ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 21, nº 1 (1967), p. 83-91
<http://www.numdam.org/item?id=ASNSP_1967_3_21_1_83_0>

THE STABILITY OF THE BOUNDARY IN A STEFAN PROBLEM

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SUMMARY - An a priori bound on the difference between the positions of the free boundaries in two Stefan problems is derived in terms of the initial conditions and the heat influxes.

1. Introduction.

A typical Stefan problem is the determination of a function $x = s(t)$, $0 < t \leq T$, and a function $u(x, t)$, $0 < x < s(t)$, $0 < t \leq T$, such that

$$
(1.1) \qquad
\begin{aligned}
& u_{xx} = u_t , \quad 0 < x < s(t), \quad 0 < t \leq T, \\
& u_x(0, t) = -a(t), \quad 0 < t \leq T, \\
& u(s(t), t) = 0, \quad 0 < t \leq T, \\
& \frac{ds}{dt}(t) = -u_x(s(t), t), \quad 0 < t \leq T,
\end{aligned}
$$

and either

$$
(1.2) \qquad s(0) = 0
$$

or

$$
(1.3) \qquad
\begin{aligned}
& s(0) = b > 0, \\
& u(x, 0) = \varphi(x) \geq 0, \quad 0 \leq x \leq b.
\end{aligned}
$$

Pervenuto alla Redazione il 9 Maggio 1966.

This research was supported in part by the Brookhaven National Laboratory and the Air Force Office of Scientific Research.

The boundary $x = s(t)$ represents the free boundary occurring at a phase change, such as the boundary between water and ice, and must be found at the same time as the temperature distribution u. Existence and uniqueness of the solution have been established for (1.1)-(1.2) and (1.1)-(1.3) by several authors [2-7] under various hypotheses on the data $a(t)$, b, and $\varphi(x)$; in fact, Kyner [7] has shown both existence and uniqueness for a nonlinear generalization of (1.1)-(1.2). It is clear that his argument also extends to the problem (1.1) (1.3).

Let us assume that $a(t)$ is a continuous, positive function for $0 \leq x \leq T$ and that, for (1.3), φ is continuously differentiable for $0 \leq x \leq b$, $\varphi'(0) = -a(0)$, $\varphi(b) = 0$, and $\varphi(x) > 0$ for $0 < x < b$. It follows from the results of Kyner (at least after trivial modification of the argument of Lemma 1 in the case (1.1)-(1.3)) that

$$0 < \frac{ds}{dt}(t), \quad |u_x(x,t)| \leq \max\left(\max_{0 \leq \tau \leq t} a(\tau), \max_{0 \leq \xi \leq b} |\varphi'(\xi)|\right) = B(t) \leq B, \quad 0 < t \leq T, \tag{1.4}$$

and

$$0 < u(x,t) < B(s(t) - x), \quad 0 < x < s(t). \tag{1.5}$$

The object of this paper is to establish an a priori estimate on the dependence of the boundary on the data $a(t)$, b, and $\varphi(x)$. The result will be stated here in terms of two problems of the form (1.1)-(1.3). Let $(s_i(t), u_i(x,t))$, $i = 1, 2$, denote the solution of (1.1)-(1.3) with data $a_i(t)$, b_i, and $\varphi_i(x)$, respectively. Assume that, for some $B > 0$,

$$\max_{i=1,2}\left(\max_{0 \leq t \leq T} a_i(t), \max_{0 \leq x \leq b_i} |\varphi_i'(x)|\right) \leq B < \infty. \tag{1.6}$$

Then, the following theorem will be proved:

THEOREM. If (s_i, u_i) is a solution of (1.1)-(1.3) for data a_i, b_i and φ_i, $i = 1, 2$, satisfying (1.6) and the conditions stated above and if $b_1 < b_2$, then the free boundaries $s_1(t)$ and $s_2(t)$ satisfy the inequality

$$|s_1(t) - s_2(t)| \leq C\left[b_2 - b_1 + \int_0^{b_1} |\varphi_1(x) - \varphi_2(x)|\,dx + \int_{b_1}^{b_2} \varphi_2(x)\,dx + \int_0^t |a_1(\tau) - a_2(\tau)|\,d\tau\right], \quad 0 \leq t \leq T, \tag{1.7}$$

where $C = C(B, T)$ is given by (2.33). In case $b_1 = 0$ or $b_1 = b_2 = 0$ the relevant terms on the right hand sides of (1.6) and (1.7) disappear; the constant $C(B, T)$ is unchanged.

A continuous dependence theorem for small T is easily obtainable from the argument of Friedman [4]; our result is global.

2. Proof of the Theorem.

It is very useful to obtain an integral relation expressing the conservation of heat by integrating the differential equation over the domain $\{0 < x < s(\tau),\ 0 < \tau < t\}$. It follows from (1.1)-(1.3) that

$$s(t) = b + \int_0^t a(\tau)\, d\tau + \int_0^b \varphi(x)\, dx - \int_0^{s(t)} u(x, t)\, dx. \tag{2.1}$$

Let

$$\begin{aligned} \alpha(t) &= \min(s_1(t),\ s_2(t)),\\ \beta(t) &= \max(s_1(t),\ s_2(t)),\\ \delta(t) &= \beta(t) - \alpha(t). \end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that, if $b_1 \leq b_2$,

$$\begin{aligned} \delta(t) \leq b_2 - b_1 &+ \left| \int_0^t \{a_1(\tau) - a_2(\tau)\}\, d\tau \right| + \left| \int_0^{b_1} \{\varphi_1(x) - \varphi_2(x)\}\, dx \right| \\ &+ \int_{b_1}^{b_2} \varphi_2(x)\, dx + \left| \int_0^{\alpha(t)} \{u_1(x, t) - u_2(x, t)\}\, dx \right| + \int_{\alpha(t)}^{\beta(t)} u_j(x, t)\, dx, \end{aligned} \tag{2.3}$$

where j is chosen so that u_j is defined for $\alpha(t) \leq x \leq \beta(t)$. Obviously, j can vary with t. The proof consists primarily in relating the last two terms to $\delta(t)$ and then estimating the solution of an integral inequality.

Note that (1.5) and (1.6) imply that

$$u_j(\alpha(t), t) = |u_1(\alpha(t), t) - u_2(\alpha(t), t)| \leq B\delta(t), \qquad 0 \leq t \leq T. \tag{2.4}$$

First, let us estimate the integral of $u_1 - u_2$ on $[0, \alpha(t)]$. Set

$$v(x, t) = u_1(x, t) - u_2(x, t) = v_1(x, t) + v_2(x, t) + v_3(x, t), \tag{2.5}$$

$$0 < x < \alpha(t), \quad 0 < t \leq T,$$

where each v_k satisfies the heat equation in the domain given and the boundary and initial conditions are chosen as follows:

$$\begin{aligned} &\frac{\partial v_1}{\partial x}(0, t) = 0, \quad 0 < t \leq T, \\ &v_1(\alpha(t), t) = 0, \quad 0 < t \leq T, \\ &v_1(x, 0) = \varphi_1(x) - \varphi_2(x), \quad 0 \leq x \leq \alpha(0) = b_1; \end{aligned} \tag{2.6}$$

$$\begin{aligned} &\frac{\partial v_2}{\partial x}(0, t) = a_2(t) - a_1(t), \quad 0 < t \leq T, \\ &v_2(\alpha(t), t) = 0, \quad 0 < t \leq T, \\ &v_2(x, 0) = 0, \quad 0 \leq x \leq b_1; \end{aligned} \tag{2.7}$$

$$\begin{aligned} &\frac{\partial v_3}{\partial x}(0, t) = 0, \quad 0 < t \leq T, \\ &v_3(\alpha(t), t) = u_1(\alpha(t), t) - u_2(\alpha(t), t), \quad 0 < t \leq T, \\ &v_3(x, 0) = 0, \quad 0 \leq x \leq b_1. \end{aligned} \tag{2.8}$$

The fact that $\alpha(t)$, being the minimum of two continuously differentiable functions with bounded derivatives, is Lipschitz continuous implies the existence of v_1, v_2, and v_3.

The integral of v_1 can be estimated as follows. The maximum principle [6] implies that $|v_1(x, t)|$ is not greater than the solution of the heat equation satisfying the first two conditions of (2.6) and given initially by $|\varphi_1(x) - \varphi_2(x)|$, and this function is in turn maximized by a solution w_1 of the heat equation in the quarter-plane $\{x, t > 0\}$ such that

$$\begin{aligned} &\frac{\partial w_1}{\partial x}(0, t) = 0, \quad 0 < t \leq T, \\ &w_1(x, 0) = \begin{cases} |\varphi_1(x) - \varphi_2(x)|, & 0 \leq x \leq b_1, \\ 0, & b < x < \infty. \end{cases} \end{aligned} \tag{2.9}$$

Thus,

$$\left| \int_0^{\alpha(t)} v_1(x, t)\, dx \right| \leq \int_0^{\infty} w_1(x, t)\, dx = \int_0^{b_1} |\varphi_1(x) - \varphi_2(x)|\, dx, \tag{2.10}$$

the equality expressing conservation of heat for w_1.

The function $|v_2|$ is maximized by the solution w_2 of the heat equation again in the quarter-plane $\{x, t > 0\}$ such that

$$\frac{\partial w_2}{\partial x}(0, t) = -|a_1(t) - a_2(t)|, \quad 0 < t \leq T, \tag{2.11}$$

$$w_2(x, 0) = 0, \quad 0 < x < \infty.$$

Thus,

$$\left| \int_0^{\alpha(t)} v_2(x, t)\, dx \right| \leq \int_0^{\infty} w_2(x, t)\, dx = \int_0^t |a_1(\tau) - a_2(\tau)|\, d\tau. \tag{2.12}$$

The estimation of the integral of v_3 requires something more than just the maximum principle, although it is again useful to apply it to obtain a simplification. Note that, as a consequence of (2.4), $|v_3|$ is dominated by w_3, where $w_3(\alpha(t), t) = B\delta(t)$ and the other two relations in (2.8) are retained. Since $\frac{\partial w_3}{\partial x}(0, t) = 0$, the domain can be reflected about the line $x = 0$ and w_3 defined for $-\alpha(t) < x < 0$ by $w_3(-x, t) = w_3(x, t)$ to obtain the solution of the heat equation for which $w_3(\pm\alpha(t), t) = B\delta(t)$ and $w_3(x, 0) = 0$ for $-\alpha(t) < x < \alpha(t)$. Then $w_3(x, t) \leq z(x, t) + z(-x, t)$, where

$$z_{xx} = z_t, \quad -\alpha(t) < x < \infty, \quad 0 < t \leq T,$$

$$z(-\alpha(t), t) = B\delta(t), \quad 0 < t \leq T, \tag{2.13}$$

$$z(x, 0) = 0, \quad -b_1 < x < \infty.$$

The function z has the well known representation [4, 6]

$$z(x, t) = \int_0^t \sigma(\tau) K_x(x, t, -\alpha(\tau), \tau)\, d\tau, \quad x > -\alpha(t), \tag{2.14}$$

where

$$K(x, t, \xi, \tau) = \frac{1}{2\pi^{1/2}(t - \tau)^{1/2}} e^{-(x-\xi^2)/4(t-\tau)} \tag{2.15}$$

and

$$K_x(x, t, \xi, \tau) = \frac{\partial K}{\partial x} = -\frac{x - \xi}{2(t - \tau)} K(x, t, \xi, \tau). \tag{2.16}$$

It follows from the standard jump relations for the fundamental solution K that

$$(2.17) \qquad z(-\alpha(t), t) = B\delta(t) = -\frac{1}{2}\sigma(t) + \int_0^t \sigma(\tau) K_x(-\alpha(t), t, -\alpha(\tau), \tau)\, d\tau.$$

It follows from (1.4) that

$$(2.18) \qquad 0 \leq \alpha(t) - \alpha(\tau) \leq B(t - \tau).$$

Hence,

$$(2.19) \qquad |\sigma(t)| \leq 2B\delta(t) + \frac{B}{2\pi}\int_0^t \frac{|\sigma(\tau)|\, d\tau}{(t-\tau)^{1/2}}.$$

Let us appeal to the following lemma [1, Lemma 2, page 380], the proof of which is obtained by applying the technique used to solve Abel integral equations.

LEMMA. If $0 \leq \varphi(t) \leq A + C\int_0^t (t-\tau)^{-1/2}\varphi(\tau)\, d\tau$, then

$$\varphi(t) \leq A\,[1 + 2Ct^{1/2}] \exp\{\pi C^2 t\}.$$

Let

$$(2.20) \qquad \|\sigma\|_t = \max_{0 \leq \tau \leq t} |\sigma(\tau)|.$$

Then the lemma implies that

$$(2.21) \quad \|\sigma\|_t \leq 2B[1 + \pi^{-1} BT^{1/2}]\, e^{BT/4\pi} \|\delta\|_t = 2\pi^{1/2} C_1(B, T) \|\delta\|_t, \quad 0 \leq t \leq T.$$

Now,

$$\int_0^{\alpha(t)} |v_3(x,t)|\, dx \leq \frac{1}{2}\int_{-\alpha(t)}^{\alpha(t)} w_3(x,t)\, dx \leq \int_{-\alpha(t)}^{\infty} z(x,t)\, dx$$

$$(2.22) \quad = \int_{-\alpha(t)}^{\infty} dx \int_0^t \sigma(\tau) K_x(x, t, -\alpha(\tau), \tau)\, d\tau = \int_0^t \sigma(\tau)\, d\tau \int_{-\alpha(t)}^{\infty} K_x(x, t, -\alpha(\tau), \tau)\, dx$$

$$\leq \frac{1}{2\pi^{1/2}}\int_0^t (t-\tau)^{-1/2} |\sigma(\tau)|\, d\tau \leq C_1(B, T)\int_0^t (t-\tau)^{-1/2} \|\delta\|_\tau\, d\tau.$$

Collecting,

$$\int_0^{\alpha(t)} |u_1(x,t) - u_2(x,t)|\,dx \leq \int_0^{b_1} |\varphi_1(x) - \varphi_2(x)|\,dx + \int_0^t |a_1(\tau) - a_2(\tau)|\,d\tau \tag{2.23}$$
$$+ C_1(B,T)\int_0^t (t-\tau)^{-1/2}\,\|\delta\|_\tau\,d\tau,$$

the integral involving the initial values disappearing if $b_1 = 0$.

Consider now the integral of u_j from $(\alpha(t), t)$ to $(\beta(t), t)$. If $\delta(t) = 0$, then $\alpha(t) = \beta(t)$ and the integral vanishes. If $\delta(t) > 0$, then two cases arise, namely when $\delta(\tau) > 0$, $0 \leq \tau \leq t$, and when there exists at least one value t_0, $0 \leq t_0 < t$, such that $\delta(t_0) = 0$. Let us treat the case for which $\delta(\tau) > 0$, $0 \leq \tau \leq t$, first. Then, the choice of j is the same for $0 \leq \tau \leq t$, and, by (2.4) and the maximum principle, u_j is dominated by the sum of two functions z_1 and z_2, where

$$\begin{aligned} &z_{1,xx} = z_{1,t}, \quad \alpha(t) < x < \beta(t), \quad 0 < t \leq T, \\ &z_1(\alpha(t), t) = z_1(\beta(t), t) = 0, \quad 0 < t \leq T, \\ &z_1(x, 0) = \varphi_j(x) = \varphi_2(x), \quad b_1 \leq x \leq b_2, \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} &z_{2,xx} = z_{2,t}, \quad \alpha(t) < x < \infty, \quad 0 < t \leq T, \\ &z_2(\alpha(t), t) = B\delta(t), \quad 0 < t \leq T, \\ &z_2(x, 0) = 0, \quad b_1 \leq x < \infty. \end{aligned} \tag{2.25}$$

Since $\varphi_2(x) > 0$ for $b_1 \leq x < b_2$, it follows from the maximum principle that $z_{1,x}(\alpha(t), t) > 0$ and $z_{1,x}(\beta(t), t) < 0$. Hence, heat is flowing out of the medium and

$$\int_{\alpha(t)}^{\beta(t)} z_1(x,t)\,dx < \int_{b_1}^{b_2} \varphi_2(x)\,dx, \quad 0 < t \leq T. \tag{2.26}$$

The estimation of the integral of z_2 can be made in exactly the same manner as that for v_3 above; consequently,

$$\int_{\alpha(t)}^{\infty} z_2(x,t)\,dx \leq C_1(B,T)\int_0^t \frac{\|\delta\|_\tau\,d\tau}{(t-\tau)^{1/2}}, \quad 0 \leq t \leq T. \tag{2.27}$$

In this case,

$$(2.28)\qquad \int_{\alpha(t)}^{\beta(t)} u_j(x,t)\,dx \leq \int_{b_1}^{b_2} \varphi_2(x)\,dx + C_1(B,T)\int_0^t \frac{\|\delta\|_\tau\,d\tau}{(t-\tau)^{1/2}},\qquad 0\leq t\leq T.$$

If $\delta(t)$ vanishes for some τ, $0\leq\tau\leq t$, let

$$(2.29)\qquad t_0 = \max\{\tau:\ 0\leq\tau<t,\ \delta(\tau)=0\}.$$

In the interval $[t_0, t]$ the choice of j is constant, and u_j is dominated by the solution z_3 of the heat equation in the region $\{\alpha(\tau)<x<\beta(\tau), t_0<\tau\leq t\}$ such that $z_3(\alpha(\tau),\tau)=B\delta(\tau)$ and $z_3(\beta(\tau),\tau)=0$; in turn, z_3 is dominated by z_4, where

$$(2.30)\qquad \begin{aligned} &z_{4,xx}=z_{4,\tau},\quad \alpha(t)<x<\infty,\quad t_0<\tau\leq t,\\ &z_4(\alpha(\tau),\tau)=B\delta(\tau),\quad t_0<\tau\leq t,\\ &z_4(x,t_0)=0,\quad \alpha(t_0)\leq x<\infty. \end{aligned}$$

It is clear that the analysis of z_4 is the same as that of z_2, except that the initial time becomes t_0. Thus,

$$(2.31)\qquad \int_{\alpha(t)}^{\beta(t)} z_4(x,t)\,dx \leq C_1(B,T)\int_{t_0}^t \frac{\|\delta\|_\tau\,d\tau}{(t-\tau)^{1/2}} \leq C_1(B,T)\int_0^t \frac{\|\delta\|_\tau\,d\tau}{(t-\tau)^{1/2}};$$

therefore, the estimate (2.28) holds in any case.

The estimates above can be applied to (2.3) to obtain

$$(2.32)\qquad \begin{aligned} \|\delta\|_t \leq b_2-b_1 &+ 2\int_0^{b_1} |\varphi_1(x)-\varphi_2(x)|\,dx + 2\int_{b_1}^{b_2}\varphi_2(x)\,dx\\ &+ 2\int_0^t |a_1(\tau)-a_2(\tau)|\,d\tau + 2C_1(B,T)\int_0^t \frac{\|\delta\|_\tau\,d\tau}{(t-\tau)^{1/2}},\qquad 0\leq t\leq T. \end{aligned}$$

Another application of the lemma completes the proof of the theorem, and

$$(2.33)\qquad C(B,T) = 2[1+4T^{1/2}C_1(B,T)]\exp\{4\pi\, C_1(B,T)^2\,T\}.$$

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