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ON THE PSEUDO-RIGIDITY OF STEIN MANIFOLDS (*)

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Suppose we have a family of domains $\{D_t\}$ in \mathbb{C}^m depending continuously on a parameter $t \in \mathbb{C}$ for $|t| < r$. Given a compact subset $K \subset D_0$, we can find an $\varepsilon > 0$ such that $K \subset D_t$ for every t with $|t| < \varepsilon$.

This fact can be formulated in a more general setting and leads to the notion of pseudo-trivial classes of local deformations of a complex space. The precise definition is given here in § 1.

The present paper is devoted to proving that any family of Stein manifolds whose parameter space is an open set in some numerical space \mathbb{C}^m gives a class of pseudo-trivial local deformations.

For Stein manifolds of dimension 1, *i. e.* for non-compact connected Riemann surfaces, this result was proved, using potential theory, by M. S. Narasimhan [3]. Our proof is a straightforward application of the theory of deformations developed by K. Kodaira and D. C. Spencer [2] modulo some minor changes to adapt it to the case of deformations of non-compact spaces.

The theorem given here is a particular case of an analogous theorem concerning 1-convex spaces (cf. [1]), but the proof of it is technically more involved. For this reason we believe it not useless to have a simple-minded proof for the particular case we have considered.

§ 1. FAMILIES OF COMPLEX SPACES.

1. Definitions. *a)* Let V_0 be a complex space ⁽¹⁾. A *deformation* of V_0 is the set of the following data:

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(1) All complex spaces will be assumed to have a countable basis for open sets.

a punctured complex space (M, m_0)

a complex space \mathcal{V}

two holomorphic maps

$$\omega : \mathcal{V} \rightarrow M, \quad i : V_0 \rightarrow \mathcal{V}$$

satisfying the following conditions :

i) the map i is an isomorphism of V_0 onto $\omega^{-1}(m_0)$

ii) for every $x \in \mathcal{V}$ there exist

a neighbourhood W of x in \mathcal{V}

a neighbourhood U of $\omega(x)$ in M

an analytic set S in an open set of some space \mathbb{C}^N

an isomorphism $\varphi : U \times S \rightarrow W$

such that $\omega \circ \varphi =$ natural projection of $U \times S$ onto U .

By condition ii) the map ω is open. If \mathcal{V} and M are complex manifolds and ω is of maximal rank at every point of \mathcal{V} , then condition ii) is always satisfied.

We will usually identify V_0 with $i(V_0) = \omega^{-1}(m_0)$.

We will say that (\mathcal{V}, ω, M) defines a *differentially trivial* deformation of V_0 if

iii) there exists a C^∞ homeomorphism $f : M \times V_0 \rightarrow \mathcal{V}$ such that $\omega \circ f =$ natural projection of $M \times V_0$ onto M .

b) Two deformations $(\mathcal{V}, \omega, M), (\mathcal{V}', \omega', M)$ of the same space V_0 over the same base (M, m_0) are said to be *equivalent* if there exists an isomorphism $\psi : \mathcal{V} \rightarrow \mathcal{V}'$ such that the following diagram is commutative :

$$\begin{array}{ccc}
 & V_0 & \\
 i \swarrow & & \searrow i' \\
 \mathcal{V} & \xrightarrow{\psi} & \mathcal{V}' \\
 \omega \searrow & & \swarrow \omega' \\
 & M &
 \end{array}$$

Two deformations $(\mathcal{V}, \omega, M), (\mathcal{V}', \omega', M)$ of the same space V_0 over the same base space (M, m_0) are said to be *locally equivalent* if there exists a neighbourhood U of m_0 in M such that the deformations $(\omega^{-1}(U), \omega, U)$

and $(\omega'^{-1}(U), \omega', U)$ are equivalent. This enables us to consider *classes of local deformations* of V_0 over (M, m_0) .

A deformation (\mathcal{V}, ω, M) of V_0 is said to be (*locally*) *trivial* if it is (*locally*) equivalent to the deformation $(M \times V_0, pr_M, M)$.

c) Let (\mathcal{V}, ω, M) be a deformation of V_0 over (M, m_0) . Let A be an open subset of V_0 . Any open subset \mathcal{A} of \mathcal{V} such that $\mathcal{A} \cap V_0 = A$ defines a deformation of the complex space A over (M, m_0) .

We will say that the deformation (\mathcal{V}, ω, M) of V_0 over (M, m_0) defines a *locally pseudo-trivial deformation* of V_0 if for every relatively compact open subset $A \subset \subset V_0$ we can find an open subset $\mathcal{A} \subset \mathcal{V}$ such that $\mathcal{A} \cap V_0 = A$, which defines a trivial deformation of A .

2. Families of complex manifolds. a) Given a deformation (\mathcal{V}, ω, M) of a complex space V_0 and a sheaf of commutative groups \mathcal{F} on \mathcal{V} , one can consider the q -th *direct image* sheaf $\mathcal{R}^q \omega(\mathcal{F})$ on M . This is the sheaf defined by the presheaf on M which associates to every open subset $U \subset M$ the group $H^q(\omega^{-1}(U), \mathcal{F})$, the restriction homomorphism being defined in an obvious way.

If \mathcal{F} is an analytic sheaf on \mathcal{V} , then the sheaves $\mathcal{R}^q \omega(\mathcal{F})$ are analytic sheaves on M .

If $\mathcal{A} \subset \mathcal{V}$ is an open subset of \mathcal{V} , we can consider the sheaf $\mathcal{F}_{\mathcal{A}} = \mathcal{F}|_{\mathcal{A}}$.

By transposition of the injection $\mathcal{A} \subset \mathcal{V}$ one obtains a homomorphism

$$\alpha : \mathcal{R}^q \omega(\mathcal{F}) \rightarrow \mathcal{R}^q \omega|_{\mathcal{A}}(\mathcal{F}_{\mathcal{A}})$$

which is a homomorphism of analytic sheaves if \mathcal{F} is an analytic sheaf on \mathcal{V} .

b) Let us now assume that \mathcal{V} and M are complex manifolds and ω a holomorphic map of maximal rank at each point of \mathcal{V} .

Since we are interested only in the local deformations of V_0 , we may assume that M is a polycylinder M_{r_0} in \mathbb{C}^m with center $m_0 = \{0\}$ and radius r_0 :

$$M_{r_0} = \{t = (t^1, \dots, t^m) \in \mathbb{C}^m \mid |t^\alpha| < r_0, \alpha = 1, \dots, m\}.$$

By definition of a deformation (condition ii)) we may find a locally finite coordinate covering of \mathcal{V} , $\mathcal{U} = \{U_i\}_{i \in I}$ with the following properties: the coordinates $(z_i^1, \dots, z_i^{m+n})$ in the coordinate patch U_i are so chosen that

α) the restriction $\omega|_{U_i}$ of ω to U_i is given by

$$\omega|_{U_i} : (z_i^1, \dots, z_i^{m+n}) \rightarrow (t^1 = z_i^{n+1}, \dots, t^m = z_i^{m+n})$$

β) for any $x \in U_i, (z_i^1, \dots, z_i^n)$ are local coordinates at x on the manifold $\omega^{-1}(\omega(x))$.

We will denote the coordinates on the coordinate patch U_i by $(z_i^1, \dots, z_i^n, t^1, \dots, t^m) = (z_i, t)$. If

$$\begin{cases} z_i^\alpha = h_{ij}^\alpha(z_j, t); t = t \\ 1 \leq \alpha \leq n \end{cases}$$

are the coordinate transformations in $U_i \cap U_j$ and if

$$\vartheta = \sum_1^m \varrho^\mu(t) \frac{\partial}{\partial t^\mu}$$

is a holomorphic vector field on M_{r_0} , then

$$\theta_{ij}^\alpha(z_i, t) = \vartheta h_{ij}^\alpha(z_j, t) = \sum_1^m \varrho^\mu(t) \frac{\partial h_{ij}^\alpha(z_j, t)}{\partial t^\mu}$$

are the components of a holomorphic vector field along the fibres in $U_i \cap U_j$.

Let Θ be the sheaf of germs of holomorphic vector fields on \mathcal{V} along the fibres. One verifies that $\{\theta_{ij}\}$ is a cocycle on the covering \mathcal{U} with values in Θ , i.e.,

$$\varrho(\mathcal{V}) = \{\theta_{ij}\} \in Z^1(\mathcal{U}, \Theta).$$

A new choice of coordinates on the covering \mathcal{U} changes the above cocycle by a coboundary. Hence if T is the sheaf of germs of holomorphic tangent vectors to M , we obtain a map:

$$\tilde{\varrho}_{r_0} : H^0(M_{r_0}, T) \rightarrow H^1(\mathcal{V}, \Theta)$$

which is linear over $H^0(M_{r_0}, \mathcal{O})$, \mathcal{O} being the sheaf of germs of holomorphic functions on M_{r_0} .

If $0 < r \leq r_0$ and $M_r = \{t \in M_{r_0} \mid |t^\alpha| < r\}$, $\mathcal{V}_r = \omega^{-1}(M_r)$, the same argument can be repeated with M_r and \mathcal{V}_r in the place of M_{r_0} and \mathcal{V} respectively. For $0 < r' < r \leq r_0$ we have an obvious commutative diagram:

$$\begin{array}{ccc} H^0(M_r, T) & \xrightarrow{\tilde{\varrho}_r} & H^1(\mathcal{V}_r, \Theta) \\ \downarrow & & \downarrow \\ H^0(M_{r'}, T) & \xrightarrow{\tilde{\varrho}_{r'}} & H^1(\mathcal{V}_{r'}, \Theta) \end{array}$$

By passing to the limit with $r \rightarrow 0$ we obtain a map :

$$\tilde{\varrho} : T_{\{0\}} \rightarrow \mathcal{R}^1 \omega(\Theta)_{\{0\}}$$

which is linear over $\bar{O}_{\{0\}}$. This is the homomorphism of Kodaira and Spencer [2].

c) We want now to prove the following

PROPOSITION 1. *Let (\mathcal{V}, ω, M) be a deformation of the complex manifold V_0 . If $\tilde{\varrho} = 0$, then (\mathcal{V}, ω, M) defines a locally pseudo-trivial deformation of V_0 .*

PROOF. $\alpha)$ Every element $\varrho \in T_{\{0\}}$ is of type

$$\varrho = \sum_1^m \varrho^\mu \frac{\partial}{\partial t^\mu}$$

with $\varrho^\mu \in \bar{O}_{\{0\}}$. By the assumption $\tilde{\varrho} = 0$ there exists r , $0 < r \leq r_0$ and on each $U_i \cap \mathcal{V}_r$ m holomorphic vector fields along the fibres :

$$\theta_{\mu i}(z_i, t) = (\theta_{\mu i}^1(z_i, t), \dots, \theta_{\mu i}^n(z_i, t)) \quad 1 \leq \mu \leq m$$

such that, for $\theta_{\mu ij}(z_i, t) = \frac{\partial h_{ij}(z_i, t)}{\partial t^\mu}$, one has

$$\theta_{\mu ij}(p) = \theta_{\mu j}(p) - \theta_{\mu i}(p) \quad 1 \leq \mu \leq m$$

for any $p \in U_i \cap U_j \cap \mathcal{V}_r$ ⁽¹⁾.

This is expressed by the formulas :

$$(1) \quad \frac{\partial h_{ij}^\alpha(z_j, t)}{\partial t^\mu} = \sum_\beta \theta_{\mu j}^\beta(z_j, t) \frac{\partial h_{ij}^\alpha(z_j, t)}{\partial z_j^\beta} - \theta_{\mu i}^\alpha(h_{ij}(z_j, t), t).$$

$\beta)$ Let (ξ_i, t) be a new system of coordinates on $U_i \cap \mathcal{V}_r$ and let

$$\begin{cases} \xi_i^\alpha = k_{ij}^\alpha(\xi_j, t); & t = t \\ 1 \leq \alpha \leq n \end{cases}$$

(1) Note that if a 1-cocycle on a covering \mathcal{U} of a space X with values in a sheaf of commutative groups induces a coboundary on a refinement of the covering \mathcal{U} , then it is also a coboundary on \mathcal{U} (\mathcal{U} locally finite).

be the corresponding coordinate transformations. Let

$$z_i^\alpha = g_i^\alpha(\xi_i, t)$$

be the expression of the old coordinates in terms of the new in $U_i \cap \mathcal{V}_r$. If \mathcal{V} defines a locally trivial deformation of V_0 , then the new coordinates ξ_i can be so chosen that

i) for $t = 0$ then

$$g_i^\alpha(\xi_i, 0) = \xi_i^\alpha$$

ii) $\frac{\partial k_{ij}^\alpha}{\partial t^\mu} \equiv 0$ for $1 \leq \alpha \leq n$ and $1 \leq \mu \leq m$

provided r is sufficiently small.

γ) From the identity in $U_i \cap U_j \cap \mathcal{V}_r$

$$g_i^\alpha(k_{ij}(\xi_j, t), t) = h_{ij}^\alpha(g_j(\xi_j, t), t)$$

we obtain by differentiation with respect to t^μ :

$$\begin{aligned} 0 &= \frac{\partial}{\partial t^\mu} \{g_i^\alpha(k_{ij}(\xi_j, t), t) - h_{ij}^\alpha(g_j(\xi_j, t), t)\} = \\ &= \sum_{\beta} \frac{\partial g_i^\alpha}{\partial \xi_i^\beta} \frac{\partial k_{ij}^\beta}{\partial t^\mu} + \frac{\partial g_i^\alpha}{\partial t^\mu} - \sum_{\beta} \frac{\partial g_j^\beta}{\partial t^\mu} \frac{\partial h_{ij}^\alpha}{\partial z_j^\beta} - \frac{\partial h_{ij}^\alpha}{\partial t^\mu}. \end{aligned}$$

Hence if condition ii) of β) is satisfied, we obtain a relation of type (1) with $\theta_{\mu i}^\alpha$ replaced by $-\frac{\partial g_i^\alpha}{\partial t^\mu}$.

This shows that $\frac{\partial g_i^\alpha}{\partial t^\mu} + \theta_{\mu i}^\alpha$ will be a global holomorphic vector field θ_μ^α along the fibres of \mathcal{V}_r , for every μ .

δ) We introduce the following notations:

$$M_r(s) = \{(t^1, \dots, t^s) \in \mathbb{C}^s \mid |t^\alpha| < r, 1 \leq \alpha \leq s\}$$

$$I_\varepsilon(h) = \{t^h \in \mathbb{C} \mid |t^h| < \varepsilon\}.$$

Let $\mathcal{V}_r(s) = \omega^{-1}(M_r(s))$.

Let $\mathcal{U}_0 = \{U_i\}_{i \in I_0}$ be the set of those U_i such that $U_i \cap V_0 \neq \emptyset$.

Let $\mathcal{U}'_0 = \{U'_i\}_{i \in I_0}$, $\mathcal{U}^*_0 = \{U^*_i\}_{i \in I_0}$ be two other coverings of V_0 in \mathcal{V} with open sets such that:

$$U'_i \subset \subset U^*_i \subset \subset U_i \text{ for every } i \in I_0.$$

For every $i \in I_0$ we can find an $\varepsilon_i > 0$ and a solution of the system of ordinary differential equations

$$\begin{cases} \frac{\partial g_i^\alpha(\xi_i, t)}{\partial t^m} + \theta_{mi}^\alpha(g_i(\xi, t), t) = 0 \\ 1 \leq \alpha \leq n \end{cases}$$

defined for $t \in M_{r_1}(m-1) \times I_{\varepsilon_i}(m)$, where $r_1 = \frac{1}{2} r_0$, with initial values

$$\begin{cases} g_i^\alpha(\xi_i, t^1, \dots, t^{m-1}, 0) = \xi_i^\alpha \\ 1 \leq \alpha \leq n \end{cases}$$

where $\xi_i^\alpha \in U_i^* \cap V_0$ and contained in U_i .

We may also assume that the n functions g_i^α thus obtained define holomorphic coordinates in $U_i \cap \omega^{-1}(M_{r_1}(m-1) \times I_{\varepsilon_i}(m)) = U_i''$.

By virtue of γ) these new coordinate patches will satisfy the condition

$$\sum \frac{\partial g_i^\alpha}{\partial \xi_i^\beta} \frac{\partial k_{ij}^\beta}{\partial t^m} = 0 \text{ in } U_i'' \cap U_j''.$$

Therefore the coordinate transformations k_{ij} will be independent of t^m .

It follows that in the open set $\bigcup_{i \in I_0} U_i''$ there is a neighbourhood \mathcal{A} of V_0 in \mathcal{V} which can be isomorphically imbedded in the product $\mathcal{V}_{r_1}(m-1) \times \mathbb{C}$, the isomorphism being the identity on $\mathcal{V}_{r_1}(m-1)$.

ε) Now replace the family \mathcal{V} with \mathcal{A} . Then the deformation-cocycle $\varrho \left(\frac{\partial}{\partial t^{m-1}} \right)$ with respect to the new coordinates considered on \mathcal{A} will again be a coboundary. The same will be true for the restriction of this cocycle to $\mathcal{V}_{r_1}(m-1)$. By the above argument we can find a neighbourhood of V_0 in $\mathcal{V}_{r_1}(m-1)$ which can be isomorphically imbedded in the product $\mathcal{V}_{r_2}(m-2) \times \mathbb{C}$, where $r_2 = \frac{1}{2} r_1$, the isomorphism being the identity on $\mathcal{V}_{r_2}(m-2)$.

Continuing in this way we see that a neighbourhood of V_0 in \mathcal{V} can be isomorphically imbedded in the product $V_0 \times \mathbb{C}^m$, the isomorphism being the identity on V_0 . This proves our statement.

REMARK 1. Actually we have proved a little more, i. e., that in the hypothesis specified above, if $\tilde{\varrho} = 0$, there exists a neighbourhood of V_0 in \mathcal{V} which can be isomorphically imbedded into the product $V_0 \times \mathbb{C}^m$, the isomorphism being the identity on V_0 .

REMARK 2. An analogous argument applies to differentiable families of complex or differentiable manifolds. In this last case the sheaf \mathcal{O} is a fine sheaf. Hence given a complex deformation (\mathcal{V}, ω, M) of the complex manifold V_0 , a neighbourhood of V_0 in \mathcal{V} can always be differentiably imbedded in the product $V_0 \times \mathbb{C}^m$ ($m = \dim_{\mathbb{C}} M$) (with a fibre-preserving imbedding which is the identity on V_0).

§ 2. DEFORMATION OF STEIN MANIFOLDS.

3. a) Let us now assume that (\mathcal{V}, ω, M) is a local deformation of a holomorphically complete manifold V_0 over the polycylinder

$$M = M_{r_0} = \{t = (t^1, \dots, t^m) \in \mathbb{C}^m \mid |t^\alpha| < r_0, 1 \leq \alpha \leq m\}.$$

We can now prove the following

PROPOSITION 2. *Let A be a relatively compact open subset of V_0 . There exists a neighbourhood \mathcal{A} of A in \mathcal{V} with $\mathcal{A} \cap V_0 = A$ such that for any coherent sheaf \mathcal{F} on \mathcal{V} the natural homomorphism*

$$r: \mathcal{R}^q \omega(\mathcal{F})_0 \rightarrow \mathcal{R}^q \omega|_{\mathcal{A}}(\mathcal{F}|_{\mathcal{A}})_0$$

is the 0-homomorphism, when $q \geq 1$.

PROOF. α) Since we are interested only in relatively compact open subsets of V_0 , by the remark 2 at the end of proposition 1 we see that it is not restrictive to assume that \mathcal{V} is differentiably trivial. Let $f: M \times V_0 \rightarrow \mathcal{V}$ be the fibre-preserving differentiable homeomorphism which gives the differentiable triviality of \mathcal{V} .

Since V_0 is a Stein manifold, there exists on V_0 a C^∞ function $g: V_0 \rightarrow \mathbb{R}$ such that

i) the sets $B_c = \{x \in V_0 \mid g(x) < c\}$ are relatively compact in V_0 for every $c \in \mathbb{R}$

ii) the function g is strongly plurisubharmonic on V_0 , i. e., at each point $x \in V_0$ the Levi form expressed in local coordinates z^α by

$$\mathcal{L}(g) = \sum \frac{\partial^2 g}{\partial z^\alpha \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta$$

is a positive definite hermitian form (cf. [4]).

Consider on \mathcal{V} the following function :

$$\tilde{g}(\xi) = g \circ pr_{V_0} \circ f^{-1}(\xi).$$

This is a C^∞ function and if, as is permitted, we assume that $f|_{V_0}$ is the identity map, the function $\tilde{g}|_{V_0}$ coincides with the function g .

Given a compact set $K \subset V_0$ we can find a constant $a_0(K) > 0$ such that for any $a > a_0(K)$ the function

$$h_a = \tilde{g} + a\omega^* \left(\sum_1^m t^\mu \bar{t}^\mu \right)$$

has a positive definite Levi form at each point of K .

Therefore there is a neighbourhood $U(K)$ of K in \mathcal{V} such that on any point of $U(K)$ the Levi form of h_a , for any $a > a_0(K)$, is positive definite.

Let $\sup_{x \in A} g(x) = C$ and set $K = \bar{B}_{C+1}$, so that $A \subset K$, and take for \mathcal{A} the set $f(M \times A)$.

We can find $\varepsilon(K) > 0$ ($\varepsilon(K) < r_0$) such that

$$f(M_{\varepsilon(K)} \times K) \subset U(K).$$

We claim that the sets

$$\mathcal{B}_\nu = \{x \in \mathcal{V} \mid h_\nu(x) < C + 1\} \quad \nu = 1, 2, \dots,$$

form a decreasing system of neighbourhoods of B_{C+1} in \mathcal{V} .

In fact, for any ν , $\mathcal{B}_\nu \cap V_0 = B_{C+1}$. Moreover if $c = \inf_{x \in V_0} g(x)$, one has

$$\mathcal{B}_\nu \subset f\left(\frac{M_{|C|+|c|}}{\nu} \times B_{C+1}\right).$$

If $\frac{|C|+|c|}{\nu} < \varepsilon(K)$, the sets \mathcal{B}_ν are relatively compact in \mathcal{V} , the function h_ν is strongly plurisubharmonic on \mathcal{B}_ν and the sets $\{h_\nu(x) < \delta\}$ are relatively compact in \mathcal{B}_ν if $\delta < C + 1$. It follows that for these values of ν the sets \mathcal{B}_ν are 1-complete manifolds, i. e., holomorphically complete.

β) Now let $\theta \in \mathcal{R}^q \omega(\mathcal{F})_0$; the class θ is defined by an element

$$\theta \in H^q(\omega^{-1}(M_\sigma), \mathcal{F})$$

where $\sigma > 0$ is sufficiently small.

Let ν be a positive integer, greater than $\frac{|C|+|c|}{\varepsilon(K)}$, such that

$$\mathcal{B}_\nu \subset \omega^{-1}(M_\sigma).$$

We can find a positive number $\varepsilon < \sigma$ such that

$$\omega^{-1}(M_\varepsilon) \cap \mathcal{A} \subset \mathcal{B}_\nu.$$

The element

$$r(\theta) \in \mathcal{R}^q|_{\mathcal{A}} (\mathcal{F}|_{\mathcal{A}})_0$$

is defined by the image of θ under the natural homomorphism

$$H^q(\omega^{-1}(M_\sigma), \mathcal{F}) \rightarrow H^q(\omega|_{\mathcal{A}}^{-1}(M_\varepsilon), \mathcal{F}|_{\mathcal{A}}).$$

On the other hand the triangle of restriction homomorphisms

$$\begin{array}{ccc} H^q(\omega^{-1}(M_\sigma), \mathcal{F}) & \rightarrow & H^q(\omega|_{\mathcal{A}}^{-1}(M_\varepsilon), \mathcal{F}|_{\mathcal{A}}) \\ & \searrow & \nearrow \\ & & H^q(\mathcal{B}_\nu, \mathcal{F}|_{\mathcal{B}_\nu}) \end{array}$$

is commutative.

Since \mathcal{B}_ν is holomorphically complete, $H^q(\mathcal{B}_\nu, \mathcal{F}|_{\mathcal{B}_\nu}) = 0$, for $q \geq 1$.

This shows that $r(\theta) = 0$.

b) We can now prove the following

THEOREM. *Every local deformation (\mathcal{V}, ω, M) of a holomorphically complete manifold V_0 over an open neighbourhood M of the origin in \mathbb{C}^m is a pseudo-trivial deformation.*

PROOF. By virtue of proposition 1 it is enough to show that for any relatively compact open subset $A \subset\subset V_0$ we can find a neighbourhood \mathcal{A} of A in \mathcal{V} , with $\mathcal{A} \cap V_0 = A$, such that the homomorphism $\tilde{\varrho}_{\mathcal{A}}$ of Kodaira and Spencer for the family $(\mathcal{A}, \omega|_{\mathcal{A}}, \omega(\mathcal{A}))$ is the zero homomorphism.

If r is the restriction homomorphism

$$\mathcal{R}^1 \omega(\theta)_0 \xrightarrow{r} \mathcal{R}^1 \omega|_{\mathcal{A}}(\theta|_{\mathcal{A}})_0,$$

then we have the factorisation $\tilde{\varrho}_{\mathcal{A}} = r \circ \tilde{\varrho}$.

Choosing \mathcal{A} as in proposition 2 we see that $r = 0$; hence $\tilde{\varrho}_{\mathcal{A}} = 0$ as we wanted.

c) *Application.* Given a compact complex manifold V let us denote by $d(V)$ the minimal number of Stein manifolds by which V can be co-

vered. If (\mathcal{V}, ω, M) is a family of deformations of compact complex manifolds, $\mathcal{V} = \{V_t\}_{t \in M}$, then $d(V_t)$ is an upper semicontinuous function of t for $t \in M$.

This fact can also be proved directly, using part of the argument given in *a*).

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