi_1^{m-i} \Xi_2^i = \varphi_{in-1}, \qquad (i = 0, \ldots, m),$$

but also a symbolical equivalent for ∇,

$$\nabla = \Xi_2 \frac{\partial}{\partial \Xi_1}.$$

Suppose there is given a product P,

$$P = (\Xi\,\Xi')^p (\Xi\,\Xi'')^q \ldots (\Xi'\,\Xi'')^r \ldots \Xi_{f(x)}^s \,\Xi'^t_{f(x)} \ldots,$$

wherein the number of symbols Ξ, Ξ',... is g, the degree of P in φ_{0n-1}, φ_{1n-1}, ... φ_{mn-1}, and that the sum of all the exponents which affect a chosen symbol Ξ of P is m. Assume P to be invariant under a transformation of f_M, for which there exist the following cogrediencies,

$$(\Xi_2, -\Xi_1) \text{ Co. } (\partial/\partial f_2, -\partial/\partial f_1) \text{ Co. } (f_1(y), f_2(y)) \text{ Co. } (y_1, y_2) \text{ Co. } (x_1, x_2).$$

Then an operator Δ of § I can be chosen in the form,

(4) $$\Delta = \left(f_2(y) \frac{\partial}{\partial \Xi_1} - f_1(y) \frac{\partial}{\partial \Xi_2}\right)^m \equiv \left(-\left(f(y) \frac{\partial}{\partial \Xi}\right)\right)^m.$$

Then,

$$-\left(f(y) \frac{\partial}{\partial \Xi}\right)(\Xi\,\Xi') = \Xi'_{f(y)},$$
$$-\left(f(y) \frac{\partial}{\partial \Xi}\right) \Xi_{f(x)} = (f(x)\, f(y)) = f_1(x) f_2(y) - f_2(x) f_1(y).$$

Hence ΔP is free from the symbols Ξ_1, Ξ_2; $\Delta^2 P$ is free from Ξ_1, Ξ_2, Ξ_1', Ξ_2', while,

$$\Delta^{g-1} P = K(f(x)\, f(y))^E \,\Xi_{f(y)}^{(g)M-F}\, \Xi_{f(x)}^{(g)F},$$

K being a numerical constant. The ground-quantic is thus determined uniquely by $\Delta^{g-1}P$, as

$$f_M = \Xi_{f(y)}^{(g)^{M-F}} \Xi_{f(x)}^{(g)^F} \Big]_{y_1=x_1,\ y_2=x_2}.$$

The covariant $(f(x), f(y))^E$ shows that P is a combinant [4] of f_1, f_2 and its factor $(xy)^E$ shows that the variables x_1, x_2 are subject to the general linear algebraic transformation Q of (1).

IV. - Modular instances and theory.

When the equalities dealt with in the above sections are changed to identical congruences with respect to a prime modulus p, much of the formulary remains intact. The group G (or $G(p)$) being the homogeneous total group (mod. p), two independent universal covariants exist from which cogrediencies may be obtained. These are due to DICKSON, as follows [5]

$$L = x_1^p x_2 - x_1 x_2^p,$$
$$N = x_1^{p(p-1)} + x_1^{(p-1)(p-1)} x_2^{p-1} + x_1^{(p-2)(p-1)} x_2^{2(p-1)} + \dots + x_2^{p(p-1)}.$$

The determinant (xy), with (y_1, y_2) Co. (x_1, x_2), and y_i arbitrary, is the all-inclusive form of universal covariant when the group is the general linear algebraic Q (cf. (1)), in which case y_i may be considered an arbitrary function of x_1, x_2. If the group is special, e. g. $G(p)$, y_i, in an invariant (xy), will be a particular function of x_1, x_2.

For construction of Δ all cogrediencies are obtainable from universal covariants. From L, $y_i = x_i^p$ Co. x_i, whence is obtained,

$$\Delta = x_2^{pe} \frac{\partial}{\partial a_0} - \binom{e}{1} x_2^{p(e-1)} x_1^p \frac{\partial}{\partial a_1} + \dots + (-1)^e x_1^{pe} \frac{\partial}{\partial a_e},$$

$(f = a_0 x_1^e + a_1 x_1^{e-1} x_2 + \dots)$. Other operators may be obtained by writing N in the form (xy). It may be written in either of two forms,

$$N = \begin{vmatrix} x_1^{(p-1)i+1}, & x_2^{(p-1)i+1} \\ \psi_1 / x_2^{(p-1)i}, & \psi_2 / x_1^{(p-1)i} \end{vmatrix}, \qquad (i = 0 \text{ or } 1;\ p > 2),$$

in which,

$$\psi_1 = -\left(x_1^{[p-(p+1)/2](p-1)} x_2^{[(p+1)/2](p-1)-1} + x_1^{[p-(p+1)/2-1](p-1)} x_2^{[(p+1)/2+1](p-1)-1} + \dots + x_2^{p(p-1)-1}\right)$$
$$\psi_2 = x_1^{p(p-1)-1} + x_1^{(p-1)(p-1)-1} x_2^{p-1} + \dots + x_1^{[p-(p+1)/2+1](p-1)-1} x_2^{[(p+1)/2-1](p-1)}.$$

The cogrediency under $G(p)$,

(5) $$(\psi_{1i},\ \psi_{2i}) \text{ Co. } (x_1^{(p-1)i+1},\ x_2^{(p-1)i+1}),$$

leads to the operator Δ_i invariant with respect to the quantic,

$$f_e = a_0 x_1^e + a_1 x_1^{e-1} x_2 + \dots + a_e x_2^e,$$

viz.,

$$\Delta_i = \psi_{2i}^e \frac{\partial}{\partial a_0} - \binom{e}{1} \psi_{2i}^{e-1} \psi_{1i} \frac{\partial}{\partial a_1} + \dots + (-1)^e \binom{e}{e} \psi_{1i}^e \frac{\partial}{\partial a_e},$$

$(\psi_{1i} = \psi_1 / x_2^{(p-1)i},\ \psi_{2i} = \psi_2 / x_1^{(p-1)\,i};\ i = 0,\ 1)$.

There is no simple formula analogous to P which would represent definitively all concomitants of f under $G(p)$, but particular cases raise important questions. Consider the formal invariant, $(p=3,\ e=2)$, first brought to light by DICKSON,

$$R = a_0^2 a_2 + a_0 a_2^2 + a_0 a_1^2 + a_1^2 a_2 - a_0^3 - a_2^3.$$

We find,

$$\Delta_i R = (x_1^2 + x_2^2)^2 [(a_0^2 + a_1^2 - a_0 a_2)\, x_2^{2(3-2i)} + (a_0 a_1 + a_1 a_2)\, x_1^{(3-2i)} x_2^{(3-2i)} + (a_1^2 + a_2^2 - a_0 a_2)\, x_1^{2(3-2i)}],$$

$$\Delta_i^2 R = -(x_1^2 + x_2^2)^4 [a_0 x_2^{4(3-2i)} + a_1 x_2^{3(3-2i)} x_1^{3-2i} + a_1 x_2^{3-2i} x_1^{3(3-2i)} + a_2 x_1^{4(3-2i)}].$$

The quantities in brackets will be covariants of the whole group $G(3)$ after transformation by $x_1 = x_2'$, $x_2 = x_1'$, but the group of $x_1^2 + x_2^2$ is the orthogonal q (or q'),

$$q: \quad x_1 = \mu x_1' - \nu x_2', \quad x_2 = \nu x_1' + \mu x_2', \qquad q': \quad x_1 = \mu x_1' - \nu x_2', \quad x_2 = -\nu x_1' - \mu x_2',$$

where μ, ν are integral residues (mod. 3).

The bracketed covariant in $\Delta_i^2 R$ is reducible if $i=0$, and if $i=1$ it is the ground-form,

$$f_4 = a_0 x_1^4 + a_1 x_1^3 x_2 + a_1 x_1 x_2^3 + a_2 x_2^4,$$

and different from $f_2 = a_0 x_1^2 + 2 a_1 x_1 x_2 + a_2 x_2^2$. Note that $f_2 = \left[x \dfrac{\partial}{\partial x^3}\right] f_4$.

Operators $\Delta_{\gamma\delta}$ analogous to (3) in § 2 exist for arbitrary quantics f_e whose orders can be composed from the orders of L and N,

$$e = \gamma p(p-1) + \delta(p+1), \qquad (\gamma,\ \delta \text{ integers}), \tag{6}$$

for example,

$$\Delta_{10} = \frac{\partial}{\partial a_0} + \frac{\partial}{\partial a_{p-1}} + \frac{\partial}{\partial a_{2(p-1)}} + \dots + \frac{\partial}{\partial a_{p(p-1)}}.$$

The above theory in which special situations are emphasized, prepares for the following,

PROPOSITION. - *For the case of a typical, special group of transformations, viz. $G(p)$, to base a new theory of fundamental systems upon the total set of ground-quantics.*

The quantic f_e will be as employed above, $a_0, \ldots, a_e$ being arbitrary variables, and the term concomitant, where not otherwise defined, will mean an invariant or covariant, modulo p, in $a_0, \ldots, a_e, x_1, x_2$. Equalities, except where further explained, will be congruences, modulo p, existing identically in the a_i, x_j. Occasionally, as a reminder, the modulus will be indicated. The Proposition is solved by means of a series of lemmas.

LEMMA 1. - *Every covariant f, (mod. p), of degree unity in the coefficients a_i of f_e can be derived from one of degree >1 in the a_i by a succession of operations by a Δ of the modular type.*

Let the coefficients of f be $\beta_0, \ldots, \beta_c$. They are linear in the a_i. Form any definite algebraic concomitant R of f of degree $g>1$ in the β_j. Then R is also a concomitant of f_e under $G(p)$. By § 3, $(n=1)$, there exists a Δ_1, and

$$\Delta_1^{g-1} R = J(xy)^b f_{yr}, \qquad (J \text{ an integer; } f_y \text{ a polar}).$$

The constant J is of non-essential type and may always be discarded but no essential numerical factor ϱ in R can be discarded. If $\varrho \equiv 0$ (mod. p), R vanishes and a concomitant of f different from R must be chosen. The operator Δ_1 is of the form,

$$\Delta_1 = y_2^c \frac{\partial}{\partial \beta_0} - \binom{c}{1} y_2^{c-1} y_1 \frac{\partial}{\partial \beta_1} + \ldots + (-1)^c \binom{c}{c} y_1^c \frac{\partial}{\partial \beta_c},$$

however,

$$\frac{\partial}{\partial \beta_j} = \frac{\partial a_0}{\partial \beta_j} \frac{\partial}{\partial a_0} + \frac{\partial a_1}{\partial \beta_j} \frac{\partial}{\partial a_1} + \ldots + \frac{\partial a_e}{\partial \beta_j} \frac{\partial}{\partial a_e}, \qquad (j=0, \ldots, c).$$

The derivatives $\partial a_i / \partial \beta_j$ are numerical. They are not all zero, else Δ_1 as an algebraic operator would be illusory, which it never is. Substituting the values of the $\partial/\partial\beta_j$, Δ_1 becomes an operator Δ in the $\partial/\partial a_i$ for which, both algebraically and modulo p.

$$\Delta^{g-1} R / J(xy)^b]_{y_k = x_k} = f, \qquad \text{q. e. d.}$$

Note that, if the algebraic concomitant chosen as R is of the functional form,

$$\varphi(a_0^p, a_1^p, \ldots, a_e^p),$$

it should be replaced, in this theory, by

$$R = \varphi(a_0, a_1, \ldots, a_e).$$

This is possible, since, under the induced group,

$$(a_0^p, a_1^p, \ldots, a_e^p) \text{ Co. } (a_0, a_1, \ldots, a_e).$$

LEMMA 2. - *Every concomitant D, modulo p of the formal type, of f_e, is a simultaneous algebraic concomitant of the set consisting of L, N, and* *the totality of linearly independent formal modular concomitants,* $l_1, l_2, \ldots,$ *of the first degree in* $a_0, \ldots, a_e$.

The proof results from the succession of algorithms (*a*), (*b*),..., (*f*). Let D be of degree g in $a_0, \ldots, a_e$.

(*a*) We are to prove first the converse of Lemma 1, that D of Lemma 2 can be differentiated to an $l_i \neq 0$ by successive operations by a Δ operator of one of two types, the first being,

$$\Delta(z) = y_2(z)^e \frac{\partial}{\partial a_0} - \binom{e}{1} y_2(z)^{e-1} y_1(z) \frac{\partial}{\partial a_1} + \ldots + (-1)^e y_1(z)^e \frac{\partial}{\partial a_e},$$

where $y_i(z)$ is an arbitrary function of z_1, z_2 particularizable only in such ways as will make the cogrediency,

$$(z_1, z_2) \text{ Co. } (y_1(x), y_2(x)) \text{ Co. } (x_1, x_2),$$

in any case, one which represents the total group, $G(p)$ (mod. p), as distinguished from a subgroup. If $\Delta(z)D = 0$, it is because this polynomial in x_1, x_2 vanishes identically [6]. If $y_1 = z_1^{p^t}$, $y_2 = z_2^{p^t}$, with t any chosen integer >0, and if C_j is any coefficient of the covariant D, ($C_j = D$ if D is an invariant), then,

$$\Delta(z)\, C_j = z_2^{ep^t} \frac{\partial C_j}{\partial a_0} - \binom{e}{1} z_2^{(e-1)p^t} z_1^{p^t} \frac{\partial C_j}{\partial a_1} + \ldots + (-1)^e z_1^{ep^t} \frac{\partial C_j}{\partial a_e},$$

vanishes identically. Hence C_j is free from all a_i, (or contains all of its a_i in the form $a_i^{\alpha p}$. Cf. (*b*)), except possibly the a_i for which $\binom{e}{i}$ is divisible by p. Thus the hypothesis, $\Delta(z)D = 0$ is untenable with y_1, y_2 as assumed, except possibly in certain special cases. In the special case, choose $\Delta_1(z)$ instead of $\Delta(z)$, $\Delta_1(z)$ being the result of substituting $\partial/\partial a_0$, $\partial/\partial a_1, \ldots,$ for x_1^e, $x_1^{e-1} x_2, \ldots,$ respectively, in the covariant,

$$\Psi = (zx)(z^p x)(z^{p^2} x) \ldots (z^{p^{e-1}} x).$$

No coefficient of a term of Ψ contains the modulus. Hence in the special cases being considered, $\Delta_1(z)\, C_j = 0$ reduces $\Delta_1(z)D = 0$ to an absurdity (Cf. (*b*) however).

Let $\Delta(z)D \neq 0$, for some y_1, y_2 considered. Then (²),

$$\text{(7)} \qquad \Delta(z)D = d_1, \qquad \Delta(z) d_1(z, x) = d_2 \neq 0.$$

Continuing we reach,

$$\Delta(z) d_{g-2}(z, x) = d_{g-1} = K_1 A(z, x)\, a_z^r b_x^s \neq 0,$$

$a_z^r\, b_x^s$ being a symbolical product equal to a form of degree one in $a_0, \ldots, a_e$,

---

(²) The $\Delta(z)$ $\Delta(z)D$ is found most easily by operating the formal square of $\Delta(z)$ upon D. The expression K_1 is constant or (if $r = s = 0$) an invariant of degree unity in $a_0, \ldots, a_e$.

and $A(z, x)$ a universal covariant which becomes a polynomial in L, N when $z_k = x_k$, the possibility $A(x, x) = 0$ not being excluded if $A(z, x)$ contains the factor (zx). We have $a_x^r\, b_x^s = l_i \neq 0$. Inversely $a_z^r\, b_x^s$ is formally a term of the r-th polar of l_i, but, since $a_z^r\, b_x^s$ is a covariant, it is the whole polar of l_i.

(*b*) There is one and only one class of exceptions to (*a*), viz. when D has the functional form, like

$$\varphi(a_0^p, \ldots) = [a^p \partial/\partial a]^h \psi(a_0, \ldots), \qquad (0 < h < g+1),$$

of a formal modular concomitant of f_e taken simultaneously with a quantic,

$$f_{te} = a_0^{pt} x_1^e + a_1^{pt} x_1^{e-1} x_2 + \ldots + a_e^{pt} x_2^e .$$

Represent the process of algebraic transvection between two quantics A, B, by $\{A, B\}^r$. Then, using only known elementary lemmas in number theory, such as

$$\binom{e\,p^t}{i\,p^t} \equiv \binom{e}{i} \quad (\text{mod. } p),$$

we find,

$$f_{te} \equiv \{f_e^{p^t}, L_t^e\}^{ep^t} \quad (\text{mod. } p) \qquad (L_t = x_1 x_2^{p^t} - x_2 x_1^{p^t}).$$

Since L_t is reducible in terms of L and N, f_{te} is in the algebraic system of the set $(L, N, l_1, l_2, \ldots)$. There is a like formula for F_{tc}, F_c being any covariant of order c. It is known also that N is an algebraic covariant of L, being given by the formula,

$$\{\ldots\{\{L, L\}^2, \overbrace{L\}, L\}, \ldots, L\}}^{p-2} \equiv N \quad (\text{mod. } p).$$

Our series of differential equations which ended with $\varDelta(z) d_{g-2}(z, x) \neq 0$ will now end before any $d_i(z, x)$ linear in $a_0, \ldots, a_e$ is reached, but not before we reach a form D' which is a concomitant of f_{te} alone. Replace, in $\varDelta$, $\partial/\partial a_i$ by $\partial/\partial a_i^{p^t}$, $(i = 0, \ldots, e)$ and call the new operator $\varDelta(\partial/\partial a_i^{p^t})$. It will reduce D' to a covariant linear in $a_0^{p^t}, \ldots, a_e^{p^t}$ and the latter will be an algebraic transvectant of one of the l_i. Integrating back to D' the latter is seen to be an algebraic concomitant of the set $(L, N, l_1, l_2, \ldots)$. Using $\varDelta(\partial/\partial a_i)$ again we can integrate from D' to D, but the nature of the integration processes is to be described in (*d*). There is an obvious extension of this algorithm to the case where D is a simultaneous concomitant of any number of forms f_{te} including f_e.

(*c*) The maximum order in x_1, x_2 of an irreducible formal modular concomitant D is $p^2 - 1$. In the writer's paper in the Annals of Mathematics volume 19, this theorem was not stated in its full generality, but in an earlier paper, a more general view-point had been discussed. In relation (1) of the Annals article, if $m > p^2$, so there is an excess in the number of coefficients of φ_1, φ_2 over the number in f_m, the excess number of coefficients can be arbitrarily assigned if the congruences afterward available are left consistent. Solution of the congruences determine φ_1, φ_2 in terms of which f_m, $m>p^2-1$, is reduced. Also D is reduced if its order is $>p^2-1$. For,

$$D'=(\lambda\mu)^k D, \quad D==L\varphi_1+N\varphi_2, \quad D'=L'\varphi_1'+N'\varphi_2'.$$

Hence $\varphi_2'=(\lambda\mu)^j\varphi_2$, (x_1, x_2) being an integral root of L. But if φ_2 is thus modular and homogeneous in x_1 and x_2 it is a formal covariant.

(*d*) Let $\Delta(z)$ be as originally chosen in (*a*). Then the partial differential equation in $a_0,\ldots, a_e$,

$$\Delta(z)d_{g-2}(z, x)=d_{g-1}(z, x),$$

can be solved for d_{g-2} as the dependent variable. The solution is known to exist and to be a concomitant. The process of solution can be regarded as unique, is analytic, and in effect algebraic (Cf. § V). The equalities are equations, the modulus not being applied either in forming equations (7) or in integrating them. Then we can integrate the equation,

$$\Delta(z)d_{g-3}(z, x)=d_{g-2}(z, x).$$

Continuing we can solve for D from,

$$\Delta(z)D=d_1(z, x).$$

The argument where D' is involved has already been stated. The main point is that D is thus obtained by algebraic processes based originally upon an l_i taken with L and N, (Cf. (*f*)).

(*e*) Some details concerning integration processes may be mentioned. We have always LAGRANGE's method, but it need not be used. If the formal modular concomitant D of f_e, with which we start is also an algebraic concomitant of f_e, D is a linear combination of symbolic monomials of the type of T,

$$T=(ab)^\varrho(ac)^\sigma\ldots(bc)^\tau\ldots a_x^q\, b_x^s\ldots, \quad (f_e=a_x^e=b_x^e=\ldots).$$

Now,

$$\Delta T=\left(y_2\frac{\partial}{\partial a_1}-y_1\frac{\partial}{\partial a_2}\right)^e T=k(bc)^\tau(bd)^r\ldots b_x^s\ldots(xy)^q\, b_y^\varrho\, c_y^\sigma\ldots,$$

and inversely if the right hand quantic kT_1 is given and T is unknown in this relation, we can solve for T. It is a term of the transvectant ([3]),

$$t_1=\{T^{(1)}, a_x^e\}^{e-q},$$

([3]) The letters k, β, γ, δ represent constants.

$T^{(1)}$ being what T_1 becomes when $(xy)^q$ is deleted and y_j changed to x_j, $(j=1, 2)$. Hence by a known theorem, T is a linear expression in algebraic transvectants,

$$T=\sum_{i=1,2,\ldots} h_i \{ T^{(i)}, a_x^e \}^{r_i} = \sum_i h_i t_i, \qquad (h_i \text{ numerical}),$$

the forms $T^{(i)}$ being obtained from the leading one, $T^{(1)}$, by convolution. If d_1, in our equation $\Delta(z)D=d_1$, is a sum; $d_1=\Sigma\beta T_1$, D is a sum $\Sigma\gamma T=\Sigma\Sigma\delta t_i$. This formulary is applicable to all equations (7) if D is algebraic.

Let D in $\Delta(z)D=d_1$ be formal modular without being regarded, at first, as algebraic. Some expressions based on modular cogrediencies will occur in the analysis which we were able to dispense with in the discussion just given. Among these are the polars,

$$P'=x_1^{pt}\frac{\partial F}{\partial x_1}+x_2^{pt}\frac{\partial F}{\partial x_2}=m\{L_t, F\}^1 \quad (\text{mod. } p),$$

m being the order of F, but the transvectant is in the algebraic system of the set $(L, N, l_1, l_2, \ldots)$ if F is, $(m \neq 0 \pmod{p})$. In the work of calculating any $\{A, B\}^r$, the last step is to change y_j into x_j. By means of the change y_j equal to x_j^{pt}, $(j=1, 2)$, we obtain a modular transvectant $\mu=\{A, B\}_{pt}^r$, but the writer has proved that any μ is a linear expression in modular polars of algebraic transvectants, the domain of the expression to include L and N [7].

The cogrediencies following (and certain contragrediencies),

$$(a_0^{pt}, a_1^{pt}, \ldots, a_e^{pt}) \text{ Co. } (a_0, a_1, \ldots, a_e),$$

under the induced linear group, are represented simply by certain pure invariants due to W. L. G. WILLIAMS [9]. The induced group is special but, for it, we can copy the invariants designated by WILLIAMS as the fundamental invariants of the arbitrary linear form in $q=e+1$ variables under the total linear group whose coefficients are marks of the $GF(p^n)$. In the present case $n=1$, and his system becomes, for us, any q invariants whose leading terms are respectively,

$$a_0 a_1^p a_2^{p^2} a_3^{p^3} \ldots a_e^{p^e}, \quad a_1^{ps} a_2^{p^2 s} a_3^{p^3 s} \ldots a_e^{p^e s}, \ldots, \quad a_e^{p^e s}, \qquad (s=p-1).$$

The invariants meant are now evidently simultaneous invariants, for certain indicated values of t, of forms f_{te} which we have constructed by algebraic transvection between f_e and L and N.

Transvection, as is well known, is the definitive process for construction of complete systems of algebraic concomitants of any set of binary quantics.

(*f*) The analytic integration of equations (7) described in (*d*), is, in effect, algebraic and is a master [4] process which includes implicitly all of the

[4] MACMAHON: *Combinatory Analysis*, Ch. II.

formularies which we have described in (*e*). At first D seems to be derivable algebraically only from l_1, l_2,..., taken with the universal covariants ([5]) of the two cogredient sets z_i, x_i. However the variables are a_0,..., a_e, in Δ; z_1, z_2, x_1, x_2 figuring as constants. The final integral D is free from z_1, z_2 and is unaffected by any admissible ([6]) particularization, as $z_i = x_i^{p^\nu}$, ν an arbitrary integer. Hence, in fact, D is an algebraic concomitant of the set $(L, l_1, l_2,...)$. Since D is derivable algebraically from this set, D had the algebraic ([7]) character *ab initio* and was a simultaneous algebraic concomitant of this ground-set $(L, l_1, l_2,...)$, q. e. d.

LEMMA 3. - *The number of linearly independent concomitants, modulo p, of f_e, of degree unity in a_0,..., a_e, is finite. Consequently the system of all formal modular concomitants of f_e is finite.*

It is known that the algebraic system of any finite set $(L, N, l_1, l_2,...)$ is finite. Hence we need to prove only the first part of the Lemma 3.

If V,

$$V: \quad x_1 = \gamma_1 x_1' + \gamma_2 x_2', \quad x_2 = \delta_0 x_1' + \delta_1 x_2',$$

is the arbitrary transformation from the total group $G(p)$, its poles are the roots of the congruence,

$$\delta_0 x_1^2 + (\delta_1 - \gamma_1) x_1 x_2 - \gamma_2 x_2^2 = \delta_0 (x_1 - r_1 x_2)(x_1 - r_2 x_2) \equiv 0 \pmod{p}.$$

The r_i are integral residues for some transformations V and Galoisian imaginaries for other V, according to the current integral values of δ_0, δ_1, γ_1, γ_2. With this understanding r_1, r_2, may be treated as parameters. The forms ϱ_1, ϱ_2 are covariants under V;

$$\varrho_1 = x_1 - r_1 x_2, \qquad \varrho_2 = x_1 - r_2 x_2,$$

and the relations exist,

$$\varrho_1' = a_1^{-1} \varrho_1, \qquad \varrho_2' = a_2^{-1} \varrho_2,$$
$$2a_1 = \gamma_1 + \delta_1 + (r_1 - r_2)\delta_0, \qquad 2a_2 = \gamma_1 + \delta_1 - (r_1 - r_2)\delta_0, \qquad a_1 a_2 = \gamma_1 \delta_1 - \gamma_2 \delta_0.$$
$$W: \quad x_1 = (r_2 \varrho_1 - r_1 \varrho_2)/(r_2 - r_1), \qquad x_2 = (\varrho_1 - \varrho_2)/(r_2 - r_1).$$

Transformed by W, f_e becomes a polynomial in $e+3$ concomitants known as invariant elements, viz., ϱ_1, ϱ_2 and $e+1$ invariants linear in a_0,..., a_e, (the coefficients of f_e'), I_0,..., I_e.

If l is a typical first degree covariant described in Lemma 3, l is isobaric (mod. $p-1$), and its covariant relation under W exhibits it as a rational integral

([5]) KRATHWOHL: Amer. Journal of Math., vol. 36 (1914), p. 449.

([6]) A value z_i is admissible if z_i Co. x_i and z_i contains an arbitrary parameter.

([7]) Paragraph (*f*) (clarified by *e*) advances the argument mainly by the truism that a concomitant derived by integration of (7) is derived by an algebraic process.

polynomial in $\varrho_1, \varrho_2, I_0,\dots, I_e$. As such it will simplify back to its primary form as a concomitant in $x_1, x_2, a_0,\dots, a_e$, with the aid of FERMAT'S congruence (generalized),

$$r_i^{p^2} - r_i \equiv 0 \pmod{p}, \qquad (i=1, 2).$$

This as a structural law defines a system of polynomials in $\varrho_1, \varrho_2, I_j$ in the sense of HILBERT'S basis theorem [8] Among all polynomials l built according to this law, there exists a finite basis-set $(l_1,\dots, l_\nu)$ such that the typical l of degree unity can be written,

$$l = P_1 l_1 + P_2 l_2 + \dots P_\nu l_\nu, \tag{8}$$

wherein the P_i are also polynomials in the invariant elements but do not, unless by additional proof, simplify as aforesaid to concomitants in $x_1, x_2, a_0,\dots, a_e$. But, since all l are linear in $a_0,\dots, a_e$, no P_i involves $a_0,\dots, a_e$. When (8) is given its expression in $\varrho_1, \varrho_2, I_j$, the P_i will be term-wise invariant expressions in ϱ_1, ϱ_2 alone. Since

$$l' = (r_2 - r_1)^h l,$$

a relation exists,

$$P_1'(r_2-r_1)^{\mu_1} l_1 + \dots + P_\nu'(r_2-r_1)^{\mu_\nu} l_\nu = (r_2-r_1)^h (P_1 l_1 + \dots + P_\nu l_\nu). \tag{9}$$

But the a_i can now be given particular values without affecting the P_i. Hence, from (8), we can have ν linear equations to determine the P_i in forms free from r_1, r_2. From their expressions in ϱ_1, ϱ_2 or from (9) they are covariants. Hence they are universal covariants and, if not constant, are polynomials in L and N, q. e. d.

We have proved therefore that a formal modular fundamental system of f_e is an algebraic simultaneous system of a terminating ground-set of quantics, $(L, N, l_1, l_2,\dots)$. The algebraic system is got by transvection (8) and the ARONHOLD symbolism, with algorithm I, § IV, Lemma 2, (e), suffices if we write $L = \lambda_x^{(1)} \dots \lambda_x^{(p+1)}$, $N = \nu_x^{(1)} \dots \nu_x^{(p(p-1))}$, certain combinations of symbols $\lambda_1^{(j)}, \lambda_2^{(j)}$, and also of $\nu_1^{(k)}, \nu_2^{(k)}$, being zero. If $e \geqslant p^2$, $\beta < \alpha+1$,

$$f_e = N^\alpha \eta_e + N^{\beta-1} L x + \dots + N L^{\gamma-1} \psi + L^\delta \omega \pmod{p},$$

whence it is seen that the l_i include the forms $\eta_e, x,\dots, \omega$. There are also various types of first degree covariants in case $e < p^2$.

In § 6 we show that some covariants, at least, among the $\eta_e, x,\dots, \omega$, are algebraic covariants of the set (L, N, f_e). On the question, whether the latter

(8) The converse statement forces the conclusion that any transvectant which does not turn out to be a formal modular concomitant has p as an essential factor, because, to make L a ground-quantic is to limit the group to $G(p)$.

set would always suffice as a ground-set, we say that we obtain a typical l_i from $a_z^r\, b_x^s$ of (7) by changing z_j into x_j, $(j=1, 2)$. The result is $B=a_x^r\, b_x^s$, and,

$$\Delta(z)B=E(z, x) \neq 0,$$

$E(z, x)$ being a universal covariant in the two cogredient sets (z_1, z_2), (x_1, x_2), and $E(x, x)$ a polynomial in L and N. With $E(z, x)$ expressed as a polynomial in fundamental universal covariants, the integral B of the equation is determined algebraically from these covariants and f_e alone. Since the variables in the equation are $a_0, \ldots, a_e$; B is unaffected by any admissible expression of z_i in terms of x_i. Hence B is a simultaneous algebraic concomitant of the set (f_e, L), and is obtainable by transvection between f_e and L.

Finally every formal modular concomitant (mod. p) of f_e is a simultaneous algebraic concomitant of f_e and L.

V. - Polynomial solutions in general of linear, partial differential equations. Almost invariantive functions.

The usual general methods of solving a linear, partial differential equation of the first order often leads to a solution in such transcendental form as is not easy to convert into a rational, integral polynomial even when it is known that a solution exists in polynomial form. We prove however that if the variables may be restricted, each to a segment of variation, any solution may be approximately expressed by a polynomial. If the latter is then a covariant, it is an almost-covariant (ε — covariant) of a determinate ground-function. The idea of an ε — covariant can be illustrated simply if we add infinitesimal increments to the coefficients of $f_e=a_0\, x_1^e+\ldots$, replacing a_i by $a_i+\delta_i a_i$, $(i=0, \ldots, e)$. A covariant I of f_e then receives an increment, and I itself will be an ε — covariant of,

$$f_e+\delta f_e=(a_0+\delta_0 a_0)x_1^e+(a_1+\delta_1 a_1)x_1^{e-1}x_2+\ldots+(a_e+\delta_e a_e)x_2^e.$$

Let,

$$\theta=f(r, a_1, \ldots, a_e)$$

be a function which is expansible into a power series absolutely convergent in a region ϱ of r — values.

Let

$$\theta=F_e(r, a_1, \ldots, a_e),$$

be a given covariant of f subject to a transformation q. This problem exists: To determine f and q with r restricted to ϱ. We reduce this question, relating to functions, also to a problem on polynomials.

If $f(r, a_1, \ldots, a_e)$ be expanded into a power series by ordinary methods, the history of the a_i may be largely lost but the requirement of convergency restricts to some region the variation of each a_i. We first prove the following,

THEOREM. - *Any e—parameter manifold (for example the solution of a differential equation),*

$$\theta = F_e(r, a_1, \ldots, a_e), \tag{10}$$

r being restricted to ϱ *and each* a_i *properly restricted, can be expressed rationally in e parameters, approximately, and the limits of error of the approximation will be bounded and controllable.*

A preliminary discussion of postulates is given.

Regions, fields, and frontiers. In the polar plane, two fixed angles (θ) and the two corresponding finite radii (r) delimit a region ϱ_0 shaped like a keystone. Let C be a continuous segment which passes from the lower to the upper bounding arc, (e_1, e_2), without intersecting the bounding radii. Assume C to be single valued with respect to θ, each θ between the radii being a coordinate of one and only one point of C. The radial width of ϱ_0 will be taken to be ϱ, the range of variation of r. We say then that C is properly contained in ϱ_0. A set of curves $C_1, \ldots, C_t$, $(t > 1)$, properly contained (p. c.) in the ϱ_0 of minimum angular width, and radial width ϱ, is to be called a field τ.

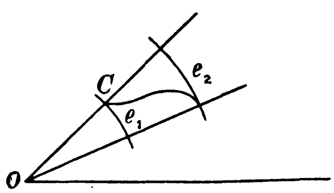

Fig. 1.

Proof of the theorem. If a curve $\theta = F(r)$ has a continuous branch p. c. in ϱ_0, n determinations (points) in ϱ_0, viz., d_i: (θ_i, r_i), $(i = 1, \ldots, n)$, are necessary and sufficient to determine $F(r)$ in the approximate form, (r on ϱ),

$$\theta = \alpha r^{n-1} + \beta r^{n-2} + \ldots + \varkappa. \tag{11}$$

The exactness of the approximation depends upon the number and law of distribution of the d_i on the branch. The law may be described by a symbol $l_0 = (\lambda_1, \ldots, \lambda_n)$, the λ_j being the respective apothems from O when perpendiculars are drawn from the d_i to the polar axis. The inaccuracy of the representation of $\theta = F(r)$ by (11) depends upon our choice of the points or, as we may say, is bounded, and the frontier is the point-set $(d_1, \ldots, d_n)$.

Let T_1 be a one-parameter system (field) of curves, $\theta = F_1(r, a_1)$, all p. c. in ϱ_0. The curve-set obtained by assigning q values to a_1, (r on ϱ), may be represented by

$$\theta = \sum_{i=1,n} a_{ij} r^{n-i}, \qquad (j = 1, \ldots, q). \tag{12}$$

Assume that the a_{ij}, $(j=1,\dots, q)$, are the determinations of a function ζ_i of μ,

$$\zeta_i=\zeta_{i1}\mu^{q-1}+\zeta_{i2}\mu^{q-2}+\dots+\zeta_{iq},$$

for assigned values $\mu=\mu^{(1)},\dots,\ \mu=\mu^{(q)}$, $(\mu^{(g)}\neq\mu^{(h)})$. With none but obvious restrictions these values of μ may be chosen arbitrarily, for, whatever set $\mu^{(g)}$ is chosen, the parameters ζ_{ij} are given by CRAMER'S rule. Hence the one-parameter system,

$$(13)\quad \theta=\sum_{i=1,n}(\zeta_{i1}\mu^{q-1}+\zeta_{i2}\mu^{q-2}+\dots+\zeta_{iq})r^{n-i}, \qquad [=\varphi(\zeta_{i_1i_2};\ \mu;\ r);\ i_1=i],$$

with μ continuous from the smallest to the largest $\mu^{(g)}$, is in approximate coincidence with T_1 or with a sub-field of T_1. The inaccuracy (limits of error) of the coincidence depends upon the law l_1 of distribution of the curve-set (12) upon T_1, the curve-set being therefore the frontier of the bounded inaccuracy.

Before we generalize (13), by induction, we consider certain details of the generalization from $\theta=F_1(r, a_1)$ to the two-parameter case $\theta=F_2(r, a_1, a_2)$. Assume a ϱ_0 which p. c. ∞^2 curves $\theta=F_2$, and assign a well-spaced set of q_2 values $a_2^{(1)},\dots, a_2^{(q_2)})$ in succession to a_2. Represent each single parameter family obtained, in a form (13). We can use (13) itself after attaching the superscript k to each $\zeta_{i_1i_2}$, k to take the integral values $1,\dots, q_2$. Referring to continuous variation of a real parameter from an abscissa a to an abscissa b as a segment (a, b), we have seen that the set $\mu^{(g)}$ may be chosen from any segment consistent with convergency. Hence when,

$$(14)\qquad \theta=F_2(r, a_1, a_2^{(1)}),\dots,\quad \theta=F_2(r, a_1, a_2^{(q_2)}),$$

are represented rationally as stated, the sets like $\mu^{(g)}$, corresponding to the respective $a_2^{(1)},\dots, a_2^{(q_2)}$, may all be chosen from the same segment. Hence one and the same parameter μ, (viz., μ_1) can be used in all q_2 rational representations, which then become

$$(15)\qquad \theta=\varphi_1(\zeta_{i_1i_2}^{(k)};\ \mu_1;\ r), \qquad (k=1,\dots, q_2;\ \varphi_1=\varphi).$$

We can now generalize to the case of e parameters. Let the curves of the manifold,

$$\theta=F_e(r, a_1,\dots, a_e),$$

be all p. c. in a ϱ_0. Assign to a_e, q_e particular values $a_e^{(1)},\dots, a_e^{(q_e)}$, thus obtaining q_e determinations in $e-1$ parameters, of $\theta=F_e$. The hypothesis of induction is that each determination, in ϱ_0, has a rational representation. The determinations are,

$$(16)\qquad \theta=F_e(r, a_1,\dots, a_{e-1}, a_e^{(1)}),\dots,\quad \theta=F_e(r, a_1,\dots, a_{e-1}, a_e^{(q_e)}),$$

and the respective rational representations are,

$$\theta = \sum_{i_1=1,n} \sum_{i_2=1,q_1} \cdots \sum_{i_e=1,q_{e-1}} \zeta^{(l)}_{i_1 \ldots i_e} \mu_1^{q_1-i_2} \mu_2^{q_2-i_3} \cdots \mu_{e-1}^{q_{e-1}-i_e} r^{n-i_1}, \quad (l=1,\ldots, q_e), \tag{17}$$

equation (17) being (13) when $l=1$, $e=2$. The abbreviation of (17) is,

$$\theta = \varphi_{e-1}(\zeta^{(l)}_{i_1 \ldots i_e};\ \mu_1, \ldots, \mu_{e-1};\ r), \qquad (l=1,\ldots, q_e).$$

The q_e determined numbers $\zeta^{(l)}_{i_1 \ldots i_e}$, $(l=1,\ldots, q_e)$, may be regarded as q_e determinations of a function $f_{i_1 \ldots i_e}$ of μ_e,

$$f_{i_1 \ldots i_e} = \zeta_{i_1 \ldots i_e 1} \mu_e^{q_e-1} + \zeta_{i_1 \ldots i_e 2} \mu_e^{q_e-2} + \cdots + \zeta_{i_1 \ldots i_e q_e}$$

for the respective assignments $\mu_e = \mu_e^{(1)}, \ldots, \mu_e = \mu_e^{(q_e)}$, chosen from a segment independent of the segments preempted by previous μ_i, $(i<e)$. Replacing in (17), $\zeta^{(l)}_{i_1 \ldots i_e}$ by $f_{i_1 \ldots i_e}$, we get,

$$\theta = \varphi_e(\zeta_{i_1 \ldots i_{e+1}};\ \mu_1, \ldots, \mu_e;\ r). \tag{18}$$

Allowing μ_e in (18) to vary continuously over its segment from the least to the greatest numbers of the set $\mu_e^{(k)}$ we obtain not merely the q_e particular equations (17), but the whole manifold,

$$\theta = F_e(r, a_1, \ldots, a_e),$$

as defined in ϱ_0, within an inaccuracy (limits of error) which depends upon the law l_e of the frontier manifold-set (16), q. e. d.

If $\theta = F_e$ is a general solution of a partial differential equation like (7), $\theta = \varphi_e$ is an almost accurate determination of the solution in polynomial form, with r, $\mu_1, \ldots, \mu_e$ restricted to segments.

If F_e is a covariant, as originally described, φ_e is an ε—covariant. If φ_e is made homogeneous in the μ_j by the adjunction of one μ, as μ_0, φ_e has the form of a covariant of a binary quantic. Assuming that, under q,

$$(\zeta_0, \zeta_1, \ldots, \zeta_e) \text{ Co. } (\mu_0, \mu_1, \ldots, \mu_e),$$

and that g is the degree of φ_e in $\mu_0, \ldots, \mu_e$, it follows that, except for ε—terms,

$$\Delta^{g-1} \varphi_e = \left(\zeta_0 \frac{\partial}{\partial \mu_0} + \zeta_1 \frac{\partial}{\partial \mu_1} + \cdots + \zeta_e \frac{\partial}{\partial \mu_e}\right)^{g-1} \varphi_e,$$

if not zero, is one of a set of admissible ground-functions $f(r, \mu_0, \ldots, \mu_e)$, possibly multiplied by a universal covariant of q, while $\Delta^g \varphi_e$ (if $\neq 0$) is such a universal covariant, from which q may be determined.

We can also generalize (17) so the number of r-variables, $(r_0, r_1, \ldots, r_n)$, is $n+1$ instead of two.

VI. - A formal modular system algebraically interpreted.

An important recent problem on complete systems has been the direct one of determining the fundamental covariants of various orders of quantics,

$$f_e = a_0 x_1^e + a_1 x_1^{e-1} x_2 + \ldots + a_e x_2^e,$$

with arbitrary coefficients, under the total linear group $G(p)$, p prime, of order,

$$(p^2 - p)(p^2 - 1).$$

When $p=2$ the group, of order 6, may be represented as follows;

$$x_1 = \lambda_1 x_1' + \mu_1 x_2', \qquad x_2 = \lambda_2 x_1' + \mu_2 x_2' \quad (\text{mod. } 2),$$

and its independent universal covariants are,

$$L = x_1^2 x_2 + x_1 x_2^2, \qquad Q = x_1^2 + x_1 x_2 + x_2^2, \quad (=N).$$

As a construction process the method given in § IV, where such a system is seen to be an algebraic simultaneous system of $(L, N, l_1, l_2, \ldots, l_\nu)$, is hardly satisfactory because some of the l_i are of high order in x_1, x_2 and moreover a purely algebraic method neglects useful number-theoretic facts, (9) but, after we have determined the simultaneous system (mod. 2) of (f_1, f_2, f_3), we shall express the fundamental covariants in terms of algebraic transvectants and thus bring to light specifically the unity between the invariant theory of quantics under the algebraic group (1), and that under a modular group. It becomes clear that contributions to the modular theory have greatly advanced general invariant theory.

The algebra here is a type of algebra of the odd numbers. Nature makes the separation into even and odd in various ways. Some species of simple flowers have nearly always an even number of petals, others an odd number.

Method for seminvariant systems.

A fundamental system of seminvariants of (f_1, f_2, f_3) is found as the culmination of an inductive process based on the simultaneous system of universal covariants (mod. 2), of the three transformations S_1, S_2, S_3,

$$S_1: \ x_1 = x_1' + x_2', \ \ x_2 = x_2', \qquad S_2: \ y_1 = y_1' + y_2', \ \ y_2 = y_2',$$
$$S_3: \ z_1 = z_1' + z_2', \ \ z_2 = z_2'.$$

(9) Dickson elaborated a Theory of Classes in invariant theory which is primarily number-theoretic. He brought it to a high degree of generality Cf. [5].

The system of (S_1, S_2) had been previously determined and consists of [5],

$$(19)\quad x_2,\quad y_2,\quad \psi_1=x_1^2+x_1x_2,\quad \psi_2=y_1^2+y_1y_2,\quad (xy)=x_1y_2+x_2y_1.$$

The writer has shown previously that the following trilinear is irreducible,

$$\Omega=x_1y_2z_1+x_1y_1z_2+x_2y_1z_1+x_2y_1z_2.$$

THEOREM. - *A fundamental system of universal covariants, (mod. 2), of* (S_1, S_2, S_3) *is composed of,*

$$(20)\quad x_2,\quad y_2,\quad z_2,\quad \psi_1,\quad \psi_2,\quad \psi_3=z_1^2+z_1z_2,\quad (xy),\quad (xz),\quad (yz),\quad \Omega.$$

The proof is as follows. A covariant may be assumed homogeneous in the three pairs of variables, and of the form,

$$F=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jki}(x_1^{l-i}x_2^i)(y_1^{m-j}y_2^j)(z_1^{n-k}z_2^k),$$

$(a_{jki}=0$ or $1)$. The transformed of F by (S_1, S_2, S_3) is

$$F'=\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jki}\left[\sum_{\lambda}\binom{l-i}{\lambda}x_1^{l-i-\lambda}x_2^{i+\lambda}\right]$$
$$\left[\sum_{\mu}\binom{m-j}{\mu}y_1^{m-j-\mu}y_2^{j+\mu}\right]\left[\sum_{\nu}\binom{n-k}{\nu}z_1^{n-k-\nu}z_2^{k+\nu}\right],$$

$(\lambda=0,\dots, l-i;\ \mu=0,\dots, m-j;\ \nu=0,\dots, n-k)$. The expression for $F'-F$ may be obtained by deletion of the zero value of each letter λ, μ, ν, from F'. Since $F'-F\equiv 0$ (mod. 2), identically in the variables, we obtain, as the necessary and sufficient conditions for the covariancy of F, a set of linear congruences in the a_{jki}. These congruences do not determine the coefficients a_{jki} uniquely.

The coefficient of $y_1^m z_1^n$ in $F'-F$ is,

$$\binom{l}{1}a_{000}x_1^{l-1}x_2+\left[\binom{l}{2}a_{000}+\binom{l-1}{1}a_{001}\right]x_1^{l-2}x_2^2+\dots,$$

hence $a_{000}\equiv 0$ (mod. 2) except possibly when l is even. Likewise $a_{000}\equiv 0$ except possibly when l, m, n are all even.

Let n be even, $n=2a$. Then F can be written as,

$$(21)\qquad F=\varphi(x_1, x_2;\ y_1, y_2)z_1^{2a}+\psi z_2$$

φ being a covariant of (S_1, S_2) and a polynomial in the quantics (19) alone. Therefore,

$$F=\varphi\psi_3^a+(\psi+x)z_2\quad \text{(mod. 2)},$$

and $\psi+x$, $(=\Delta_1)$, is a covariant of (S_1, S_2, S_3) of the next lower order $n-1$ in z_1, z_2; (All equalities are congruences mod. 2).

To obtain the analogous reduction formula for the case of an odd n, $(=2a+1)$ we note that, since a_{000} is then zero, φ contains no term without one of the variables x_2, y_2 as a factor. Hence,

$$\varphi = x_2\varphi_1 + y_2\varphi_2 + (xy)\varphi_3,$$

and $\varphi_1, \varphi_2, \varphi_3$ are covariants. Therefore F is of the form,

$$\begin{aligned} F &= [x_2z_1\varphi_1 + y_2z_1\varphi_2 + (xy)z_1\varphi_3]\psi_3^a + (\psi+\xi)z_2 \\ &= [(xz)\varphi_1 + (yz)\varphi_2 + \Omega\varphi_3]\psi_3^a + \Delta_2 z_2 \quad (\text{mod. } 2), \end{aligned}$$

in which Δ_2 is a covariant of order $n-1$ in z_1, z_2. If the same schemes of reduction are applied to the residuals, Δ_1, Δ_2, successively, the orders in z_1, z_2 are ultimately reduced to zero, so that F is reduced completely to a rational integral polynomial in the covariants (20), q. e. d.

By the same methods of reduction, or by direct verification, the following syzygies may be established.

$$(22) \quad \begin{cases} (xy)^2 + x_2y_2\,(xy) + y_2^2\psi_1 + x_2^2\psi_2 = 0, \\ (xz)^2 + x_2z_2\,(xz) + z_2^2\psi_1 + x_2^2\psi_3 = 0, \\ (yz)^2 + y_2z_2\,(yz) + z_2^2\psi_2 + y_2^2\psi_3 = 0, \\ \Omega^2 + z_2\,(xy)\Omega + z_2^2\psi_1\psi_2 + (xy)^2\psi_3 = 0. \end{cases}$$

It follows that no reduced covariant of (S_1, S_2, S_3) need involve any properly simultaneous fundamental covariant to a higher power than the first.

The seminvariants of (f_1, g_1, f_2).

Consider the set,

$$f_1 = a_0x_1 + a_1x_2, \quad g_1 = b_0x_1 + b_1x_2, \quad f_2 = c_0x_1^2 + c_1x_1x_2 + c_2x_2^2.$$

The seminvariants of any one of these quantics are its invariants under the transformation S_1. The three induced transformations under S_1, are

$$\begin{gathered} \sigma_1: \ a_0' = a_0, \quad a_1' = a_0 + a_1, \qquad \sigma_2: \ b_0' = b_0, \quad b_1' = b_0 + b_1, \\ \sigma_3: \ c_0' = c_0, \quad c_1' = c_1, \quad c_2' = c_0 + c_1 + c_2. \end{gathered}$$

Writing $c_0 = 0$ temporarily, we note that $\sigma_1, \sigma_2, \sigma_3$, respectively, are identical in structure with S_1, S_2, S_3. Therefore if

$$F(a_0, a_1, b_0, b_1, c_0, c_1, c_2),$$

is the general seminvariant of the set (f_1, g_1, f_2) we have,

$$F(a_0, a_1, b_0, b_1, 0, c_1, c_2) = \Phi_1(a_0, b_0, c_1, P_1, P_2, P, D_1, D', D, W'),$$

wherein,

$$P_1 = a_1^2 + a_0 a_1, \quad P_2 = b_1^2 + b_0 b_1, \quad P = c_2^2 + c_1 c_2, \quad D_1 = a_1 b_0 + a_0 b_1, \quad D' = a_1 c_1 + a_0 c_2,$$
$$D = b_1 c_1 + b_0 c_2, \quad W' = a_1 b_0 c_2 + a_1 b_1 c_1 + a_0 b_1 c_2 + a_0 b_1 c_1.$$

The quantic Φ_1 is integral in its arguments. The quantics P, D', D, W' are not seminvariants of the induced transformations σ_i, but they are, respectively, residues (mod. c_0), of the following seminvariants,

$$P_3 = P + c_0 c_2, \quad D_2 = D' + a_1 c_0, \quad D_3 = D + b_1 c_0, \quad W = W' + c_0(a_1 b_0 + a_1 b_1).$$

Hence there exists a relation of the following type,

$$F(a_0, a_1, b_0, b_1, c_0, c_1, c_2) = \Phi_1(a_0, b_0, c_1, P_1, P_2, P_3, D_1, D_2, D_3, W)$$
$$+ c_0 \Psi_1(a_0, a_1, b_0, b_1, c_0, c_1, c_2) \quad (\text{mod. } 2),$$

and evidently Ψ_1 is a seminvariant of order, in c_0, c_1, c_2, one less than the order of F in c_0, c_1, c_2.

We can apply this reduction to Ψ_1 and its successors $\Psi_2, \Psi_3, \ldots$ until a vanishing Ψ_n is reached, or as long as a Ψ_k contains c_0, and as long as the last Φ_i being considered contains one or more of P, D', D, W'. If this Φ_i contains none of these four expressions, (even though it may contain the invariant c_1), the process of reduction can no longer be used to reduce the order in c_0, but then the last Ψ_i is a seminvariant which involves only a_0, a_1, b_0, b_1, c_1, and is reducible in terms of $a_0, b_0, c_1, P_1, P_2, D_1$. Therefore F is reduced, that is, we have proved;

THEOREM. - *A fundamental system of seminvariants of* (f_1, g_1, f_2) *is composed of the eleven quantics,*

$$a_0, b_0, c_0, c_1, P_1, P_2, P_3, D_1, D_2, D_3, W.$$

The seminvariants of (f_1, f_2, g_2).

Assume,

$$f_1 = a_0 x_1 + a_1 x_2, \qquad f_2 = b_0 x_1^2 + b_1 x_1 x_2 + b_2 x_2^2, \qquad g_2 = c_0 x_1^2 + c_1 x_1 x_2 + c_2 x_2^2.$$

The transformations induced by S_1 are,

$$s_1: \ a_0' = a_0, \quad a_1' = a_0 + a_1, \qquad s_2: \ b_0' = b_0, \quad b_1' = b_1, \quad b_2' = b_0 + b_1 + b_2,$$
$$s_3: \ c_0' = c_0, \quad c_1' = c_1, \quad c_2' = c_0 + c_1 + c_2.$$

With $b_0=0$, s_1, s_2, s_3 are, except for notation, identical in structure with $\sigma_1, \sigma_2, \sigma_3$, respectively; therefore if

$$G(a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2)$$

is the arbitrary seminvariant of (f_1, f_2, g_2), we have,

$$G(a_0, a_1, 0, b_1, b_2, c_0, c_1, c_2)=H_1(a_0, b_1, c_0, c_1, p_1', p_2', p_3', d_1', d_2', d_3', z'),$$

wherein abbreviations have the following meanings,

$$p_1'=a_1^2+a_0a_1, \quad p_2'=b_2^2+b_1b_2, \quad p_3'=c_0c_2+c_1c_2+c_2^2,$$
$$d_1'=a_1b_1+a_0b_2, \quad d_2'=a_1c_0+a_1c_1+a_0c_2, \quad d_3'=b_2c_0+b_2c_1+b_1c_2,$$
$$z'=a_1b_1c_2+a_1b_2c_1+a_0b_2c_2+a_0b_2c_1+a_1b_1c_0+a_1b_2c_0.$$

The latter seven quantics are respectively residual, (mod. b_0), to the following seminvariants of (f_1, f_2, g_2),

$$p_1=p_1', \quad p_2=p_2'+b_0b_2, \quad p_3=p_3', \qquad d_1=d_1'+a_1b_0, \quad d_2=d_2', \quad d_3=d_3'+b_0c_2,$$
$$z=z'+b_0(a_1c_1+a_0c_2+a_1c_2).$$

Hence,

$$G(a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2)=H_1(a_0, b_1, c_0, c_1, p_1, p_2, p_3, d_1, d_2, d_3, z)$$
$$+b_0G_1(a_0, a_1, b_0, b_1, b_2, c_0, c_1, c_2),$$

and G_1 is a seminvariant of degree in b_0 one less than the degree of G in b_0. Reducing G_1 and the other succeeding residuals by the same process, we reach a residual seminvariant G_n which is of degree 0 in b_0 Therefore, except when $G_{n+1}=0$, G_n does not contain p_2, d_1, d_3, or z, that is, a repetition of the argument with which the preceding section closes gives the following,

THEOREM. - *A fundamental system of seminvariants of* (f_1, f_2, g_2) *is composed of,*

$$a_0, \quad b_0, \quad b_1, \quad c_0, \quad c_1, \quad p_1, \quad p_2, \quad p_3, \quad d_1, \quad d_2, \quad d_3, \quad z.$$

The seminvariants of (f_1, f_2, f_3).

We now consider the ground-quantics of the main Problem,

$$f_1=a_0x_1+a_1x_2, \quad f_2=b_0x_1^2+b_1x_1x_2+b_2x_2^2, \quad f_3=c_0x_1^3+c_1x_1^2x_2+c_2x_1x_2^2+c_3x_2^3.$$

The transformations induced by S_1 are,

$$t_1: \quad a_0'=a_0, \quad a_1'=a_0+a_1, \qquad t_2: \quad b_0'=b_0, \quad b_1'=b_1, \quad b_2'=b_0+b_1+b_2,$$
$$t_3: \quad c_0'=c_0, \quad c_1'=c_0+c_1, \quad c_2'=c_0+c_2, \quad c_3'=c_0+c_1+c_2+c_3.$$

If $c_0=0$, t_1, t_2, t_3 are identical with s_1, s_2, s_3, respectively, except for notation in the case of t_3. If $F(c_0)$ is the arbitrary seminvariant of (f_1, f_2, f_3),

$$F(0)=\Phi(a_0, b_0, b_1, c_1, e, q_1', q_2', q_3', e_1', e_2', e_3', \alpha'),$$

wherein e is the invariant $e=c_1+c_2$, and

$$q_1'=a_1^2+a_0a_1, \quad q_2'=(b_0+b_1+b_2)b_2, \quad q_3'=(c_1+c_2+c_3)c_3,$$
$$e_1'=a_0b_2+a_1(b_0+b_1), \quad e_2'=a_0c_3+a_1(c_1+c_2), \quad e_3'=b_2(c_1+c_2)+(b_0+b_1)c_3,$$
$$\alpha'=a_1b_1c_3+a_1b_2c_2+a_0b_2c_3+a_0b_2c_2+a_1b_1c_1+a_1b_2c_1+a_1b_0c_2+a_0b_0c_3+a_1b_0c_3.$$

These quantics respectively are residual (mod. c_0) to the seven seminvariants of (f_1, f_2, f_3),

$$q_1=q_1', \quad q_2=q_2', \quad q_3=q_3'+c_0c_3,$$
$$e_1=e_1', \quad e_2=e_2'+a_1c_0, \quad e_3=e_3'+b_2c_0, \quad \alpha=\alpha'+a_1b_0c_0.$$

The letter c_1 is exceptional in Φ, and, for this reason, we write Φ explicitly as far as c_1 is concerned and obtain,

$$F(0)=G_0+G_1c_1^2+\dots+G_vc_1^{2v}+c_1(H_0+H_1c_1^2+\dots+H_uc_1^{2u}) \quad (\text{mod. } 2).$$

All G, H functions are like

$$G_0=G_0(a_0, b_0, b_1, e, q_1', q_2', q_3', e_1', e_2', e_3', \alpha').$$

In the latter value of $F(0)$ we next make the replacements,

$$\begin{pmatrix} a_0, b_0, b_1, e, q_1', q_2', q_3', e_1', e_2', e_3', \alpha' \\ a_0, b_0, b_1, e, q_1, q_2, q_3, e_1, e_2, e_3, \alpha \end{pmatrix},$$

also c_1^{2j}, $(j=0, 1,\dots)$, is replaced by λ^j, λ being the seminvariant $\lambda=c_1^2+c_0c_1$. The grand result is,

$$F(c_0)=I'+c_1J+c_0K \quad (\text{mod. } 2), \tag{23}$$

where I' and J are seminvariants reduced in terms of,

$$a_0, \quad b_0, \quad b_1, \quad e, \quad q_1, \quad q_2, \quad q_3, \quad e_1, \quad e_2, \quad e_3, \quad \alpha, \quad \lambda.$$

We transform the congruence (23) by (t_1, t_2, t_3) and obtain,

$$F(c_0)=I'+(c_0+c_1)J+(K+K_1)c_0 \quad (\text{mod. } 2),$$

whence $K_1=J$, that is, the seminvariant J is the increment of a polynomial K in the nine coefficients of f_1, f_2, f_3, under (t_1, t_2, t_3). The terms of such an increment can be grouped so each group contains, as a factor, one of the following expressions; a_0, b_0+b_1, c_0, e $(=c_1+c_2)$. Hence,

$$F(c_0)=I'+a_0c_1J_1+(b_0+b_1)c_1J_2+ec_1J_3+c_0K',$$

wherein J_1, J_2, J_3 are reduced seminvariants. Consider next the new seminvariants, (new except $\varDelta$),

$$\varDelta = c_0c_3 + c_1c_2, \qquad \varGamma = a_0c_1 + a_1c_0, \qquad E = (b_0+b_1)c_1 + b_2c_0.$$

Since $\varDelta = c_1e + \lambda + c_0c_1 + c_0c_3$,

$$F(c_0) = I' + \varGamma J_1 + EJ_2 + (\varDelta + \lambda)J_3 + c_0L_1, \qquad (= M_1 + c_0L_1). \tag{24}$$

All letters on the right are now reduced seminvariants excepting L_1. Though L_1 is not reduced it, also, is a seminvariant. Its order in $c_0, \ldots, c_3$ is one less than the order of $F(c_0)$. We apply an analogous reduction to L_1, $L_2, \ldots$ in succession. If a vanishing L_n is reached before the order in c_0 is reduced to zero, $F(c_0)$ is reduced in terms of fundamental seminvariants. If we reach an $L_k \neq 0$ free from c_0, and $L_{k+1} \neq 0$, M_{k+1} will not contain $\varDelta$, λ, $\varGamma$, E, q_3, e_2, e_3 or α, otherwise L_{k+1} would be of order -1 in c_0, which is absurd. Hence M_{k+1} is a polynomial in e whose coefficients are seminvariants of f_1, f_2, whence $F(c_0)$ is reduced. Therefore,

THEOREM. - *A fundamental system of seminvariants of* (f_1, f_2, f_3) *is composed of the quantics,*

$$a_0,\ b_0,\ b_1,\ c_0,\ e,\ \lambda,\ q_1,\ q_2,\ q_3,\ e_1,\ e_2,\ e_3,\ \alpha,\ \varDelta,\ \varGamma,\ E. \tag{25}$$

Syzygies.

The first five of the following syzygetic relations were given in previous papers by the author. Of the rest, the last is the most complicated. Its derivation was based upon the last relation (22), supplemented by some empiricism.

$$a_0^2 + q_1 + Q' = 0, \qquad (Q' = a_0^2 + a_0a_1 + a_1^2),$$
$$a_0^3 + a_0Q' + L' = 0, \qquad (L' = a_0q_1),$$
$$b_0^3 + b_0^2b_1 + b_0q + k = 0, \qquad (q = q_2 + b_0b_1 + b_0^2,\ k = b_0q_2),$$
$$c_0^3 + c_0^2e + c_0I + K = 0, \qquad (I = c_0^2 + c_0e + q_3,\ K = c_0q_3),$$
$$\lambda^2 + \lambda(c_0^2 + c_0e + I + e^2) + (c_0^2 + c_0e + I + \varDelta)(c_0e + e^2) + I\varDelta + \varDelta^2 + g_1 = 0,$$

where g_1 is an invariant,

$$e_1 + a_0b_1 + j_1 = 0, \qquad \{j_1 = a_0(b_1 + b_2) + a_1(b_0 + b_1)\},$$
$$e_2 + a_0e + j_2 = 0, \qquad e_3 + b_0e + c_0b_1 + j_3 = 0,$$

where j_1, j_2, j_3 are invariants,

$$\varGamma^2 + \varGamma a_0c_0 + \lambda a_0^2 + (a_0^2 + Q')c_0^2 = 0,$$
$$E^2 + E(b_0 + b_1)c_0 + \lambda(b_0^2 + b_1^2) + (b_0^2 + b_0b_1 + q)c_0^2 = 0.$$

For the quantics f_1, g_1, f_2 there exists the syzygy,

$$W^2 + W(a_0 b_0 c_0 + a_0 D_3) + P_1(b_0^2 c_0^2 + D_3^2) + a_0^2 c_0 c_1 P_2 + a_0^2 P_2 P_3 = 0,$$

and for f_1, f_2, g_2, the syzygy,

$$z^2 + z(a_0 b_0 c_1 + a_0 b_1 c_0 + a_0 d_3) + p_1(b_0^2 c_1^2 + b_1^2 c_0^2 + d_3^2) + a_0^2 c_0 c_1 p_2 + \\ + a_0^2 b_0 b_1 p_3 + a_0^2 p_2 p_3 + a_0^2 b_0 c_1 d_3 = 0.$$

For f_1, f_2, f_3 there exsists the syzygy,

$$\alpha^2 + \alpha\,\{ (b_0 + b_1)\Gamma + a_0 b_0 e + a_0 b_1 c_0 + a_0 e_3 \} + q_1 \{ E^2 + E(b_0 c_0 + b_1 c_0) \\ + (b_0^2 c_0 + b_0 b_1 c_0) e + b_0^2 e^2 + b_0 b_1 c_0^2 + (b_0 c_0 + b_1 c_0) e_3 + e_3^2 \} + a_0^2 b_0 e_3 e \\ + a_0 b_0 e_3 \Gamma + (a_0^2 \Delta + a_0^2 c_0 e + a_0 c_0 \Gamma + a_0 c_0 e_2) q_2 + a_0^2 b_0 b_1 q_3 + a_0^2 q_2 q_3 = 0.$$

The arbitrary seminvariant of the set (f_1, f_2, f_3) can be reduced, by means of these syzygies, to a finite, rational and integral polynomial f which is linear in each fundamental seminvariant α, λ, Γ, E, quadratic in a_0, b_0, c_0, and free from q_1, q_2, q_3, e_1, e_2, e_3. The coefficients of f are arbitrary polynomials in the invariants (26). It is not claimed that this set is a fundamental system of invariants.

$$(26) \qquad \Delta, \quad e, \quad q, \quad Q', \quad L', \quad k, \quad I, \quad K, \quad b_1, \quad g_1, \quad j_1, \quad j_2, \quad j_3.$$

Technique of covariant reduction.

Any formal covariant modulo 2, of a set f_m, f_n,...,

$$K_s = C_0 x_1^s + C_1 x_1^{s-1} x_2 + \dots + C_s x_2^s,$$

gives rise to a scale of concomitants,

$$K_s, \quad w^i K_s, \quad (i = 1, \dots, s) \qquad w^j [K_s], \quad (j = 0, 1), \\ < K_s >, \quad \{ K_s \}, \quad \left(w = x_1^2 \frac{\partial}{\partial x_1} + x_2^2 \frac{\partial}{\partial x_2} \right).$$

These exist for all orders s; (Def., $[g] = g$ if g is linear), where [10],

$$\{ K_s \} = C_0 x_1^2 + < K_s > x_1 x_2 + C_s x_2^2, \qquad < K_s > = C_1 + C_2 + \dots + C_{s-1}, \\ [K_s] = (C_0 + < K_s >) x_1 + (< K_s > + C_s) x_2,$$

and if s is odd, $(s = 2a + 1)$, the scale contains also a cubic,

$$\{ \overline{K_s} \} = C_0 x_1^3 + I_1 x_1^2 x_2 + I_2 x_1 x_2^2 + C_s x_2^3, \quad (i_j \neq i_k \neq 0 \text{ or } s; \; I_1 + I_2 = < K_s >),$$

([10]) « Bull. Amer. Math. Soc. », vol. 30 (1924), p. 135. Also, « Trans. Amer. M. S. », vol. 17 (1916). The models of the copied forms $\{ \overline{K_s} \}$ are tabulated on p. 550.

$(I_1 = C_{i_1} + \dots + C_{i_a},\ I_2 = C_{i_{a+1}} + \dots + C_{i_{2a}})$. If S is a seminvariant and S_1 is its conjugate under the substitution on the coefficients of f_1, f_2, f_3,

$$t = (a_0 a_1)(b_0 b_2)(b_1)(c_0 c_3)(c_1 c_2),$$

a necessary and sufficient condition that S should have the property that it can lead a covariant is, that $S + S_1'$ should be an invariant, where S_1' is the increment of S_1 under (t_1, t_2, t_3). In fact, if S leads a K_s, it leads,

$$R(S) = \{ K_s \} = Sx_1^2 + \varphi x_1 x_2 + S_1 x_2^2,$$

φ being the scale invariant The relation which expresses the covariancy of $R(S)$ gives $S + S_1' = \varphi$. With S given we can construct $R(S)$ uniquely.

If S and T are seminvariants which lead covariants K_s and A respectively, A being cubic, and if H is a cubic led by ST, H is reducible in any set which contains A, $\{A\}$, φ, $[K_s]$, L and invariants. The reduction formula is,

$$H = [K_s]\{ A \} + \varphi A + \psi L \quad (\text{mod. } 2), \qquad ([K_s] = (S + \varphi)x_1 + (\varphi + S_1)x_2),$$

ψ being an invariant.

If S and T are seminvariants which lead respective quadratic covariants, viz.,

$$\{ C \} = Sx_1^2 + \varphi x_1 x_2 + S_1 x_2^2, \qquad \{ D \} = Tx_1^2 + x x_1 x_2 + T_1 x_2^2,$$

and G is a quadratic led by ST, G is reducible in any set which contains $\{C\}$, $\{D\}$, $[C]$, $[D]$, φ, x, and Q. The formula is,

$$G = [C][D] + \{ C \} x + \{ D \} \varphi + \varphi x Q \quad (\text{mod. } 2).$$

Some of the syzygetic relations among leading seminvariants, implied by congruences of the latter type, are consequences of a class of secondary syzygies among the universal covariants of (S_1, S_2, S_3), (Cf. (20)). The following are such syzygies,

$$\begin{aligned}
&(xy)(yz) + \psi_2 x_2 z_2 + \Omega y_2 = 0,\\
&(xy)(xz) + (xy)x_2 z_2 + \psi_1 y_2 z_2 + \Omega x_2 = 0,\\
&(xz)(yz) + (yz)x_2 z_2 + \psi_3 x_2 y_2 + \Omega z_2 = 0,\\
&(xz)\Omega + (yz)\psi_1 z_2 + (xy)\psi_3 x_2 = 0,\\
&(xy)\Omega + (xz)\psi_2 x_2 + \psi_2 x_2^2 z_2 + (yz)\psi_1 y_2 + \Omega x_2 y_2 = 0,\\
&(yz)\Omega + (xz)\psi_2 z_2 + \psi_2 x_2 z_2^2 + (xy)\psi_3 y_2 + \Omega y_2 z_2 = 0.
\end{aligned}$$

In the known expansion, (wherein $\beta < a + 1$, $s > 3$),

$$K_s = N^a \eta_s + N^{\beta-1} L x + \dots + N L^{\gamma-1} \psi + L^\delta \omega \quad (\text{mod. } 2),$$

η_s is always quadratic if s is even. If s is odd, $\eta_s = \{\overline{K_s}\}$, and is cubic. We find [11],

$$\eta_4 = C_0x_1^2 + (C_1 + C_2 + C_3)x_1x_2 + C_4x_2^2 = \{\{K_4N,\ LN\}^4,\ L\}^2 + N\{K_4N,\ N^3\}^6,$$
$$\eta_6 = C_0x_1^2 + (C_1 + C_2 + C_3 + C_4 + C_5)x_1x_2 + C_6x_2^2$$
$$= \{\{K_6,\ LN\}^4,\ L\}^2 + N\{K_6,\ N^3\}^6,$$

$$(27)\quad \eta_5 = C_0x_1^3 + (C_1 + C_2)x_1^2x_2 + (C_3 + C_4)x_1x_2^2 + C_5x_2^3 = \{K_5,\ L^2\}^4 + \{K_5,\ N\}^2,$$

$$(28)\quad \eta_7 = C_0x_1^3 + (C_1 + C_2 + C_4)x_1^2x_2 + (C_3 + C_5 + C_6)x_1x_2^2 + C_7x_2^3$$
$$= \{\{K_7,\ L^2\}^4,\ L^2\}^4 + \{\{K_7,\ L^2\}^4,\ N\}^2.$$

We have proved that $\{\overline{K_s}\}$ is always a simultaneous algebraic covariant of the set (L, N, K_s), but the problem of generalization of the transvectant formulary is formidable. We have also,

$$\{K_s\} = \{\{\overline{K_s}\},\ L\}^2,\quad [K_s] = \{\{\overline{K_s}\},\ N\}^2,\quad \langle K_s\rangle = \{\{K_s\},\ N\}^2.$$

The following are among the algebraic formulae of interest when the modulus is p,

$$\{K_s,\ L\}^r = C_{p+s-2r+1},\qquad (L = x_1^px_2 - x_1x_2^p).$$

In view of the special form of L, if $C_{p+s-2r+1}$ is known and K_s is unknown, this equation can be solved for K_s by the methods of § IV, Lemma 2, (*e*). We have also,

$$\{K_s,\ N\}^1 = K_sL^{p-2},\qquad \{K_s,\ N\}^2 = D_{p+s-1}L^{p-3},$$
$$\{K_s,\ N\}^r = E_{(r-1)p+s-r+1}L^{p-r-1},$$

E being of especial note when its order is divisible by $p-1$; $s = a(p-1)$.

Covariants led by prime seminvariants.

A seminvariant with no invariant factor, which is not a product of two seminvariants may be said to be prime. We construct covariants modulo 2 led by each prime case of the typical term of *f*. Such a case is a, and (cf. the syzygies),

$$R(a) = ax_1^2 + Ax_1x_2 + a_1x_2^2,$$

---

[11] The transvectant is always defined as having been freed from the numerical g. c. d. of its terms as given by the transvectant operator.

exists as a covariant. Here a_1 is the conjugate of a under t, and Λ is the scale invariant,

$$\begin{aligned}\Lambda = & a_0b_1c_0+a_0b_2c_0+a_1b_1c_0+a_0b_0c_1+a_0b_0c_2+a_0b_2c_1+a_0b_2c_3+a_1b_1c_1+a_0b_1c_2\\ & +a_1b_0c_0+a_1b_0c_2+a_1b_2c_1+a_1b_2c_2+a_0b_1c_3+a_1b_0c_3+a_1b_1c_3 .\end{aligned}$$

The covariants $[X_4]$, $Q[X_4]$, (where X_4 is $R(a)$), are led by $a+\Lambda$. If a cubic M, led by Λ, exists, $M+Q[X_4]$ will be a cubic led by a.

LEMMA. - *There exists a cubic covariant which has Λ as its leading coefficient.*

In outline the proof is as follows. If a function primed represents the function's increment with respect to (t_1, t_2, t_3), so that $a+a_1'=\Lambda$, and if

$$M=\Lambda x_1^3+\Lambda_1x_1^2x_2+\Lambda_2x_1x_2^2+\Lambda x_2^3,$$

is a covariant, the relation expressing covariancy gives $\Lambda_1'=\Lambda_2'=\Lambda$, because $\Lambda_1+\Lambda_2=\Lambda$ and $\Lambda'=0$. To construct M we partition Λ into its two parts λ_1, λ_2 conjugate with respect to t and try λ_1 for Λ_1 and λ_2 for Λ_2, but it is found that $\lambda_1'=\lambda_2'\neq\Lambda$. The λ_1', λ_2' have six terms in common with Λ and four other terms. The problem then is to choose by inspection a sum Σ of terms $a_i\,b_j\,c_k$ such that $\lambda_1'+\Sigma'=\lambda_2'+\Sigma'=\Lambda$, for then, $\lambda_1+\Sigma=\Lambda_1$, $\lambda_2+\Sigma=\Lambda_2$. Then a tedious process of inspection gave the self-conjugate Σ,

$$\Sigma=a_0b_0c_0+a_1b_0c_3+a_0b_2c_3+a_1b_2c_2+a_0b_1c_3+a_0b_2c_0+a_1b_0c_0+a_0b_0c_1+a_1b_1c_0+a_1b_2c_3 .$$

Hence M is a covariant, q. e. d., with

$$\begin{aligned}\Lambda_1= & a_0b_0c_0+a_1b_0c_3+a_0b_1c_0+a_0b_0c_2+a_0b_2c_1+\\ & +a_1b_1c_1+a_1b_2c_2+a_0b_1c_3+a_1b_0c_0+a_1b_2c_3,\\ \Lambda_2= & a_0b_0c_0+a_0b_2c_0+a_1b_1c_3+a_1b_2c_1+a_1b_0c_2+\\ & +a_0b_1c_2+a_0b_0c_1+a_1b_1c_0+a_0b_2c_3+a_1b_2c_3 .\end{aligned}$$

Let $K_6=f_1f_2f_3$; then M, as an algebraic covariant is,

$$\begin{aligned}M= & \{(\{\{K_6, LN\}^4, L\}^2+\{K_6, N^3\}^6N), L\}^2N+\{K_6, LN\}^4\\ & +(\{f_3, L\}^3L+\{\{f_3, L\}^2, L\}^2N+f_3)\{f_1, \{f_2, L\}^2\}^1+\{\{\{K_6, LN\}^4, L\}^2\\ & +\{(\{\{f_3, L\}^2, L\}^2N+f_3)\{f_1, \{f_2, L\}^2\}^1, L\}^2, N\}^2L.\end{aligned}$$

In view of (28) the formulae (30), (31) below are expressed algebraically. If I_j is an arbitrary integral polynomial in the invariants of the set (26), now to be augmented by Λ, the typical term of the f considered above can be written as,

(29) $$r_j=I_j\,a_0^{m_1}b_0^{m_2}c_0^{m_3}a^{n_1}\lambda^{n_2}\Gamma^{n_3}E^{n_4},\qquad (m_i=0, 1, \text{or } 2;\ n_k=0 \text{ or } 1).$$

By the two main reduction algorithms described above, a non-linear, (as to x_1, x_2), covariant led by r_j/I_j is reducible in terms of covariants led by prime cases of r_j/I_j, but this principle will be amplified further later. We now give tables of fundamental covariants. Each covariant is led by the prime seminvariant above it.

TABLE I: *Fundamental covariants of* (f_1, f_2, f_3) *under* G_6.

			Cubic order			
a_0	b_0+b_1	c_0	$\lambda+e^2+\Delta$	Γ	$E+b_1e$	α
Qf_1	$Q[f_2]$	f_3	$Q[R(\lambda)]$	$Q[R(\Gamma)]$	$Q[R(E)]$	$M+Q[X_4]$
			Quadratic order			
a_0	b_0	c_0	λ	Γ	E	α
wf_1	f_2	$\{f_3\}$	$R(\lambda)$	$R(\Gamma)$	$R(E)$	$R(\alpha)$
			Linear order			
a_0	b_0+b_1	c_0+e	$\lambda+e^2+\Delta$	Γ	$E+b_1e$	$\alpha+\Lambda$
f_1	$[f_2]$	$[f_3]$	$[R(\lambda)]$	$[R(\Gamma)]$	$[R(E)]$	$[R(\alpha)]$

Covariants led by invariants.

All quantics of order >3 are reducible modulo 2. There exists a cubic covariant led by e, viz., $f_3+Q[f_3]$. Hence we have cubics led by E, and by $\lambda+\Delta$. Each of the tabulated linear covariants is led by a binomial, a seminvariant plus an invariant, the latter being zero in two cases. If we change any factor of r_j into such a binomial, without really altering f, we merely rearrange the terms of f, but, if it is done to express f in terms of leaders of linear covariants of table III (below), it may bring unidentified scale invariants into the absolute term of f. The latter invariants are centrals of covariants like,

$$\{f_1 f_2^2 f_3^2 R(\Gamma)\} = a_0 b_0^2 c_0^2 \Gamma x_1^2 + \dots . \tag{30}$$

But the absolute term, in our linear covariant, must vanish identically or we would have a linear covariant led by an invariant, an impossibility. The principle involved is involved more explicitly in the following discussion.

To lead a covariant cubic, an invariant must not contain any term which is left unaltered by the substitution t. In general f is a nonhomogeneous polynomial in a_0, b_0, c_0, α, λ, Γ, E. It has an absolute term r_0 free from these seminvariants being therefore an arbitrary polynomial in forms of the set (26) augmented by Δ.

Now, $r_0 = G + W$, and W is the part whose terms contain as factors, one or more invariants of the set, Δ, k, g_1, e, j_1, j_2, j_3, L', K. In Table II are given. cubic covariants led by these respective invariants. Multiply these cubics by arbitrary polynomials in the fourteen invariants and add, and we have a cubic g whose first term is Wx_1^3. The part G is a polynomial in Δ, q, I, b_1, Q' exclusively. Each of these invariants has a term which is left unaltered by t and these respective terms are different. Hence if r_0 leads a cubic h, $G = 0$, and,

$$h = g + LJ \quad (\text{mod. } 2), \qquad (J \text{ invariant}).$$

The covariant $\{f_3, f_3\}^2$, in Table II is a Hessian (with no numerical factor excepting unity), and Δ_2 is the irreducible covariant,

$$\Delta_2 = ((b_0 + b_2)x_1 + (b_0 + b_1)x_2)((b_1 + b_2)x_1 + (b_0 + b_2)x_2).$$

The last term of the formula opposite g_1 is itself a covariant led by $c_0^2 \lambda$, and therefore reducible. Note that $I_j Q$ is a quadratic led by I_j.

TABLE II. - *Covariants led by invariants.*

Cubic order

Leader	Covariant
k'	$[f_2]\Delta_2 \qquad (k' = k + b_1 q)$
Δ	M
e	$Q[f_3] + f_3$
L'	$Q'Qf_1 + f_1^3$
j_1	$Q[f_1 f_2] + f_1 f_2$
j_2	$Q[f_1 f_3] + f_1 \{f_3\}$
j_3	$Q[f_2 f_3] + b_1 Q[f_3] + b_1 f_3 + \{\overline{f_2 f_3}\}$
K	$Q[ef_3\{f_3\} + f_3\{f_3, f_3\}^2] + [e\{f_3\} + \{f_3, f_3\}^2]\{f_3\} + \Delta Q[f_3] + e^2 f_3$
g_1	$Q[(e\{f_3\} + \{f_3, f_3\}^2)(ef_3 + [f_3]\{f_3\})] + \{\overline{(e\{f_3\} + \{f_3 . f_3\}^2)(ef_3 + [f_3]\{f_3\})}\}$

A fundamental system of covariants of (f_1, f_2, f_3), (mod. 2).

A fundamental system is a set of irreducibles which suffice for the reduction of all orders of arbitrary covariants, that is, in the present theory, for the reduction of all covariants of orders 1, 2, 3 whose leading coefficients are in the form f of which r_j of (29) is a typical term. If a covariant ξ is led by f, we can construct a covariant of like order for each term of f (led by the term).

The sum η of the latter covariants is a covariant and, since $\xi-\eta$ has the factor x_2,

$$\xi=\eta+XL, \qquad (X \text{ an invariant or } 0),$$

and ξ is reduced. The principle is due to DICKSON. The case of the term r_0 of f was treated above. The complete story of the reduction of a covariant of order 2 or 3 led by a composite r_j $(j \neq 0)$ involves two steps. First, construct a covariant of the given order led by r_j/I_j. It will be multiplied by I_j later. Formula (30) shows the method of construction of a quadratic of led by r_j/I_j or one of its factors. If we require a cubic, the formula will be of the type,

$$\{\overline{f_1 f_2^2 f_3^2 R(\Gamma)}\} = a_0 b_0^2 c_0^2 \Gamma x_1^3 + \ldots\ldots \tag{31}$$

Second, factor r_j/I_j into two factors S, T and construct a covariant of the order led by each factor. The covariant whese leader is r_j/I_j is then reducible by the first two algorithms (italicized) in our Paragraph on reduction techniques. Repeat the process for the covariants led by S, T. Continue until the two factors required each time become prime seminvariants. The covariant whose leader is r_j will then have been reduced in terms of covariants of Tables I, II, universal covariants, invariants, and covariants of the first order.

Consider the fundamental covariants of order unity in x_1, x_2. Every quantic of Table III below is irreducible in view of the Paragraph on syzygies and all such are contained in the Table. If we should adjoin to the seminvariant system (25) a complete system of invariants [9], some new syzygies probably would result which would reduce some forms of Table III.

The aggregate of irreducibles from the finite number of covariants given in Tables I, II, III, (with L, N, Λ, and invariants (26)), *is a fundamental system of formal covariants* (mod. 2) *of the set* (f_1, f_2, f_3).

TABLE III. - *Fundamental covariants of order unity.* - With $R(\lambda)=X_1$, $R(\Gamma)=X_2$, $R(E)=X_3$, $R(a)=X_4$, the irreducible covariants of the first order in x_1, x_2 are those given by the formula,

$$[f_1^i\, f_2^j\, f_3^k\, \delta],$$

where i, j, k assume the values 0, 1, or 2, repetitions being permitted, and δ runs through the set 1, X_1, X_2, X_3, X_4, X_1X_2, X_1X_3, X_1X_4, X_2X_3, X_2X_4, X_3X_4, $X_1X_2X_3$, $X_1X_2X_4$, $X_1X_3X_4$, $X_2\,X_3\,X_4$, $X_1\,X_2\,X_3\,X_4$.

* * *

A method for fundamental invariants, not limited to those of (26), can be based upon the theorem stated in the last line of § IV.

Among the generalizations which are possible, beyond the more special situations which we have described in this paper, is a theorem that the invariant theory of a set of polynomials in any number of variables which are subject to linear transformations, is a correlation between two sets of invariant hyperplanes. On the algebraic side, Hilbert's basis theorem is the leading principle of the correlation. (Cf. § II).

The principles of § IV can be made to apply to the concomitants of any special linear group, used in place of $G(p)$, and these methods are the only ones of this generality that have been found.

VII. - Special processes (*i*, *ii*, *iii*).

(*i*) - **Erosion of a solid afloat in a stream.**

A suggestion for special inverse invariant processes may be obtainable from some geometrical configuration of a structure definitely described by analytic properties. Some segment, for example, may be given by a formula and when the segment is assumed invariant under alterations of the configuration, some other geometric element may become the dependent variable in a type of equation. Solution of the equation will determine a function from its invariant.

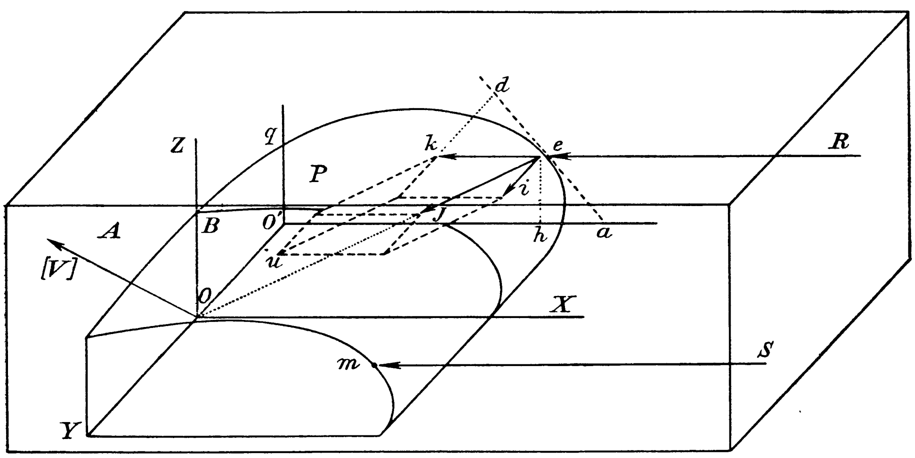

Fig. 2.

Accordingly, consider a liquid in the form of a freely-flowing canal-stream A in which there is a solid B of such density that its floating level is at a depth δ. Since the upper part of a stream moves faster than the lower part, the stream A will exert a shear upon the solid B.

PROBLEM. - *To find the equation of the solid's surface considered to be the natural erosion product of the action.*

HYPOTHESIS. - *The solid B has a vertical plane of symmetry (the XZ plane), whose horizontal elements are stream-lines of the flow of the liquid. The plane sections of the figure, (as P), are perpendicular to the Y-axis, (OO').*

If stream and solid were at rest the condition for equilibrium is that any pressure vector through the center of mass (taken as O) is counterbalanced by an equal and opposite vector through O. The shearing pressure on B may be described as follows. The pressure R at e, against a small square area of an interposed vertical plane parallel to YOZ, exerted by the cube C of the liquid, of which the square is one face, is poportional to C's momentum. Both mass M and velocity v, of C, are functions of a depth variable θ ($=hO'e$); $M=g(\theta)$, $v=f(\theta)$, and we consider θ only between definite limits $\theta_1<\theta<\theta_2$. Hence,

$$R=af(\theta)g(\theta), \qquad (=aH(\theta)).$$

The effective pressure on B at e is the projection $T=eu$, of the vector R, upon the normal to the surface. Thus T is expressed in terms of R and the angle φ between R and eu. Taking eu as the diagonal of a parallelopiped of forces, we resolve T into components, viz., ek along R produced in P; ei in P along the normal to the curve of section eq; ej along the radial line Oe. The line Oe is r in the polar equation of the surface and $O'e$ is r in the polar equation of eq.

We resolve similarly at m, the point of B symmetrical to e, the parallel vector $S=R$. Then the resultant of the pair of components through O is a single vector V which reaches forward in plane XZ, and the resultant of all vectors V, for all pairs (e, m) of B, allowed by θ's limits, is a single vector $[V]$ which pulls B forward without rotation. At e there remain the normal component ei, and the component ek parallel to OX. Both of these vectors have a shearing effect and the total shear due to R is their resultant in P.

With P, (that is, OO'), fixed, and $\theta_1<\theta<\theta_2$, the angle φ, the vector R, and therefore ek, are single-valued functions of θ. Hence, to any degree of accuracy, depending upon s,

$$ek=a\theta^{s-1}+b\theta^{s-2}+\dots+l, \quad (=G(\theta)),$$

where $a, \dots, l$ are numerical.

INVARIANT PROPERTY. - *We assume (what is in accordance with fact) that, as θ varies, the segment dk, parallel to the vector ei, remains invariant.*

We can then determine a differential equation of the curve eq, which involves $G(\theta)$ among its coefficients. From the shape of eq for different distances OO', the shape of the surface can be determined. With O' as origin we have,

$$\angle aO'e=\theta, \quad O'e=r, \quad \angle O'ea=\psi,$$
$$\tan\psi=rd\theta/dr, \quad dk=\gamma \quad \text{(constant)}.$$

Hence,

$$dk = ek \cos(\theta + \psi - \pi/2) = G(\theta) \sin(\theta + \psi) = \gamma,$$

or,

$$G(\theta)\left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right) = \gamma\sqrt{\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]}. \tag{32}$$

Here $G(\theta)$ may figure as an arbitrary function and γ as an arbitrary constant. The variables are not separable but a solution $r = F(\theta)$ exists, and $F(\theta)$ is an instance of a function determined from one of its invariants.

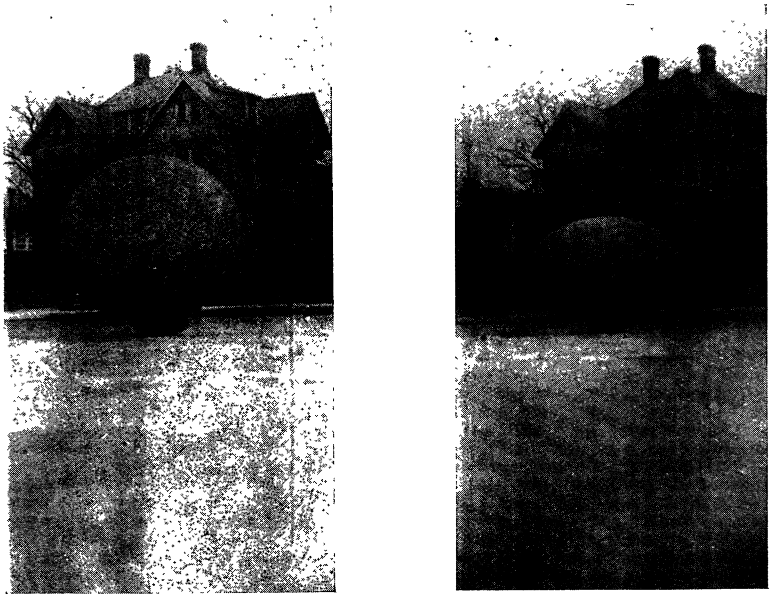

Fig. 3.

GLACIERS. - Our statement of the present problem describes the flow of a deep glacier because the bottom layer of a glacier is under such pressure that its composition is that of semi-crystalized water [10], [11] and is capable of flowing slowly. A boulder afloat within the bottom layer, assumed to contain sand which aids erosion, will be ground to a form which is represented in cross-section by the solution of equation (32). Since the writer has seen a number of such boulders from moraines of the great glacier in Maine, we were able to use their cross-sections as a solution $r = F(\theta)$ of (32), whence we determine $G(\theta)$. These boulders, which are of granite, are true ellipsoids with equation,

$$x^2/a^2 + y^2/c^2 + z^2/b^2 = 1, \qquad (a > b > c),$$

The photographs show side view and edge view of one, of dimensions $a=7$ in., $b=6$ in., $c=4.4$ in. approximately [12]. After $G(\theta)$ had been determined as above, we generalized it and reversed the equation (32), solving for a generalized form of $r=F(\theta)$.

Thus the curve *eq* was identified as an ellipse experimentally. We can readily show that the flow of a stream will hold some principal plane of such an ellipsoid in coincidence with the plane XZ, it being a question of chance as to which principal plane it is. Hence we can choose the least axis (c) on $O'Y$, whence *eq* is,

$$r=I/(a^2 \sin^2 \theta+b^2 \cos^2 \theta)^{1/2}=F(\theta), \qquad (I \text{ constant}).$$

Substituting this $F(\theta)$ in (32) we get,

$$G(\theta)=\gamma(\sqrt{[a^4 \sin^2 \theta+b^4 \cos^2 \theta]})/b^2 \cos \theta.$$

Now, the essential symmetries, special cases, and property of being an increasing function of θ, of this $G(\theta)$, will be preserved if we replace it by,

$$(33) \qquad G(\theta)=\gamma(\sqrt{[a^4 \sin^{4p-2} \theta+b^4 \cos^{4p-2} \theta]})/b^2 \cos^{2p-1}\theta, \qquad (p>1/2).$$

Using this form in (32) the integral is,

$$(34) \qquad r=J/(b^2 \cos^{2p} \theta \pm a^2 \sin^{2p} \theta)^{1/2p}=F(\theta), \qquad (J \text{ constant}).$$

This is an (ab) section of a surface,

$$(35) \qquad E: \quad x^{2p}/a^2+y^{2p}/c^2 \pm z^{2p}/b^2=1, \qquad (a \geqq b \geqq c).$$

In the case of a glacier, the upper sign in (34), ($p=1$), gives the ellipsoid as a product of erosion, an the lower sign gives hyperboloids. The latter are the shapes of the hills which can survive beneath a deep glacier.

The transformations. - The form of $G(\theta)$ in (33) is independent of the size of the solid and depends only on the parameters γ, p, a/b. Hence the most fundamental transformation leaving dk invariant is the ternary linear transformation Γ which carries an ellipsoidal E into an ellipsoidal E' and leaves the axes of E' proportional respectively tho those of E. Thus under Γ,

$$\Gamma: \quad x=\lambda_1 x'+\mu_1 y'+\nu_1 z', \quad y=\lambda_2 x'+\mu_2 y'+\nu_2 z', \quad z=\lambda_3 x'+\mu_3 y'+\nu_3 z',$$

we should have,

$$(36) \qquad lx^{2p}+my^{2p}+nz^{2p}=K(lx'^{2p}+my'^{2p}+nz'^{2p}),$$
$$(l=1/a^2, \; m=1/c^2, \; n=\pm 1/b^2, \; K \text{ constant}).$$

If we assume (36) to be an identity we can solve for some of the λ_i, μ_i, ν_i,

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(12) « Found on Cape Ann », Mass. by MR. W. H. BARKER of Lansdowne, Pa.

(six in the case $p=1$), in terms of the rest as independent parameters. We have carried out the work for $p=1$, as shown in what follows.

We note first that the determination of all binary transformations,

$$t: \quad x=\lambda_1 x'+\mu_1 z', \quad z=\lambda_2 x'+\mu_2 z', \qquad (\lambda\mu)\neq 0,$$

for which,

$$x^2/a^2+z^2/b^2=L(x'^2/a^2+z'^2/b^2), \qquad (L \text{ constant}),$$

is a special solution for transformations which leave dk invariant, that is, which leave the ellipse eq relatively invariant. We readily determine t, as,

$$t: \quad x=\frac{1}{a}\,[\sqrt{(La^2-\mu_1^2 b^2)}]x'+\mu_1 z', \quad z=\frac{b^2}{a^2}\,\mu_1 x'-\frac{1}{a}\,[\sqrt{(La^2-\mu_1^2 b^2)}]\,z'.$$

When we substitute, from Γ in (36), we obtain; $(p=1)$,

$$(37) \qquad \begin{cases} l\lambda_1^2+m\lambda_2^2+n\lambda_3^2=Kl, & l\lambda_1\mu_1+m\lambda_2\mu_2+n\lambda_3\mu_3=0, \\ l\mu_1^2+m\mu_2^2+n\mu_3^2=Km, & l\lambda_1\nu_1+m\lambda_2\nu_2+n\lambda_3\nu_3=0, \\ l\nu_1^2+m\nu_2^2+n\nu_3^2=Kn, & l\mu_1\nu_1+m\mu_2\nu_2+n\mu_3\nu_3=0, \end{cases}$$

To derive elimination formulas which will be free from extraneous factors the following plan must be pursued. Eliminate λ_3 between the first, fourth and fifth equations (37), leaving five equations in eight unknowns, three being free from λ_2. Eliminate λ_2 and we then have,

$$\begin{aligned}\lambda_1=\sqrt{Kmn}(\mu_2\nu_3-\mu_3\nu_2)/\{ & lm\mu_1^2\nu_2^2+lm\mu_2^2\nu_1^2+ln\mu_1^2\nu_3^2+ln\mu_3^2\nu_1^2+mn\mu_2^2\nu_3^2 \\ & +mn\mu_3^2\nu_2^2-2lm\mu_1\mu_2\nu_1\nu_2-2ln\mu_1\mu_3\nu_1\nu_3-2mn\mu_2\mu_3\nu_2\nu_3\}^{1/2},\end{aligned}$$

together with enough auxiliary equations to eliminate μ_2, μ_3, ν_3 from λ_1, leaving the latter a function of μ_1, ν_1, ν_2. The solution may then be completed readily and is as follows, when expressed in the form of a transformation T;

$$\begin{aligned}
x &= \frac{\sqrt{\Delta_2}}{\sqrt{m\Delta_2}}\,x'+\mu_1 y'+\nu_1 z', \\
y &= \frac{l\mu_1\Delta_2\sqrt{mK}-lm\nu_1\nu_2\sqrt{\Delta_1}}{(-lm\nu_1^2+mnK)\sqrt{m\Delta_2}}\,x'+\frac{lm\mu_1\nu_1\nu_2+\sqrt{mK\Delta_1}}{lm\nu_1^2-mnK}\,y'+\nu_2 z', \\
z &= \frac{(l^2m\nu_1^3-lmnK\nu_1+lm^2\nu_1\nu_2^2)\sqrt{\Delta_1}-lm\mu_1\nu_2\Delta_2\sqrt{mK}}{(-lm\nu_1^2+mnK)\Delta_2\sqrt{m}}\,x'+ \\
&\qquad +\frac{lmnK\mu_1\nu_1-l^2m\mu_1\nu_1^3-lm^2\mu_1\nu_1\nu_2^2-m\nu_2\sqrt{mK\Delta_1}}{(lm\nu_1^2-mnK)\sqrt{\Delta_2}}\,y'+\frac{\sqrt{\Delta_2}}{n}\,z'.
\end{aligned}$$

In this result the essential irrationalities, assumed real, are $\sqrt{m\Delta_1}$, $\sqrt{m\Delta_2}$ in which

$$\begin{aligned}\Delta_1 &= lmn\mu_1^2\nu_2^2+l^2n\mu_1^2\nu_1^2+l^2m\nu_1^4+lm^2\nu_1^2\nu_2^2-2lmnK\nu_1^2-m^2nK\nu_2^2-ln^2K\mu_1^2+mn^2K^2, \\ \Delta_2 &= -ln\nu_1^2-mn\nu_2^2+n^2K.\end{aligned}$$

If s stands for the alterations under which dk is invariant, dk is relatively invariant under sT.

(ii) - Functional relations between the masses of the planets in a system.

It is known that the integral,

$$r=f(\theta, c_1, c_2),$$

of the equation of central orbits,

$$d^2u/d\theta^2+u=[F(1/u)]/\gamma^2u^2, \qquad (u=1/r, \ F=\text{force}), \tag{38}$$

involves, as a parameter, the mass m of the moving body, here assumed to have the dimensions of a planet. A finite segment of the orbit can therefore be represented by an equation,

$$\theta=\Delta_0(m)r^{n-1}+\Delta_1(m)r^{n-2}+...+\Delta_{n-1}(m), \quad (=\Delta(r, m)). \tag{39}$$

The $\Delta_i(m)$ are rational, integral quantics with real coefficients [12],

$$\Delta_i(m)=e_{0i}m^{s-1}+e_{1i}m^{s-2}+...+e_{s-1i},$$

in which m has a range of continuous variation but so small that the orbital perturbations caused by the variation are not sufficient to render the orbit unstable. The parameter m cannot be varied *ab initio* over the masses of the sequence of planets in a central system.

For any value θ in (39) the distances from the center of force o to the respective planets are given by a generalized form of BODE'S formula, viz.,

$$q_{j+2}=g(q_2/q_1)^j+h, \qquad (j=0,..., k),$$

$g=q_1(q_2-q_3)/(q_1-q_2)$, $h=(q_1q_3-q_2^2)/(q_1-q_2)$, q_i being the distance from o to planet number i, and q_1, q_2, necessarily given by observational measurements. It is obviously an algebraic simplification to subtract h from the roots of $\Delta(r, m)$, and multiply the roots of the result by $1/g$. These processes lead to equations of fictitious or auxiliary orbits,

$$\theta=\gamma_0(m)r^{n-1}+\gamma_1(m)r^{n-2}+...+\gamma_{n-1}(m), \quad (=\Gamma(r, m)), \tag{40}$$

each of which is moved outward by one space if we multiply the roots of $\Gamma(r, m)$ by $z=q_2/q_1$. Here θ should be allowed only a small variation which, by a choice of axes, may be held near to zero. With z known we will therefore have the following as an orbital system;

$$z^{(n-1)e}\theta=\gamma_0(m)r^{n-1}+z^e\gamma_1(m)r^{n-2}+...+z^{(n-1)e}\gamma_{n-1}(m), \quad (e=0,..., k). \tag{41}$$

The independence of these equations can be emphasized by using a different letter m in each; also a different r. Relations between the different masses can be shown to exist by means of (41), but more simply as follows. If we form an equation like (39) for the orbit q_{e+2}, its auxiliary will have an equation,

$$(42) \qquad \theta = \lambda_0(m')r^{n-1} + \lambda_1(m')r^{n-2} + \dots + \lambda_{n-1}(m'), \quad (= \Lambda(r, m')).$$

Since the equation for the mass m' is unique and λ_v may be assumed to be of the same order as γ_v we have,

$$(43) \qquad \lambda_v(m') = z^{(v+1-n)e}\gamma_v(m), \qquad (v = 0, \dots, n-1),$$

whence the functional relation between m' and m must be linear, that is, homographic in general,

$$H(m, m'): \quad m = \frac{a_1 m' + a_2}{\beta_0 m' + \beta_1}.$$

If λ_v is the same function as γ_v, (43) is a relation of covariancy of γ_v under direct transformation by H. Therefore a homographic relation between the masses of any two planets, correlating their respective orbital equations, will exist if an expansion (41), of generality sufficient to represent the system of orbits, can be formed, having covariants of H as the γ_v.

The key to this construction is the periodic homography. The period must be greater than the number of planets in the system. The homography H will be periodic, and will have the period w mentioned concurrently, if the parameters satisfy one of the following relations. Each $H(m, m')$ generates a formal cyclic group.

$$\beta_1 = -a_1, \qquad w = 2,$$
$$a_2 = -(a_1^2 + a_1\beta_1 + \beta_1^2)/\beta_0, \qquad w = 3,$$
$$a_2 = -(a_1^2 + \beta_1^2)/2\beta_0, \qquad w = 4,$$
$$a_2 = -[3a_1^2 + 4a_1\beta_1 + 3\beta_1^2 - (a_1 + \beta_1)^2\sqrt{5}]/2\beta_0, \qquad w = 5,$$
$$a_2 = -(a_1^2 - a_1\beta_1 + \beta_1^2)/3\beta_0, \qquad w = 6.$$

The following propositions are important, though, for the sake of brevity, proofs are omitted.

A homography of period three combined with one of period five is a homography of period fifteen, thus some higher orders, also, are represented in the list.

If a set of real numbers m_1, $m_2, \dots$, are permuted in a simple cycle by a homography in which the parameters are complex imaginaries, then m_1, $m_2, \dots$ can also be permuted cyclically by a homography H whose parameters are real. Hence only real homographies need be considered here.

If a homography is periodic with period > 2, its poles are imaginary.

Covariants of periodic homographies. - Iteration of an $H(p', p)$ of period

w gives a set S of linear fractional functions which are permuted cyclically by the transformation. When $w=3$ the set is,

$$S:\quad p,\quad [\alpha_1 p-(\alpha_1^2+\alpha_1\beta_1+\beta_1^2)/\beta_0]/(\beta_0 p+\beta_1),\quad [-\beta_1 p-(\alpha_1^2+\alpha_1\beta_1+\beta_1^2)/\beta_0]/(\beta_0 p-\alpha_1).$$

When the cycle of w functions is known, the quantic in homogeneous indeterminates x_1, x_2 which has the functions as roots, is a covariant under direct transformation by $H(x_1'/x_2', x_1/x_2)$, since the transformation on x_1/x_2 induces a cogredient transformation on the roots, which are therefore merely permuted. With $w=3$, the covariant is,

$$\begin{aligned} P=&(\beta_0 p-\alpha_1)(\beta_0 p+\beta_1)x_1^3+[-\beta_0^2 p^3+3(\alpha_1^2+\alpha_1\beta_1+\beta_1^2)p-(\alpha_1^3-\beta_1^3)/\beta_0]\,x_1^2x_2\\ &+[\beta_0(\alpha_1-\beta_1)p^3-3(\alpha_1^2+\alpha_1\beta_1+\beta_1^2)p^2+(\alpha_1^2+\alpha_1\beta_1+\beta_1^2)^2/\beta_0^2]\,x_1x_2^2\\ &+[\alpha_1\beta_1 p^3+(\alpha_1^3-\beta_1^3)p^2/\beta_0-(\alpha_1^2+\alpha_1\beta_1+\beta_1^2)^2 p/\beta_0^2]\,x_2^3, \end{aligned}$$

and the transformations are,

$$R:\quad x_1=[\alpha_1 x_1'-(\alpha_1^2+\alpha_1\beta_1+\beta_1^2)x_2'/\beta_0]\,\zeta,\quad x_2=(\beta_0 x_1'+\beta_1 x_2')\zeta.$$

where ζ is arbitrary. From $RP=P'$ we find,

$$P=-(\alpha_1+\beta_1)D\zeta^3P',$$

$D\zeta^3$ being the determinant of R. We can assume $-(\alpha_1+\beta_1)D\zeta^3=z^{-1}$, (Cf. (41)), i. e. choose the modulus of P for z. We also require an absolute covariant. Let P_1, P_2 be two covariants of type P, order w, and modulus M, of $H(x_1'/x_2', x_1/x_2)$ of period w. Then, by polynomial approximation,

$$Q(x)=P_1(x)/P_2(x),\qquad (x=x_1/x_2),$$

can be expressed in the form,

$$x_2^w Q(x)=a_0x_1^w+a_1x_1^{w-1}x_2+\ldots+a_wx_2^w.$$

In the finite region delimited by the determinations $(Q_i, x_{(i)})$ used in calculating $Q(x)$, the latter is an absolute ε—covariant of H.

We have now determined (40) so the γ_v will have the required invariant properties (43) under a periodic $H(x, x')$. When (42) is the same as F, (c_i arbitrary constants),

$$F:\quad \theta=c_0Q(x)^{n-1}r^{n-1}+c_1Q(x)^{n-2}P(x)r^{n-2}+\ldots+c_{n-1}P(x)^{n-1}.$$

then (41) is the same as H^eF, except for ε—terms, H^e being the e-th iteration of H, q. e. d.

All masses of the planetary system can be obtained from the second mass by the iteration of a properly calculated $H(m, m')$, but the first, obtainable from the inverse of H, should not be used as a basis for the calculation of H. The main practical object is the determination of H.

The system of moons (planets) of Saturn.

We chose the system of moons of Saturn as the planetary system most likely to give accurate results as a numerical instance of the above theory. That is, the generalized Bodeian law for the distances of these moons (from Saturn), gives, with the value $z=1\cdot3$ as the Bodeian ratio-number, accurate values of the distances. [12] This system is the best known illustration of the Bodeian principle.

We substituted successively in the equation,

$$\beta_0 mm' + \beta_1 m' - a_1 m - a_2 = 0,$$

for (m, m'), (xa Enceladus, ya Tethys), (ya Tethys, $\cdot187a$ Dione), ($\cdot187a$ Dione, $\cdot4a$ Rhea), and solved for the ratios a_1/β_0, a_2/β_0, β_1/β_0. This gives the homography,

$$H(m', m): \quad m' = \frac{Ama + Ba^2}{Cm + Da},$$

$$A = -\cdot187y^2 - \cdot213xy + \cdot1496y - \cdot0139876,$$
$$B = \cdot4xy^2 - \cdot039831y^2 - \cdot109769xy + \cdot0139876x,$$
$$C = -y^2 + \cdot587y - \cdot213x - \cdot034969,$$
$$D = xy^2 - \cdot374xy + \cdot0748x - \cdot039831y.$$

Beginning with $m = \cdot4a$ (Rhea) we iterated this H four times, that is, across four Bodeian intervals, from Rhea to Titan and equated the result to 21 (Titan). The unit of mass here is $a/100000$, a being the mass of Saturn. Numbers such as $\cdot187a$ (Dione) were treated as if exact but, as this computation progresses, no decimal number should be cut off, as sufficiently accurate, short of fourteen decimal places. We thus obtain an equation, $J=0$, of order 12 in (x, y). Knowing that the mass of Enceladus is near $\cdot025a$, we construct a series of solutions (x, y) of $J=0$ with x in the vicinity of $\cdot025a\cdot$One such solution, substituted in $H(m', m)$, would give to H its desired numerical form, except that H itself imposes certain arithmetical restrictions. All results of its iteration, in numerical form, should be positive numbers. Hence C and D should be opposite, and C and A the same in sign. Also the zero $-aB/A$ and the singularity $-aD/C$ should be near together in value, for if any m' produced by iteration falls within the interval I between the zero and the singularity, the next m' will be negative. It is soon seen that the value $\cdot025a$, which astronomers have published for xa (Enceladus), is the most favorable x-value in our series of solutions (x, y) of $J=0$. The latter equation then becomes an octic in y, the pertinent solution of which is $ya = \cdot137241a$ (Tethys). The uniqueness of the resulting H in the vicinity (x, y) considered on $J=0$, appears to be a case of an important theorem yet unproved. Our final result for H is,

$$H_1(m', m): \quad m' = \frac{\cdot 22906831ma - \cdot 05887998a^2}{2 \cdot 14313749m - \cdot 44087723a}.$$

The poles of H_1 are imaginary as the theory requires.

Substituting $m' = \cdot 025a$ (Enceladus) in H_1 and solving for m we get, $m = \cdot 2727112a$, (Mimas). The results obtained by iterating H_1 across the seventeen Bodeian intervals from Mimas to Phoebe are shown in Table IV wherein the numbers in parentheses are the mass-values given previously in astronomical tables.

TABLE IV. - *The mass-values of the moons of Saturn.*

Mimas	Enceladus	Tethys	Dione	Rhea	Lucretia *
$\cdot 2727112a$ ($\cdot 007a$)	$\cdot 025a$ ($\cdot 025a$)	$\cdot 137241a$ ($\cdot 11a$)	$\cdot 187a$ ($\cdot 187a$)	$\cdot 4a$ ($\cdot 4a$)	$\cdot 0786482a$
Clio *	**Minerva ***	**Titan**	**Hyperion**	**Terpsichore ***	**Faustina ***
$\cdot 1500575a$	$\cdot 2054483a$	$21 \cdot a$ ($21 \cdot 277a$)	$\cdot 1066207a$	$\cdot 1622444a$	$\cdot 23308a$ Revised to $\cdot 264a$
Japetus	**Clymene ***	**Kala ***	**Theresa ***	**Tyche ***	**Phoebe**
$\cdot 0127615a$	$\cdot 1353155a$	$\cdot 1848087a$	$\cdot 369278a$	$\cdot 0733445a$	$\cdot 148328a$

The asterisks designate moons which no longer exist or have not been discovered.

The small revision of the mass of Faustina should be explained. The mass given by H_1^{11} is $m' = \cdot 23308a$ and this value is on the interval I between the singularity and the zero of H_1; $I(\cdot 205716a, \cdot 25704a)$. Trial calculations show that this impasse cannot be corrected by small alterations alone of the mass of Enceladus ($\cdot 025a$). It is a result of the several possibilities for small errors which we have mentioned but especially of the fact that we have treated the mass of Dione and that of Rhea as if they were exact. Accordingly we arbitrarily replaced the value $\cdot 23308a$ by $\cdot 264a$ thus moving the mass-value for Faustina a little way out of the interval I. It is perhaps fortunate, since we cannot expect perfect accuracy, that, after eleven iterations, we have a method of introducing a corrective alteration that is both definite and much restricted. Wishing to avoid conjectures, we do not account for the fact that the mass-value obtained for Mimas is too large.

The fact that a unique homography so accurately produces the mass-values of this planetary system appears to be a contradiction of the idea that chance has played any essential part in the cosmic creative processes, as far as the sizes of the planets of a central system are concerned [15].

(*iii*) - Invariants in the theory of orbits.

Further variety in the program of inverse processes in invariant theory may be obtained and this will be evident from our third special problem. The question next considered is that of the determination of the equation of the orbit of a celestial body from its invariants of stability. We derive the equation of the arbitrary stable arc of finite length. Most of our illustrations were drawn from the theory of the comets. The problem of the invariance of periods of revolution is considered.

The stabilizing invariants.

A curve is said to be a stable orbit of a celestial mass if the curve's position and form are such that the central forces will constrain the mass to move on the curve. Since least action is a vital principle in central motion we regard the form of the orbital curve, and not the form of the force-function F in (38), as the element expected to have essential simplicity [13]. In a previous paper [12], I have derived the most general form of F in order that the orbit should be stable. It is,

$$F = G_n(r) = \gamma^2\lambda^2\,[2p(r)^2 - rp(r)p'(r) + r^2/\lambda^2]/r^5, \tag{44}$$

$(p' = \partial p/\partial r,\ p(r) = ar^{n-1} + br^{n-2} + \dots + k)$, r being the distance of the astral body from the center of force. Another definition of a stable arc would be, any integral curve of (38) when F has the value (44). For increasing integral values of n, G_n is an increasingly accurate gravitational formula. The number a decreases rapidly as n increases. We can say that G_5 is « practically accurate » for astrophysical theory, and,

$$\begin{aligned} G_5(r) = \gamma^2\lambda^2\,[-\{\,2a^2r^8 + 3abr^7 + (b^2+2ac)r^6 + (ad+bc)r^5\,\} \\ + (be+cd)r^3 + (d^2+2ce+1/\lambda^2)r^2 + 3der + 2e^2]/r^5 = \gamma^2\lambda^2 A_5(r). \end{aligned} \tag{45}$$

It is important that the potential $rG_5(r)$ is nearly reproduced, with sign reversed, by the reciprocation $r = 1/r'$.

The segment of the orbit, for which (44) is valid as the force, has the equation,

$$\int dr/p(r)=\lambda\theta+\lambda_1, \qquad (\lambda,\ \lambda_1 \text{ constant}), \tag{46}$$

but (39), which we express more simply as,

$$C: \quad \theta=\nu(m)r^{n-1}+\xi(m)r^{n-2}+\ldots+\pi(m)r+\varrho(m),$$

can also be the equation of this segment. If we make (46) and C identical, we obtain $n-2$ rational expressions in $\nu,\ldots,\pi$, the vanishing of which are necessary and sufficient conditions that C should be stable. These expressions,

$$C_{in}, \qquad (i=1,\ldots,n-2), \text{ (Cf. } R_4,\ p.\ 303),$$

are invariants and may be called invariants of stability. When $n=5$,

$$C_{15}=4\,[16\nu O^4-36\nu\xi O^2\pi+16\nu^2 O\pi^2+9\nu\xi^2\pi^2]/\pi^5\lambda$$
$$C_{25}=[48\xi O^4-108\xi^2 O^2\pi+96\nu\xi O\pi^2+27\xi^3\pi^2-32\nu O^3\pi-16\nu^2\pi^3]/\pi^5\lambda$$
$$C_{35}=[32O^5-96\xi O^3\pi+48\nu O^2\pi^2+54\xi^2 O\pi^2-24\nu\xi\pi^3]/\pi^5\lambda.$$

THEOREM. - *The equation of any stable finite arc, ($n=5$), can be expressed in the form,*

$$\theta=y\left[\frac{O(m)^3}{\pi(m)^2}\right]r^4+x\left[\frac{O(m)^2}{\pi(m)}\right]r^3+O(m)r^2+\pi(m)r+\varrho(m), \tag{47}$$

where $(x,\ y)$ are the (cartesian) coordinates of a common intersection of the three curves,

$$\begin{cases} (9x^2-36x+16y+16)y=0, \\ 27x^3-108x^2+48x+96xy-16y^2-32y=0, \\ 54x^2-96x-24xy+48y+32=0. \end{cases} \tag{48}$$

In proof of this theorem, the elimination of ν between $C_{25}=0$, and $C_{35}=0$, gives,

$$243\xi^5-729\xi^4\left(\frac{O^2}{\pi}\right)-108\xi^3\left(\frac{O^2}{\pi}\right)^2+1728\xi^2\left(\frac{O^2}{\pi}\right)^3-1728\xi\left(\frac{O^2}{\pi}\right)^4+512\left(\frac{O^2}{\pi}\right)^5=0.$$

Hence, $\xi=xO^2/\pi$ and,

$$243x^5-729x^4-108x^3+1728x^2-1728x+512=0. \tag{49}$$

Then $\nu=yO^3/\pi^2$, in which,

$$y=(27x^2-48x+16)/(12x-24).$$

Substitution of these values ν, ξ in C_{i5}, $(i=1, 2, 3)$, gives (48). The equations from (46) onward hold within the field (Y) of approximation which is assumed when C is derived by polynomial approximation. The powers of O/π, which are involved as factors against the expressions (48) were cancelled. They are small for some units of distance and large for other units, while the set (48) is unaltered by a change of units. That is, the equations hold regardless of the factors, q. e. d.

A solution of (48) is $x=\cdot 5934$, $y=\cdot 1763$, this x being a root of (49). If we use $n=6$, the equation of C will be (47) with the term $r^5 z_1 O(m)^4/\pi(m)^3$ added to the right hand side. Then (x, y, z_1) is the common intersection of four algebraic surfaces, and the new (x, y) will be a little more accurate than the (x, y) of (48).

To find the substitutions for which the C_{in} are invariant, we may use the generators of the algebraic Q, (Cf. (1), § I), as given by DICKSON

$$V:\ \ r_1=-r_2',\ \ r_2=r_1';\qquad T_h:\ \ r_1=r_1'+hr_2',\ \ r_2=r_2';$$
$$S_k:\ \ r_1=r_1',\ \ r_2=kr_2',\qquad\qquad (k\neq 0).$$

When these in succession are tried on the homogeneous form of C, it is found that only S_k leaves C_{in} invariant, q. e. d.

The arbitrary phase of the ground-function (47) depends upon the arbitrariness of O, π, ϱ. There is but one intersection of (48).

Equation C is in the rational or parabolic form. We shall be able to compare it with the general functional form in certain cases.

Can an ellipse which is very eccentric be the permanent orbit of a comet?

HALLEY'S is a typical comet, and its orbit has been considered to be an ellipse of large eccentricity, which is described to a close approximation by the following data. (The ellipse of the data is a little longer than that of the comet). We refer to the ellipse of the data as Y, and we can proceed as if the orbit had been transferred to the ecliptic.

(50) Unit of distance 22046875 miles; semi-axes $a=80$, $b=20$,
Perihelion distance $D=a(1-\sigma)=2\cdot 5403$, $(=56006000$ miles$)$,
Aphelion distance $r_1'=157\cdot 459667$, $(=3471493600$ miles$)$,
Eccentricity $\sigma=[\sqrt{(a^2-b^2)}]/a=\frac{1}{4}\sqrt{15}=\cdot 9682458$,
Equation $r=5/(1-\cdot 9682458\cos\theta)$.

The aphelion is on the positive polar axis and has the coordinates (r_1', θ_1'), viz., $(80+20\sqrt{15}, 0)$. The intersection of Y with the mean orbit of Jupiter; (Mean distance from the Sun 483000000 miles), is the point Q_2: (r_2', θ_2'), viz., $(21\cdot 90786, \cdot 648351)$. We shall prove the following,

LEMMA 1. - *An arc of the ellipse Y, chosen in the vicinity of the intersection of Y with the orbit of Venus, is not a stable arc.*

Five points on Y in the vicinity noted are P_i: $(r_i, \theta_i^{\text{rad.}})$, as follows; (The mean distance from the Sun to Venus being taken as 67000000 miles),

$$
(51) \qquad \left\{
\begin{array}{lll}
P_0: & r_0=2\cdot 9023382, & \theta_0=2\cdot 4135092=138^0\,17'\,2'', \\
P_1: & r_1=2\cdot 9690874, & \theta_1=2\cdot 3552685=134^0\,56'\,49\cdot 1'', \\
P_2: & r_2=3\cdot 0389794, & \theta_2=2\cdot 3002372=131^0\,47'\,37\cdot 9'', \\
P_3: & r_3=3\cdot 1122409, & \theta_3=2\cdot 2477901=128^0\,47'\,19\cdot 9'', \\
P_4: & r_4=3\cdot 1891221, & \theta_4=2\cdot 1974667=125^0\,54'\,20\cdot 2''.
\end{array}
\right.
$$

The equation,

$$\theta = \nu r^4 + \xi r^3 + O r^2 + \pi r + \varrho,$$

determined by these points is,

$$(52) \qquad \theta = \cdot 44686 r^4 - 5\cdot 93300 r^3 + 29\cdot 83118 r^2 - 67\cdot 84843 r + 61\cdot 39025.$$

It is the equation of the arc P_0P_4. In order to determine whether it is a case of the type-form (47) we calculate the results,

$$y O^3/\pi^2 = 1\cdot 01667, \qquad x O^2/\pi = -7\cdot 78302.$$

Comparing these numbers with the respective coefficients of r^4 and r^3 in (52) we conclude that the discordance is too great and that it cannot be corrected. No ellipse of large eccentricity will satisfy this test for stability.

If a comet travelling on an ellipse Y is considerably perturbed in the region corresponding to its intersection with the orbit of Venus, it will not reestablish Y as its orbit. The question therefore arises, what closed curve, possible as the orbit of a comet, possesses the property of stability throughout its length? [14].

The central force which engages a comet.

We obtain the answer to this question from the standpoint of formula (44) which includes the central force for the case of any stable central motion of a single mass.

Whether a comet is an inert remnant of the primordial nebula (SIMON NEWCOMB) or a composition within which some chemical action, involving an effect of illumination, is taking place (W. W. CAMPBELL), or a cloud which has integrated from the substance of a « planetary nebula » surrounding, distantly, the Sun, it is a reasonable conjecture that its motion is mechanically equivalent to a mere flight of particles which influence each other gravitationally only in a measure insufficient to destroy the stability of central motion of any constituent particle.

Let the origin O (Fig. 1) be the center of the force (Sun). Project all of the aforesaid particles of the comet by dropping a perpendicular from each to the plane of the orbit of the nucleus. The feet of these perpendiculars will trace central orbits, each having the property of stability, and a chosen group of these orbits will be properly contained in a region ϱ_0. The equation of a chosen orbit C_1 in ϱ_0 is given by the formula (13), that is,

$$\theta = \varphi(\zeta_{i_1 i_2};\ \mu;\ r). \tag{53}$$

Here r is continuous over a range $\varrho > l > 0$. and μ is a number of a point-set on a segment $(\mu^{(1)}, \ldots, \mu^{(q)})$.

In the perturbational field τ immediately surrounding C_1, μ may be regarded as the mass m of a particle of which C_1 is the orbit Cf. (40). We choose the lower bounding radius, of ϱ_0, as the polar axis, and assume

POSTULATE 1. - *The angular width θ_0 of ϱ_0 is of such smallness that $s\ \theta_0 = 0$ within (Y), s being taken from the transformation T_0 which returns a perturbed orbit of τ to its primary C_1, (Cf. R_1, p,* 301),

$$T_0:\quad \theta' = \theta,\quad r' = r + sp(r),\quad (s \doteq 0,\quad p(r) = ar^{n-1} + br^{n-2} + \ldots + k).$$

We note that $G(r)$ of (44) was derived by means of T_0, and secondly that, with ϱ conveniently extensive, the effect of angular narrowing of ϱ_0 is to make the orbital arcs in ϱ_0 like eccentric ellipses focal at O rather than approximate circles centered at O.

The formal inverse of (53) is now, $m = \psi(r, \theta)$. Expressing ψ on ϱ by polynomial approximation, $(\theta \doteq 0,\ \theta^2 = 0$ in $\varrho_0)$,

$$\begin{aligned} m &= (A + \theta A_1) r^{n-1} + (B + \theta B_1) r^{n-2} + \ldots + (K + \theta K_1) \\ &= h(r) + \theta h_1(r), \qquad\qquad (A, \ldots, K_1 \text{ constant}). \end{aligned} \tag{54}$$

In the central force-function for a stable orbit C_1, viz.,

$$G(r) = \gamma^2 \lambda^2 [2p(r)^2 - rp(r)p'(r) + r^2/\lambda^2]/r^5 = \gamma^2 \lambda^2 \Lambda_n(r), \qquad (p' = \partial p / \partial r),$$

we may now use $h(r) + \theta h_1(r)$ for $p(r)$, that is, $h(r)$ for $p(r)$, since $s\theta_0 = 0$, and $\theta < \theta_0$ in ϱ_0. We then find,

$$G(r) = \gamma^2 \lambda^2 [m\ \{ 2m - 4\theta h_1(r) + \theta r h_1'(r) \}]/r^5 + \gamma^2/r^3. \tag{55}$$

POSTULATE 2. - *The mass m is small, such that the bracket expression in* (55) *is zero, in* (Y)

In point of fact the mass of a whole comet may be small enough to satisfy this postulate, but we can assume this only for the mass m of one of its particles. Moreover, when advantage results from the broader interpretation, we can replace r by r_1+w in T_0, and in (54), giving a new p, a new set $A,\ldots, K_1$, and therefore a new h and h_1. The first G is reproduced with r_1 in place of r and (55) becomes $G(r_1)$. Stability of (46) is preserved by this transformation if the positive w is properly limited above. Therefore,

THEOREM. - *The central force, as it acts upon the diffuse mass called a comet, is equal in* (Y), *to the formula of the inverse cube,* $J(r)=\gamma^2/(r-w)^3$.

The transcendental equation of the orbit.

When V/r_1^3 is substituted for the arbitrary $F(1/u)$ in (38), i. e., in

$$D(r_1, \theta, F)=d^2u/d\theta^2+u-F(1/u)/\gamma^2u^2=0, \quad (u=1/r_1),$$

the equation can be integrated and the result is,

$$r=1/(xe^{\alpha\theta}+\eta e^{-\alpha\theta})+w, \qquad [=1/\Gamma(\theta, \alpha, x, \eta)+w, \quad e=2\cdot 71828\ldots]. \tag{56}$$

Arbitrary constants are x, η, α, the latter being a function of V/γ^2. This orbit is a double spiral (R_2, p. 244). In the vicinity of the origin it consists of two oppositely directed whirls, each approaching asymptotically the small circle $r=w$, this circle being a species of ROCHE limit. In the outer part of the plane the curve is a single oval. Its aphelion point may be, in fact, at infinity but we give attention only to cases where the aphelion Ω_1 is at a finite distance from O. We shall assume $O\Omega_1$ as the positively directed polar axis. By definition the outer node δ of the curve (in the negative direction from the Sun) is the perinodal point and the next consecutive node v, on the positive side of the Sun is the anodal point.

LEMMA 2. - *The polar axis can always be chosen so as to be a unique line of symmetry of the curve* (56).

If we rotate the polar axis through an angle $-\beta$, the equation (56) is transformed into,

$$r=1/[(xe^{\alpha\beta})e^{\alpha\theta}+(\eta e^{-\alpha\beta})e^{-\alpha\theta}]+w.$$

Write,

$$\zeta=xe^{\alpha\beta}=\eta e^{-\alpha\beta},$$

then,

$$\beta=[\log_e(\eta/x)]/2\alpha,$$

a unique result, assuming that η, x, a are different from zero. The equation now becomes,

$$r = 1/\zeta(e^{a\theta} + e^{-a\theta}) + w, \tag{57}$$

and it is evident that the positive coordinate axis is a line of symmetry; (If (r', θ') is on the curve, $(r', -\theta')$ is also), q. e. d.

The conclusion is forced that (57), for properly chosen a, ζ, w must be an approximation of an ellipse which is focal at the Sun. A special case is where we would make (57) coincide as near as possible with the orbit of HALLEY'S, comet that is, with Y of (50). We would substitute the numerical coordinates of three points for (r, θ) in (57) and solve for a, ζ, w. These points are the coordinates (r_1', θ_1') of the aphelion Ω_1, the coordinates (r_2', θ_2') of the point where Jupiter's orbit intersects Y, and $(r_3', \theta_3') = (24{\cdot}99251, \ {\cdot}613444)$. We would then plot (57) and Y together making them coincide as nearly as their respective equations permit.

This work can be simplified some by means of a preliminary theorem about a. Let (r_2, θ_2), $(r_2, -\theta_2)$, be any two points symmetrically situated on an orbit (57), and $(r_1, 0)$ the aphelion. Then,

$$2\zeta = 1/(r_1 - w), \qquad \zeta(e^{a\theta_2} + e^{-a\theta_2}) = 1/(r_2 - w),$$

whence,

$$e^{2a\theta_2} - 2\left(\frac{r_1 - w}{r_2 - w}\right) e^{a\theta_2} + 1 = 0,$$

and we obtain a general formula for a,

$$a = \frac{1}{\theta_2} \log_e \{ [r_1 - w + \sqrt{((r_1 - w)^2 - (r_2 - w)^2)}]/(r_2 - w) \}. \tag{58}$$

The function a remains invariant when the point (r_2, θ_2) is moved along the curve.

In the determination, described above, of a special equation (57), where,

$$r_1' = 157{\cdot}459667, \qquad r_2' = 21{\cdot}90786, \qquad \theta_2' = {\cdot}648351, \quad \text{etc.},$$

we find,

$$a = 4{\cdot}267975, \qquad \zeta = {\cdot}00322685, \qquad w = 2{\cdot}51.$$

The equation of the orbit is, therefore,

$$r = 1/\{ {\cdot}00322685(e^{4\cdot 267975\theta} + e^{-4\cdot 267975\theta}) \} + 2{\cdot}51. \tag{59}$$

Figure 4 shows the curve (59) and the ellipse Y. Any arc of (59) satisfies the conditions for stability given by (47).
The equation (57) can be made to represent, closely within the solar system, an ellipse, a parabola, or a hyperbola, by proper choices of the set (α, ζ, w) [14]. More than two hundred of the observed orbits of comets could not be distin-

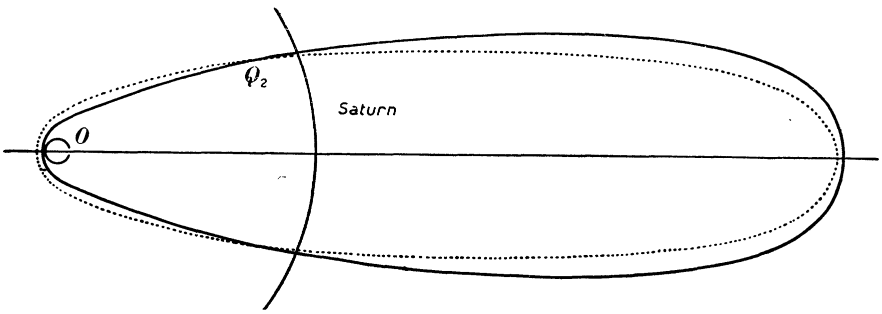

Fig. 4.

guished from parabolas. Related to the properties of the transcendental curves is the fact that the great comet of 1811, which was perturbed from its normal course by Jupiter's attraction, at the beginning of its recessive motion, approximated δv as its path. A comet must have such momentum that it will span the perinodal region as a projectile. The jump required will be of almost zero duration if the radius w is but little less than the comet's perihelial distance. The node δ is an « almost minimal » point.

The period of Halley's comet. Invariant periods.

We give a simple determination of the period of HALLEY'S comet. An equation from the known theory of central motion, (center of force at O), which involves the time variable t, is

$$dt = r^2 d\theta/\gamma.$$

The time required for a planetary body N to complete a revolution around O on a simple closed circuit $r = F(\theta)$ may therefore be written,

$$T = \frac{1}{\gamma}\int_0^{2\pi} [F(\theta)]^2 d\theta.$$

If $r = F(\theta)$ is the equation (57), the indefinite integral of r^2 is,

$$W(\theta) = \int r^2 d\theta = -\frac{1}{2\alpha\zeta^2}\left(\frac{1}{e^{2\alpha\theta}+1}\right) + \frac{2w}{\alpha\zeta}\tan^{-1} e^{\alpha\theta} + w^2\theta + H, \quad (H \text{ constant}).$$

Particularizing (57) to (59), $W(\theta)$ becomes a numerical function of θ. Accepted Astronomical data give this comet (HALLEY'S) a speed of « about one mile per second » at aphelion. We shall use this speed as ·6914 miles per second, or

$$l = 4181587 \cdot 2$$

miles in 70 days on a circular arc Ω, (center of circle at O), which is bisected by the aphelion point. Hence Ω is an arc of radius,

$$r_1' = 3471493600$$

miles. The angle subtended at O by Ω is,

$$E = \cdot 00120455 \quad \text{radians}, \qquad (= l/r_1').$$

With $a = 4 \cdot 267975$, $\zeta = \cdot 003227$, $w = 2 \cdot 51$, time-unit$=$one day, we have

$$70\gamma = 2\,W(\theta)]_0^{E/2} = 29 \cdot 87884, \qquad \gamma = \cdot 42684.$$

The period of this comet is, therefore,

$$T = \frac{2}{\cdot 42684}\, W(\theta)]_0^{\pi} = 76 \cdot 08 \quad \text{years}.$$

The next return to perihelion of this comet will be in 1986.

* * *

LAPLACE (1749-1827) emphasized the importance of the invariability of the period of the Earth's revolution around the Sun, and referred to this invariance as « one of the most remarkable phenomena in the system of the universe » [15]. Thus he was among the first to recognize an invariant; was second only to LAGRANGE who had noticed, in 1773, the invariance of the discriminant of a binary quadratic quantic under a linear transformation. No complete analytic proof of LAPLACE'S invariance has ever been given. We here prove a theorem which includes, as a special case, the invariance of the Earth's period.

We have stated that (46) is the equation of only a segment of the orbit which corresponds to $G(r)$ as the force, but, if the segment is of length above some definite minimum there will be enough discrete points upon it to determine the orbit completely. This being assumed, $G(r)$ may be regarded as the force at any point on the stable orbit. Its general equation can therefore be determined. We have only to integrate.

$$D(r, \theta, G(r)) = 0.$$

With,

$$G(r)=\gamma^2\lambda^2[2p(r)^2-rp(r)p'(r)+r^2/\lambda^2]/r^5$$
$$=\gamma^2\lambda^2[-\{(n-3)\,a^2r^{2n-2}+A_1r^{2n-3}+\ldots+A_{2n-7}r^5\}+Lr^3+Mr^2+Ur+V]/r^5,$$

and,

$$f(r)=2\lambda^2\Big[\frac{a^2}{2}\,r^{2n-2}+\frac{A_1}{2n-7}\,r^{2n-3}+\frac{A_2}{2n-8}\,r^{2n-4}+\ldots+\frac{A_{2n-7}}{1}\,r^5+Lr^3+\frac{1}{2}(M-\lambda^{-2})r^2+$$
$$+\frac{1}{3}\,Ur+\frac{1}{4}\,V+C_1r^4\Big],$$

the integral is,

$$\theta+C_2=\int dr/\sqrt{f(r)},\quad (C_1,\ C_2 \text{ arbitrary constants}). \tag{60}$$

We can now identify,

$$\theta+\lambda_1/\lambda=\int dr/\lambda p(r), \tag{61}$$

that is (46), with any arc of the orbit, assuming the latter to be a closed curve of one circuit, for, since (60), (61) are the same functional forms on the arc, we have,

$$\sqrt{f(r)}=\lambda p(r).$$

Note also that the hyperelliptic integrand in the case of any orbit, becomes rational when the orbit is stable.

We shall refer to a maximum (of r) on the segment (61) as an aphelion and to the next consecutive extreme, a minimum, if existent on the segment, as a perihelion.

LEMMA 3. - *The aphelion and perihelion distances are roots of* $p(r)=0$.

In fact, from (61), $dr/d\theta=\lambda p(r)$, and, since we have assumed the existence of the extremes, the Lemma follows from the elementary theory of such extremes.

THEOREM. - *The time of rotation from aphelion,* $(r=\sigma)$, *to perihelion,* $(r=\nu)$, *on a stable orbit as represented by* (61), *is an* ε*-invariant under the arbitrary perturbations taken as equivalent to the transformation,*

$$T_0:\quad \theta'=\theta,\qquad r'=r+sp(r),\qquad (s(=\varepsilon)\doteq 0).$$

Our proof is a follows. The equation used above to introduce the time-variable t, can be written,

$$dt=r^2dr/\gamma(dr/d\theta)=r^2dr/\gamma\lambda p(r).$$

Therefore the time of rotation from aphelion to perihelion is,

$$\tau=\frac{1}{\gamma\lambda}\int_\nu^\sigma \frac{r^2}{p(r)}\,dr.$$

When we transform τ by T_0, since σ, ν are roots of $p(r)$, the limits of the integral remain unaltered. We consequently obtain,

$$\frac{1}{\gamma\lambda}\int_{\nu}^{\sigma}\frac{r'^2}{p(r')}\,dr' = \frac{1}{\gamma\lambda}\int_{\nu}^{\sigma}\frac{r^2}{p(r)}\,dr + \frac{\varepsilon}{\gamma\lambda}(\sigma^2-\nu^2), \qquad \text{q. e. d.}$$

If the orbit is that of the Earth, ν is so near in value to σ that the ε-term is of the second order of magnitude and negligible. If the orbit is that of HALLEY'S comet there is an aphelion distance but no proper perihelion. The arc upon which the comet makes its circuit about the Sun may be said to provide a weak perihelion, $(r=\nu)$. There will remain, however, an ε-term which can be made small but not zero by decreasing ε. It is known from observations that the period of the comet is not exactly invariant but depends upon the relative positions of the planets when the comet passes through their field.

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