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ON THE PSEUDO-RIGIDITY OF STEIN MANIFOLDS(*)

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Suppose we have a family of domains $\{D_t\}$ in $\mathbf{C}^n$ depending continuously on a parameter $t \in \mathbf{C}$ for $|t| < r$. Given a compact subset $K \subset D_0$, we can find an $\varepsilon > 0$ such that $K \subset D_t$ for every t with $|t| < \varepsilon$.

This fact can be formulated in a more general setting and leads to the notion of pseudo-trivial classes of local deformations of a complex space. The precise definition is given here in § 1.

The present paper is devoted to proving that any family of Stein manifolds whose parameter space is an open set in some numerical space $\mathbf{C}^m$ gives a class of pseudo-trivial local deformations.

For Stein manifolds of dimension 1, *i. e.* for non-compact connected Riemann surfaces, this result was proved, using potential theory, by M. S. Narasimhan [3]. Our proof is a straightforward application of the theory of deformations developed by K. Kodaira and D. C. Spencer [2] modulo some minor changes to adapt it to the case of deformations of non-compact spaces.

The theorem given here is a particular case of an analogous theorem concerning 1-convex spaces (cf. [1]), but the proof of it is technically more involved. For this reason we believe it not useless to have a simple-minded proof for the particular case we have considered.

§ 1. FAMILIES OF COMPLEX SPACES.

1. Definitions. *a*) Let V_0 be a complex space (1). A *deformation* of V_0 is the set of the following data:

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(1) All complex spaces will be assumed to have a countable basis for open sets.

a punctured complex space (M, m_0)

a complex space $\mathcal{V}$

two holomorphic maps

$\omega : \mathcal{V} \to M, \qquad i : V_0 \to \mathcal{V}$

satisfying the following conditions:

i) the map i is an isomorphism of V_0 onto $\omega^{-1}(m_0)$
ii) for every $x \in \mathcal{V}$ there exist

a neighbourhood W of x in $\mathcal{V}$

a neighbourhood U of $\omega(x)$ in M

an analytic set S in an open set of some space $\mathbf{C}^N$

an isomorphism $\varphi : U \times S \to W$

such that $\omega \circ \varphi =$ natural projection of $U \times S$ onto U.
By condition ii) the map ω is open. If $\mathcal{V}$ and M are complex manifolds and ω is of maximal rank at every point of $\mathcal{V}$, then condition ii) is always satisfied.

We will usually identify V_0 with $i(V_0) = \omega^{-1}(m_0)$.

We will say that $(\mathcal{V}, \omega, M)$ defines a *differentiably trivial* deformation of V_0 if

iii) there exists a C^∞ homeomorphism $f : M \times V_0 \to \mathcal{V}$ such that $\omega \circ f =$ natural projection of $M \times V_0$ onto M.

b) Two deformations $(\mathcal{V}, \omega, M)$, $(\mathcal{V}', \omega', M)$ of the same space V_0 over the same base (M, m_0) are said to be *equivalent* if there exists an isomorphism $\psi : \mathcal{V} \to \mathcal{V}'$ such that the following diagram is commutative:

$$\begin{tikzcd} & V_0 \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ \mathcal{V} \arrow[rr, "\psi"] \arrow[dr, "\omega"'] & & \mathcal{V}' \arrow[dl, "\omega'"] \\ & M & \end{tikzcd}$$

Two deformations $(\mathcal{V}, \omega, M)$, $(\mathcal{V}', \omega', M)$ of the same space V_0 over the same base space (M, m_0) are said to be *locally equivalent* if there exists a neighbourhood U of m_0 in M such that the deformations $(\omega^{-1}(U), \omega, U)$

and $(\omega'^{-1}(U), \omega', U)$ are equivalent. This enables us to consider *classes of local deformations* of V_0 over (M, m_0).

A deformation $(\mathcal{V}, \omega, M)$ of V_0 is said to be (*locally*) *trivial* if it is (locally) equivalent to the deformation $(M \times V_0, pr_M, M)$.

c) Let $(\mathcal{V}, \omega, M)$ be a deformation of V_0 over (M, m_0). Let A be an open subset of V_0. Any open subset $\mathcal{A}$ of $\mathcal{V}$ such that $\mathcal{A} \cap V_0 = A$ defines a deformation of the complex space A over (M, m_0).

We will say that the deformation $(\mathcal{V}, \omega, M)$ of V_0 over (M, m_0) defines a *locally pseudo-trivial deformation* of V_0 if for every relatively compact open subset $A \subset\subset V_0$ we can find an open subset $\mathcal{A} \subset \mathcal{V}$ such that $\mathcal{A} \cap V_0 = A$, which defines a trivial deformation of A.

2. Families of complex manifolds. *a*) Given a deformation $(\mathcal{V}, \omega, M)$ of a complex space V_0 and a sheaf of commutative groups $\mathcal{F}$ on $\mathcal{V}$, one can consider the *q-th direct image* sheaf $\mathcal{R}^q\omega(\mathcal{F})$ on M. This is the sheaf defined by the presheaf on M which associates to every open subset $U \subset M$ the group $H^q(\omega^{-1}(U), \mathcal{F})$, the restriction homomorphism being defined in an obvious way.

If $\mathcal{F}$ is an analytic sheaf on $\mathcal{V}$, then the sheaves $\mathcal{R}^q\omega(\mathcal{F})$ are analytic sheaves on M.

If $\mathcal{A} \subset \mathcal{V}$ is an open subset of $\mathcal{V}$, we can consider the sheaf $\mathcal{F}_{\mathcal{A}} = \mathcal{F}|_{\mathcal{A}}$. By transposition of the injection $\mathcal{A} \subset \mathcal{V}$ one obtains a homomorphism

$$\alpha : \mathcal{R}^q\omega(\mathcal{F}) \to \mathcal{R}^q\omega|_{\mathcal{A}}(\mathcal{F}_{\mathcal{A}})$$

which is a homomorphism of analytic sheaves if $\mathcal{F}$ is an analytic sheaf on $\mathcal{V}$.

b) Let us now assume that $\mathcal{V}$ and M are complex manifolds and ω a holomorphic map of maximal rank at each point of $\mathcal{V}$.

Since we are interested only in the local deformations of V_0, we may assume that M is a polycylinder M_{r_0} in $\mathbf{C}^m$ with center $m_0 = \{0\}$ and radius r_0:

$$M_{r_0} = \{t = (t^1, \dots, t^m) \in \mathbf{C}^m \mid |t^\alpha| < r_0, \alpha = 1, \dots, m\}.$$

By definition of a deformation (condition ii)) we may find a locally finite coordinate covering of $\mathcal{V}$, $\mathcal{U} = \{U_i\}_{i \in I}$ with the following properties: the coordinates $(z_i^1, \dots, z_i^{m+n})$ in the coordinate patch U_i are so chosen that

α) the restriction $\omega|_{U_i}$ of ω to U_i is given by

$$\omega|_{U_i} : (z_i^1, \dots, z_i^{m+n}) \to (t^1 = z_i^{n+1}, \dots, t^m = z_i^{m+n})$$

β) for any $x \in U_i$, $(z_i^1, \dots, z_i^n)$ are local coordinates at x on the manifold $\omega^{-1}(\omega(x))$.

We will denote the coordinates on the coordinate patch U_i by $(z_i^1, \dots, z_i^n, t^1, \dots, t^m) = (z_i, t)$. If

$$\begin{cases} z_i^\alpha = h_{ij}^\alpha(z_j, t)\,;\, t = t \\ 1 \leq \alpha \leq n \end{cases}$$

are the coordinate transformations in $U_i \cap U_j$ and if

$$\mathcal{V} = \sum_1^m \mathcal{V}^\mu(t) \frac{\partial}{\partial t^\mu}$$

is a holomorphic vector field on M_{r_0}, then

$$\theta_{ij}^\alpha(z_i, t) = \mathcal{V}\, h_{ij}^\alpha(z_j, t) = \sum_1^m \mathcal{V}^\mu(t) \frac{\partial h_{ij}^\alpha(z_j, t)}{\partial t^\mu}$$

are the components of a holomorphic vector field along the fibres in $U_i \cap U_j$.

Let Θ be the sheaf of germs of holomorphic vector fields on $\mathcal{V}$ along the fibres. One verifies that $\{\theta_{ij}\}$ is a cocycle on the covering $\mathcal{U}$ with values in Θ, i.e.,

$$\varrho(\mathcal{V}) = \{\theta_{ij}\} \in Z^1(\mathcal{U}, \Theta).$$

A new choice of coordinates on the covering $\mathcal{U}$ changes the above cocycle by a coboundary. Hence if T is the sheaf of germs of holomorphic tangent vectors to M, we obtain a map:

$$\tilde{\varrho}_{r_0} : H^0(M_{r_0}, T) \longrightarrow H^1(\mathcal{V}, \Theta)$$

which is linear over $H^0(M_{r_0}, \mathcal{O})$, $\mathcal{O}$ being the sheaf of germs of holomorphic functions on M_{r_0}.

If $0 < r \leq r_0$ and $M_r = \{t \in M_{r_0} \mid |t^\alpha| < r\}$, $\mathcal{V}_r = \omega^{-1}(M_r)$, the same argument can be repeated with M_r and $\mathcal{V}_r$ in the place of M_{r_0} and $\mathcal{V}$ respectively. For $0 < r' < r \leq r_0$ we have an obvious commutative diagram:

$$\begin{array}{ccc} H^0(M_r, T) & \xrightarrow{\tilde{\varrho}_r} & H^1(\mathcal{V}_r, \Theta) \\ \downarrow & & \downarrow \\ H^0(M_{r'}, T) & \xrightarrow{\tilde{\varrho}_{r'}} & H^1(\mathcal{V}_{r'}, \Theta) \end{array}$$

By passing to the limit with $r \to 0$ we obtain a map:

$$\tilde{\varrho} : T_{\{0\}} \to \mathcal{R}^1 \omega (\Theta)_{\{0\}}$$

which is linear over $\mathcal{O}_{\{0\}}$. This is the homomorphism of Kodaira and Spencer [2].

c) We want now to prove the following

Proposition 1. *Let $(\mathcal{V}, \omega, M)$ be a deformation of the complex manifold V_0. If $\tilde{\varrho} = 0$, then $(\mathcal{V}, \omega, M)$ defines a locally pseudo-trivial deformation of V_0.*

Proof. α) Every element $\mathfrak{v} \in T_{\{0\}}$ is of type

$$\mathfrak{v} = \sum_1^m \mathfrak{v}^\mu \frac{\partial}{\partial t^\mu}$$

with $\mathfrak{v}^\mu \in \mathcal{O}_{\{0\}}$. By the assumption $\tilde{\varrho} = 0$ there exists r, $0 < r \leq r_0$ and on each $U_i \cap \mathcal{V}_r$ m holomorphic vector fields along the fibres:

$$\theta_{\mu i}(z_i, t) = (\theta^1_{\mu i}(z_i, t), \dots, \theta^n_{\mu i}(z_i, t)) \qquad 1 \leq \mu \leq m$$

such that, for $\theta_{\mu ij}(z_i, t) = \dfrac{\partial h_{ij}(z_i, t)}{\partial t^\mu}$, one has

$$\theta_{\mu ij}(p) = \theta_{\mu j}(p) - \theta_{\mu i}(p) \qquad 1 \leq \mu \leq m$$

for any $p \in U_i \cap U_j \cap \mathcal{V}_r$ (1).

This is expressed by the formulas:

$$\frac{\partial h^\alpha_{ij}(z_j, t)}{\partial t^\mu} = \sum_\beta \theta^\beta_{\mu j}(z_j, t) \frac{\partial h^\alpha_{ij}(z_j, t)}{\partial z^\beta_j} - \theta^\alpha_{\mu i}(h_{ij}(z_j, t), t). \tag{1}$$

β) Let (ξ_i, t) be a new system of coordinates on $U_i \cap \mathcal{V}_r$ and let

$$\begin{cases} \xi^\alpha_i = k^\alpha_{ij}(\xi_j, t); \quad t = t \\ 1 \leq \alpha \leq n \end{cases}$$

(1) Note that if a 1-cocycle on a covering $\mathcal{U}$ of a space X with values in a sheaf of commutative groups induces a coboundary on a refinement of the covering $\mathcal{U}$, then it is also a coboundary on $\mathcal{U}$ ($\mathcal{U}$ locally finite).

be the corresponding coordinate transformations. Let

$$z_i^\alpha = g_i^\alpha(\xi_i, t)$$

be the expression of the old coordinates in terms of the new in $U_i \cap \mathcal{V}_r$. If $\mathcal{V}$ defines a locally trivial deformation of V_0, then the new coordinates ξ_i can be so chosen that

i) for $t = 0$ then

$$g_i^\alpha(\xi_i, 0) = \xi_i^\alpha$$

ii) $\dfrac{\partial k_{ij}^\alpha}{\partial t^\mu} \equiv 0$ for $1 \leq \alpha \leq n$ and $1 \leq \mu \leq m$

provided r is sufficiently small.

γ) From the identity in $U_i \cap U_j \cap \mathcal{V}_r$

$$g_i^\alpha(k_{ij}(\xi_j, t), t) = h_{ij}^\alpha(g_j(\xi_j, t), t)$$

we obtain by differentiation with respect to t^μ:

$$0 = \frac{\partial}{\partial t^\mu}\{g_i^\alpha(k_{ij}(\xi_j, t), t) - h_{ij}^\alpha(g_j(\xi_j, t), t)\} =$$

$$= \sum_\beta \frac{\partial g_i^\alpha}{\partial \xi_i^\beta}\frac{\partial k_{ij}^\beta}{\partial t^\mu} + \frac{\partial g_i^\alpha}{\partial t^\mu} - \sum_\beta \frac{\partial g_j^\beta}{\partial t^\mu}\frac{\partial h_{ij}^\alpha}{\partial z_j^\beta} - \frac{\partial h_{ij}^\alpha}{\partial t^\mu}\,.$$

Hence if condition ii) of β) is satisfied, we obtain a relation of type (1) with $\theta_{\mu i}^\alpha$ replaced by $-\dfrac{\partial g_i^\alpha}{\partial t^\mu}$.

This shows that $\dfrac{\partial g_i^\alpha}{\partial t^\mu} + \theta_{\mu i}^\alpha$ will be a global holomorphic vector field θ_μ^α along the fibres of $\mathcal{V}_r$, for every μ.

δ) We introduce the following notations:

$$M_r(s) = \{(t^1, \ldots, t^s) \in \mathbf{C}^s \mid \; | t^\alpha | < r, 1 \leq \alpha \leq s\}$$
$$I_\varepsilon(h) = \{t^h \in \mathbf{C} \mid \; | t^h | < \varepsilon\}.$$

Let $\mathcal{V}_r(s) = \omega^{-1}(M_r(s))$.

Let $\mathcal{U}_0 = \{U_i\}_{i \in I_0}$ be the set of those U_i such that $U_i \cap V_0 \neq \emptyset$.

Let $\mathcal{U}_0' = \{U_i'\}_{i \in I_0}$, $\mathcal{U}_0^* = \{U_i^*\}_{i \in I_0}$ be two other coverings of V_0 in $\mathcal{V}$ with open sets such that:

$$U_i' \subset\subset U_i^* \subset\subset U_i \quad \text{for every } i \in I_0\,.$$

For every $i \in I_0$ we can find an $\varepsilon_i > 0$ and a solution of the system of ordinary differential equations

$$\begin{cases} \dfrac{\partial g_i^\alpha (\xi_i, t)}{\partial t^m} + \theta_{mi}^\alpha (g_i (\xi, t), t) = 0 \\ 1 \leq \alpha \leq n \end{cases}$$

defined for $t \in M_{r_1}(m-1) \times I_{\varepsilon_i}(m)$, where $r_1 = \dfrac{1}{2} r_0$, with initial values

$$\begin{cases} g_i^\alpha (\xi_i, t^1, \dots, t^{m-1}, 0) = \xi_i^\alpha \\ 1 \leq \alpha \leq n \end{cases}$$

where $\xi_i^\alpha \in U_i^* \cap V_0$ and contained in U_i.

We may also assume that the n functions g_i^α thus obtained define holomorphic coordinates in $U_i' \cap \omega^{-1}(M_{r_1}(m-1) \times I_{\varepsilon i}(m)) = U_i''$.

By virtue of $\gamma)$ these new coordinate patches will satisfy the condition

$$\Sigma \frac{\partial g_i^\alpha}{\partial \xi_i^\beta} \frac{\partial k_{ij}^\beta}{\partial t^m} = 0 \text{ in } U_i'' \cap U_j''.$$

Therefore the coordinate transformations k_{ij} will be independent of t^m.

It follows that in the open set $\bigcup_{i \in I_0} U_i''$ there is a neighbourhood $\mathcal{A}$ of V_0 in $\mathcal{V}$ which can be isomorphically imbedded in the product $\mathcal{V}_{r_1}(m-1) \times \mathbf{C}$, the isomorphism being the identity on $\mathcal{V}_{r_1}(m-1)$.

ε) Now replace the family $\mathcal{V}$ with $\mathcal{A}$. Then the deformation-cocycle $\varrho\left(\dfrac{\partial}{\partial t^{m-1}}\right)$ with respect to the new coordinates considered on $\mathcal{A}$ will again be a coboundary. The same will be true for the restriction of this cocycle to $\mathcal{V}_{r_1}(m-1)$. By the above argument we can find a neighbourhood of V_0 in $\mathcal{V}_{r_1}(m-1)$ which can be isomorphically imbedded in the product $\mathcal{V}_{r_2}(m-2) \times \mathbf{C}$, where $r_2 = {}^1/_2 r_1$, the isomorphism being the identity on $\mathcal{V}_{r_2}(m-2)$.

Continuing in this way we see that a neighbourhood of V_0 in $\mathcal{V}$ can be isomorphically imbedded in the product $V_0 \times \mathbf{C}^m$, the isomorphism being the identity on V_0. This proves our statement.

Remark 1. Actually we have proved a little more, i. e., that in the hypothesis specified above, if $\tilde{\varrho} = 0$, there exists a neighbourhood of V_0 in $\mathcal{V}$ which can be isomorphically imbedded into the product $V_0 \times \mathbf{C}^m$, the isomorphism being the identity on V_0.

REMARK 2. An analogous argument applies to differentiable families of complex or differentiable manifolds. In this last case the sheaf Θ is a fine sheaf. Hence given a complex deformation $(\mathcal{V}, \omega, M)$ of the complex manifold V_0, a neighbourhood of V_0 in $\mathcal{V}$ can always be differentiably imbedded in the product $V_0 \times \mathbf{C}^m$ ($m = dim_{\mathbf{C}} M$) (with a fibre-preserving imbedding which is the identity on V_0).

§ 2. DEFORMATION OF STEIN MANIFOLDS.

3. *a*) Let us now assume that $(\mathcal{V}, \dot{\omega}, M)$ is a local deformation of a holomorphically complete manifold V_0 over the polycylinder

$$M = M_{r_0} = \{t = (t^1, \dots, t^m) \in \mathbf{C}^m \mid \; | t^\alpha | < r_0, \; 1 \leq \alpha \leq m\}.$$

We can now prove the following

PROPOSITION 2. *Let A be a relatively compact open subset of V_0. There exists a neighbourhood $\mathcal{A}$ of A in $\mathcal{V}$ with $\mathcal{A} \cap V_0 = A$ such that for any coherent sheaf $\mathcal{F}$ on $\mathcal{V}$ the natural homomorphism*

$$r : \mathcal{R}^q \omega (\mathcal{F})_0 \longrightarrow \mathcal{R}^q \omega \,|_{\mathcal{A}} (\mathcal{F}|_{\mathcal{A}})_0$$

is the 0-*homomorphism, when* $q \geq 1$.

PROOF. α) Since we are interested only in relatively compact open subsets of V_0, by the remark 2 at the end of proposition 1 we see that it is not restrictive to assume that $\mathcal{V}$ is differentiably trivial. Let $f: M \times V_0 \longrightarrow \mathcal{V}$ be the fibre-preserving differentiable homeomorphism which gives the differentiable triviality of $\mathcal{V}$.

Since V_0 is a Stein manifold, there exists on V_0 a C^∞ function $g : V_0 \longrightarrow \mathbf{R}$ such that

i) the sets $B_c = \{x \in V_0 \mid g(x) < c\}$ are relatively compact in V_0 for every $c \in R$

ii) the function g is strongly plurisubharmonic on V_0, i. e., at each point $x \in V_0$ the Levi form expressed in local coordinates z^α by

$$\mathcal{L}(g) = \Sigma \frac{\partial^2 g}{\partial z^\alpha \partial \bar{z}^\beta} u^\alpha \bar{u}^\beta$$

is a positive definite hermitian form (cf. [4]).

Consider on $\mathcal{V}$ the following function:

$$\tilde{g}(\xi) = g \circ pr_{V_0} \circ f^{-1}(\xi).$$

This is a C^∞ function and if, as is permitted, we assume that $f|_{V_0}$ is the identity map, the function $\tilde{g}|_{V_0}$ coincides with the function g.

Given a compact set $K \subset V_0$ we can find a constant $a_0(K) > 0$ such that for any $a > a_0(K)$ the function

$$h_a = \tilde{g} + a\,\omega^*\left(\sum_1^m t^\mu \bar{t}^\mu\right)$$

has a positive definite Levi form at each point of K.

Therefore there is a neighbourhood $U(K)$ of K in $\mathcal{V}$ such that on any point of $U(K)$ the Levi form of h_a, for any $a > a_0(K)$, is positive definite.

Let $\sup\limits_{x \in A} g(x) = C$ and set $K = \bar{B}_{C+1}$, so that $A \subset K$, and take for $\mathscr{A}$ the set $f(M \times A)$.

We can find $\varepsilon(K) > 0$ ($\varepsilon(K) < r_0$) such that

$$f(M_{\varepsilon(K)} \times K) \subset U(K).$$

We claim that the sets

$$\mathcal{B}_\nu = \{x \in \mathcal{V} \mid h_\nu(x) < C + 1\} \qquad \nu = 1, 2, \dots,$$

form a decreasing system of neighbourhoods of B_{C+1} in $\mathcal{V}$.

In fact, for any ν, $\mathcal{B}_\nu \cap V_0 = B_{C+1}$. Moreover if $c = \inf\limits_{x \in V_0} g(x)$, one has

$$\mathcal{B}_\nu \subset f(M_{\frac{|C|+|c|}{\nu}} \times B_{C+1}).$$

If $\dfrac{|C|+|c|}{\nu} < \varepsilon(K)$, the sets $\mathcal{B}_\nu$ are relatively compact in $\mathcal{V}$, the function h_ν is strongly plurisubharmonic on $\mathcal{B}_\nu$ and the sets $\{h_\nu(x) < \delta\}$ are relatively compact in $\mathcal{B}_\nu$ if $\delta < C + 1$. It follows that for these values of ν the sets $\mathcal{B}_\nu$ are 1-complete manifolds, i. e., holomorphically complete.

β) Now let $\theta \in \mathcal{R}^q\,\omega\,(\mathcal{F})_0$; the class θ is defined by an element

$$\theta \in H^q(\omega^{-1}(M_\sigma), \mathcal{F})$$

where $\sigma > 0$ is sufficiently small.

Let ν be a positive integer, greater than $\dfrac{|C|+|c|}{\varepsilon(K)}$, such that

$$\mathcal{B}_\nu \subset \omega^{-1}(M_\sigma).$$

We can find a positive number $\varepsilon < \sigma$ such that

$$\omega^{-1}(M_\varepsilon) \cap \mathcal{A} \subset \mathcal{B}_\nu .$$

The element

$$r(\theta) \in \mathcal{R}^q|_{\mathcal{A}}(\mathcal{F}|_{\mathcal{A}})_0$$

is defined by the image of θ under the natural homomorphism

$$H^q(\omega^{-1}(M_\sigma), \mathcal{F}) \longrightarrow H^q(\omega|_{\mathcal{A}}^{-1}(M_\varepsilon), \mathcal{F}|_{\mathcal{A}}).$$

On the other hand the triangle of restriction homomorphisms

$$\begin{array}{ccc} H^q(\omega^{-1}(M_\sigma), \mathcal{F}) & \longrightarrow & H^q(\omega|_{\mathcal{A}}^{-1}(M_\varepsilon), \mathcal{F}|_{\mathcal{A}}) \\ \searrow & & \nearrow \\ & H^q(\mathcal{B}_\nu, \mathcal{F}|_{\mathcal{B}_\nu}) & \end{array}$$

is commutative.

Since $\mathcal{B}_\nu$ is holomorphically complete, $H^q(\mathcal{B}_\nu, \mathcal{F}|_{\mathcal{B}_\nu}) = 0$, for $q \geq 1$. This shows that $r(\theta) = 0$.

b) We can now prove the following

THEOREM. *Every local deformation* $(\mathcal{V}, \omega, M)$ *of a holomorphically complete manifold* V_0 *over an open neighbourhood* M *of the origin in* $\mathbf{C}^m$ *is a pseudo-trivial deformation.*

PROOF. By virtue of proposition 1 it is enough to show that for any relatively compact open subset $A \subset\subset V_0$ we can find a neighbourhood $\mathcal{A}$ of A in $\mathcal{V}$, with $\mathcal{A} \cap V_0 = A$, such that the homomorphism $\tilde{\varrho}_{\mathcal{A}}$ of Kodaira and Spencer for the family $(\mathcal{A}, \omega|_{\mathcal{A}}, \omega(\mathcal{A}))$ is the zero homomorphism.

If r is the restriction homomorphism

$$\mathcal{R}^1 \omega(\theta)_0 \xrightarrow{r} \mathcal{R}^1 \omega|_{\mathcal{A}}(\theta|_{\mathcal{A}})_0 ,$$

then we have the factorisation $\tilde{\varrho}_{\mathcal{A}} = r \circ \tilde{\varrho}$.

Choosing $\mathcal{A}$ as in proposition 2 we see that $r = 0$; hence $\tilde{\varrho}_{\mathcal{A}} = 0$ as we wanted.

c) *Application.* Given a compact complex manifold V let us denote by $d(V)$ the minimal number of Stein manifolds by which V can be covered. If $(\mathcal{V}, \omega, M)$ is a family of deformations of compact complex manifolds, $\mathcal{V} = \{V_t\}_{t \in M}$, then $d(V_t)$ is an upper semicontinuous function of t for $t \in M$.

This fact can also be proved directly, using part of the argument given in a).

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