ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

RAGHAVAN NARASIMHAN

A note on Stein spaces and their normalisations

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 16, nº 4 (1962), p. 327-333

<http://www.numdam.org/item?id=ASNSP_1962_3_16_4_327_0>

A NOTE ON STEIN SPACES AND THEIR NORMALISATIONS

RAGHAVAN NARASIMHAN (Bombay) (*)

§ 1. Introduction.

It is well known that every open Riemann surface is a Stein manifold. But no proof has so far appeared of the corresponding statement for complex spaces of dimension one (with arbitrary non-normal singularities) viz. that *every* (reduced) *complex space of dimension one, which has no compact irreducible components, is a Steiu space.* The object of the present note is to give a proof of the following theorem on complex spaces, of which the statement made above is a particular case in view of the fact that every normal complex space of dimension one is nonsingular (i. e. a disjoint union of Riemann surfaces).

THEOREM 1. *A* (*reduced*) *complex space X is a Stein space if and only if its normalisation X^* is a Stein space.*

A corollary to this statement is the following.

A complex space all of whose irreducible components are Stein spaces is itself a Stein space.

Of course, this statement becomes trivial if we replace « irreducible components » by « connected components ».

§ 2. Preliminaries.

Let $(X, \mathcal{H})$ be a complex space in the sense of Grauert [3] and $(X, \mathcal{O})$ the corresponding *reduced* complex space; for $x \in X$, $\mathcal{H}_x$ may contain nilpotent elements, while $\mathcal{O}_x$ does not. If $\mathcal{H}_x$ contains no nilpotent elements, then $\mathcal{H}_x = \mathcal{O}_x$.

(*) Supported in part by AF-EOAR Grant 62-35.

Let $(X, \mathcal{O})$ be a reduced complex space. We call X a Stein *space* if it is holomorph-convex [i. e., for any infinite discrete set $D \subset X$, there is a holomorphic function f for which $f(D)$ is unbounded] and if holomorphic functions separate points of X. The following theorem is well known [1].

THEOREM a. *Let $(X, \mathcal{O})$ be a paracompact reduced complex space. Then X is a Stein space if and only if for every coherent analytic subsheaf $\mathcal{J} \subset \mathcal{O}$, we have*

$$H^1(X, \mathcal{J}) = 0$$

If $(X, \mathcal{O})$ is Stein, then for any coherent analytic sheaf S, we have $H^q(X, S) = 0$, $q \geq 1$.

The following theorem can be deduced from Theorem a; see [3, § 2, Satz 3].

THEOREM b. *Let $(X, \mathcal{H})$ be an arbitrary complex space for which the corresponding reduced space $(X, \mathcal{O})$ is Stein. Let S be any coherent $\mathcal{H}$-sheaf. Then we have*

$$H^q(X, S) = 0 \text{ for } q \geq 1.$$

Let now X, Y be two reduced complex spaces and $\pi : X \longrightarrow Y$ a proper holomorphic map with discrete fibres. Let S be a coherent analytic sheaf on X and let $\pi_\nu(S)$ be the ν^{th} direct image of S under π, i. e. for any open set $U \subset Y$, we have

$$H^0(U, \pi_\nu(S)) = H^\nu(\pi^{-1}(U), S).$$

Then we have [5, Satz 27]

THEOREM c. $\pi_\nu(S) = 0$ *for* $\nu \geq 1$, $\pi_0(S)$ *is a coherent analytic sheaf on* Y.

We require also the following theorem [4, Satz 6]

THEOREM d. *Let X, Y be complex spaces, and $\varphi : X \longrightarrow Y$ a holomorphic map. Let S be an analytic sheaf on X. Suppose that for $\nu \geq 1$, we have $\varphi_\nu(S) = 0$. Then, for $\nu \geq 0$, we have*

$$H^\nu(X, S) = H^\nu(Y, \varphi_0(S)).$$

Let now $(X, \mathcal{O})$ be a reduced complex space. X is called *normal* if for any $x \in X$, the local ring $\mathcal{O}_x$ is integrally closed in its complete ring of quotients.

To every reduced complex space $(X, \mathcal{O})$ corresponds a « normalisation » $(X^*, \mathcal{O}^*) \cdot (X^*, \mathcal{O}^*)$ is a normal complex space, and there is a proper holomorphic map $\pi: X^* \longrightarrow X$ which is onto and has discrete fibres. If $\widetilde{\mathcal{O}} = \pi_0(\mathcal{O}^*)$, then for $x \in X$, $\widetilde{\mathcal{O}}_x$ is the integral closure of $\mathcal{O}_x$ and if $A \subset X$ is the singular locus of X, then $\pi | (X^* - \pi^{-1}(A))$ is an analytic isomorphism onto $X - A$. $\widetilde{\mathcal{O}}$ is a subsheaf of the sheaf of germs of meromorphic functions on X.

§ 3. Proof of Theorem 1.

Let $(X, \mathcal{O})$ be a complex space for which the normalisation $(X^*, \mathcal{O}^*)$ is Stein. Let $\mathcal{J}$ be a coherent sheaf of ideals, i. e. an analytic subsheaf of $\mathcal{O}$ on X. Let $\widetilde{\mathcal{O}} = \pi_0(\mathcal{O}^*)$ where $\pi: X^* \longrightarrow X$ is the canonical map. For $x \in X$, let $\mathcal{W}_x$ be the largest ideal in $\mathcal{O}_x$ such that $\mathcal{W}_x \cdot \widetilde{\mathcal{O}}_x \subset \mathcal{O}_x$ and let $\mathcal{W} = \bigcup_{x \in X} \mathcal{W}_x$.

Then $\mathcal{W}$ is an analytic sheaf on X; moreover, it is a *coherent* analytic sheaf on X; see [6 § 2 Prop. 9 and remark which follows Prop. 9].

Let $\mathcal{F}^*$ be the analytic inverse image on X^* of the coherent analytic sheaf $\mathcal{W}. \mathcal{J}$ (i. e. $\mathcal{F}^*$ is the tensor product of the topological inverse image of $\mathcal{W} \cdot \mathcal{J}$ and $\mathcal{O}^*$ over the topological inverse image of $\mathcal{O}$). Then $\mathcal{F}^*$ is a coherent $\mathcal{O}^*$-sheaf [4, § 2, (g)].

Let $\mathcal{F} = \pi_0(\mathcal{F}^*)$. By Theorem c, $\mathcal{F}$ is a coherent $\mathcal{O}$-sheaf. Morevoer, since $\mathcal{W} \cdot \widetilde{\mathcal{O}} = \mathcal{W} \cdot \pi_0(\mathcal{O}^*) \subset \mathcal{O}$, it follows that $\mathcal{F}$ is a subsheaf of $\mathcal{O}$ and in fact of $\mathcal{J}$. Finally we remark that by Theorem c, $\pi_\nu(\mathcal{F}^*) = 0$ for $\nu \geq 1$, so that, by Theorem d, we have

$$H^q(X^*, \mathcal{F}^*) = H^q(X, \mathcal{F}).$$

By Theorem a, we have $H^q(X^*, \mathcal{F}^*) = 0$ for $q \geq 1$, so that we conclude that $H^q(X, \mathcal{F}) = 0$ for $q \geq 1$.

We shall first prove Theorem 1 for spaces of finite dimension. Let n be the complex dimension of X, and suppose inductively that Theorem 1, has been proved for all spaces of dimension $\leq n-1$. We then assert that any closed nowhere dense analytic set Y of X is a Stein space. This follows from the following lemma, and the inductive hypothesis.

LEMMA 1. *Let $(X, \mathcal{O})$ be a reduced complex space for which the normalisation $(X^*, \mathcal{O}^*)$ is Stein. Then, for any closed analytic set $Y \subset X$, with the induced reduced structure from X, the normalisation Y^* is Stein.*

The proof will be given later.

We go back to the proof of Theorem 1 in the special case.

Let $\mathcal{I}$, $\mathcal{W}$, $\mathcal{F}$ be as above and consider the exact sequence

$$(*) \qquad 0 \to \mathcal{F} \to \mathcal{I} \to \mathcal{I}/\mathcal{F} \to 0$$

Now, since $\pi \mid X^* - \pi^{-1}(A)$ is an analytic isomorphism and, for $x \notin A$, $\tilde{\mathcal{O}}_x = \mathcal{O}_x$, we see that $\mathcal{W}_x = \mathcal{O}_x$ for $x \notin A$ and $\mathcal{F}_x = \mathcal{I}_x$ for $x \notin A$. Hence the set Y of points $x \in X$ with $\mathcal{W}_x \neq \mathcal{O}_x$ (which contains the set of points where $\mathcal{I}_x \neq \mathcal{F}_x$) is a nowhere dense analytic set in X, and so, with its reduced structure, is a Stein space. Moreover, if $\mathcal{S}$ is the restriction of $\mathcal{I}/\mathcal{F}$ to Y, then $\mathcal{S}$ is a coherent $\mathcal{H}$-sheaf, where $\mathcal{H}$ is the restriction of $\mathcal{O}/\mathcal{W}$ to Y [6, § 2, Thèorème 3]. Now, by our remark above (inductive assumption and Lemma 1), Y is a Stein space. Hence, by Theorem d, $H^q(Y, \mathcal{S}) = 0$ for $q \geq 1$. But since $H^q(Y, \mathcal{S}) \approx H^q(X, \mathcal{I}/\mathcal{F})$, we conclude that $H^q(X, \mathcal{I}/\mathcal{F}) = 0$ for $q \geq 1$. Hence, since, $H^q(X, \mathcal{F}) = 0$ for $q \geq 1$, we deduce from the exact cohomology sequence associated to (*), that $H^q(X, \mathcal{I}) = 0$ for $q \geq 1$; because of Theorem a, this concludes modulo Lemma 1 the proof of Theorem 1 in the special case when X has finite dimension.

For the proof of Lemma 1, we require the following result.

LEMMA 2. *Let X, Y be normal complex spaces (reduced) and $\pi : X \to Y$ a proper holomorphic map with discrete fibres onto Y. Then, X is Stein if and only if Y is Stein.*

PROOF. The fact that if Y is Stein, then so is X follows at once from [2, Satz B]. Conversely, suppose X Stein. We may suppose X and Y connected. Then, there is a nowhere dense analytic set $M \subset Y$ such that $\pi \mid X - \pi^{-1}(M)$ is an *unramified* covering of $Y - M$ (say with p sheets); we may suppose also that M contains the singular locus of Y. Then, if f is holomorphic on X, and, for $y \in Y - M$, $a_\nu(y)$ is the ν^{th} elementary symmetric function of the values of f at the points of $\pi^{-1}(y)$, then the $a_\nu(y)$ remain bounded as $y \to y_0 \in M$ and *since Y is normal*, can be extended to holomorphic functions a_ν on Y. Moreover, we have $f^p(x) + \sum_{\nu \geq 1} f^{p-\nu}(x) a_\nu(\pi(x)) = 0$.

It is now obvious that if $|f|$ is unbounded on a set $D \subset X$, then at least one a_ν is unbounded on $\pi(D)$. Since X is holomorphconvex, so is Y. Now Y can contain no compact analytic set T of positive dimension since $\pi^{-1}(T)$ would then be a compact analytic set of positive dimension in X, and this cannot exist since holomorphic functions on X separate points. If we use the fact that a holomorph-convex reduced complex space which contains no compact analytic sets of positive dimension is Stein (an easy consequence of [2, Satz B]), we see that Y is Stein.

PROOF OF LEMMA 1. Let $\pi : X^* \longrightarrow X$ be the natural map, and $Y^1 = \pi^{-1}(Y)$. Since Y^1 is a closed subspace of the Stein space X^*, Y^1 is Stein. Hence, by [2, Satz B], its normalization $\widetilde{Y}$ is Stein. Clearly, we have a proper holomorphic map $\varphi : \widetilde{Y} \longrightarrow Y$ which has discrete fibres. Let Y^* be the normalisation of Y and $\pi^1 : Y^* \longrightarrow Y$ the natural map. Since $\widetilde{Y}$ is normal, there exists a holomorphic map $\varphi^1 : \widetilde{Y} \longrightarrow Y^*$ such that $\pi^1 \circ \varphi^1 = \varphi$. Since, clearly φ^1 must be proper, surjective and have discrete fibres, and since $\widetilde{Y}$ is Stein, we see, by Lemma 2, that Y^* is Stein, which is Lemma 1.

To prove Theorem 1 in the general case, we proceed as follows. Let X_k, $k = 1, 2, \ldots$ be the union of the irreducible components of dimension $\leq k$ of X. The normalisation of X_k is a union of connected components of X and so is Stein. By the special case of Theorem 1 which is already proved, each X_k is Stein.

Let now D be any discrete subset of X and let $D_k = D \cap X_k$, $E_1 = D_1$ and $E_{k+1} = D_{k+1} - D_k$. Let h be a holomorphic function on D (i. e. assignment of a complex number to each point of D) and, for $k \geq 1$, h_k the restriction of h to E_k. Since X_1 is Stein, there is a holomorphic function f_1 on X_1, so that $f_1 \mid E_1 = h_1$. Clearly $E_2 \cup X_1$ is a closed subspace of X_2, so that there is, since X_2 is Stein, a holomorphic function f_2 on X_2 such that $f_2 \mid X_1 = f_1$, $f_2 \mid E_2 = h_2$. Proceeding thus, we construct f_{k+1} holomorphic on X_{k+1} so that $f_{k+1} \mid X_k = f_k$, $f_{k+1} \mid X_{k+1} = h_{k+1}$. If $f = \lim f_k$, then f is holomorphic on X and clearly $f \mid D = h$. Hence X is itself Stein, and this proves Theorem 1 in the general case.

Using Theorem 1 and Lemma 2, it is possible to prove Lemma 2 without the assumption of normality. We formulate this as a separate Theorem.

THEOREM 2. *Let X, Y be reduced complex spaces, $\pi : X \longrightarrow Y$ a proper holomorphic map onto Y. Then, if X is Stein, so is Y.*

PROOF. Since X is Stein, X contains no compact analytic sets of positive dimension. Hence every fibre of π, being a compact analytic set, is a finite set.

Let X^*, Y^* be the normalisations of X, Y respectively and $\pi_X : X^* \longrightarrow X$, $\pi_Y : Y^* \longrightarrow Y$ the corresponding projections. Let $\varphi = \pi \circ \pi_X : X^* \longrightarrow Y$. Then φ is a surjective proper holomorphic map of X^* onto Y with discrete fibres. Since X^* is normal, there is a holomorphic map $\varphi^1 : X^* \longrightarrow Y^*$ which is surjective, so that $\pi_Y \circ \varphi^1 = \varphi$. Since X is Stein, so is X^*; by Lemma 2, so is Y^*. By Theorem 1, we deduce that Y itself is Stein.

Finally we give a sketch of a direct proof for spaces with isolated singularities in particular, for spaces of one dimension. This proof has the

« merit » of not depending on the heavy machinery of direct and inverse images of analytic sheaves.

Let X be a reduced complex space with isolated singularities, A the set of singular points of X and X^* the normalisation of X. We suppose that X^* is Stein. Let $\{X_k^*\}$ be a sequence of relatively compact open sets in X^* with the following properties.

a) X_k^* is Stein, $X_k^* \subset\subset X_{k+1}^*$ and $\partial X_k^* \cap \pi^{-1}(A) = \emptyset$ [here $\pi: X^* \to X$ is the natural map].

b) X_k^* is X^*-convex, i. e. if K is a compact subset of X_k^* then $\widehat{K} = \{x \in X_k^* \mid |f(x)| \leqq \sup |f(K)|$ for all f holomorphic *in* X^* is compact.

Let $X_k = \pi(X_k^*)$. We assert that (*i*) X_k is Stein and that (*ii*) X_k is X_{k+1}-convex. It then follows that X is Stein.

PROOF OF (*i*). Since $X_k^* \subset\subset X^*$, for any f holomorphic in X_k^* which vanishes on $X_k^* \cap \pi^{-1}(A)$, there exists an integer $\lambda > 0$ so that $f^\lambda = g \circ \pi$ for some g holomorphic on X_k. Clearly we may find, for any $x_0 \in \partial X_k$, an f holomorphic on X_k^*, vanishing on $X_k^* \cap \pi^{-1}(A)$, such that $|f(y)| \to \infty$ as $y \to y_0$ if $y_0 \in \pi^{-1}(x_0) \cap \partial X_k^*$.

If λ is such that $f^\lambda = g \circ \pi$, then clearly $|g(x)| \to \infty$ as $x \to x_0$. Hence X_k is holomorph-convex. As in the proof of Lemma 2, X_k has no compact analytic sets of positive dimension and so is Stein.

PROOF OF (*ii*). If K is a compact set of X_k and $x_0 \in \partial X_k$, then, there exists f holomorphic on X_{k+1}^*, *vanishing on* $\pi^{-1}(A) \cap X_{k+1}^*$ so that, if $y_0 = \pi^{-1}(x_0)$, then $|f(y_0)| > \sup_{y \in K^*} f(y)$ where $K^* = \pi^{-1}(K) \cap X_k^*$ (note that for the existence of f, we need the fact that $\partial X_k^* \cap \pi^{-1}(A) = \emptyset$).

Choose $\lambda > 0$ so that $f^\lambda = g \circ \pi$ where g is holomorphic on X_{k+1}^*. Then $|g(x_0)| > \sup_{x \in K} |g(x)|$. Hence X_k is X_{k+1}-convex.

REFERENCES

1. H. CARTAN. *Variétés analytiques complexes et cohomologie,* Colloque sur les fonctions de plusieurs variables complexes de Bruxelles, (1953), 41-55.
2. H. GRAUERT. *Charakterisierung der holomorph-vollständigen komplexen Räume,* Math. Ann. 129 (1955), 233-259.
3. H. GRAUERT. *Ein Theorem der analytischen Garbentheorie und die Modulräume komplexer Strukturen,* Publications mathematiques de l'Inst. des Hautes Etudes Scientifiques, Paris, N⁰ 5, 1960.
4. H. GRAUERT und R. REMMERT. *Bilder und Urbilder analytische Garber,* Annals of Math. 68 (1958), 393-443.
5. H. GRAUERT und R. REMMERT. *Komplexe Räume,* Math. Ann. 136 (1958), 245-318.
6. J.-P. SERRE. *Faisceaux algébriques cohérents.* Annals of Math. 61 (1955), 197-278.

Tata Institute of Fundamental Research,
Bombay, India.