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INEQUALITIES FOR CONVEX FUNCTIONS

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In proving semicontinuity theorems for nonparametric curve integrals of the calculus of variations, Tonelli utilized linear inequalities for functions convex in a suitably strengthened sense [1, p. 405]. These inequalities were extended to the multidimensional case by Turner [2]. In the corresponding theory of optimal control, it is appropriate to consider functions on proper subsets of Euclidean space. We here extend the results of Tonelli and Turner to this situation.

1. **Preliminary results.**

We collect here definitions and standard results for later use. With the exception of the existence of a supporting plane, all results below are elementary.

A set S in Euclidean space E_m is called convex if, for any points $s, s' \in S$ and any real number α, $0 < \alpha < 1$,

$$\alpha s + (1 - \alpha) s' \in S.$$

If a convex set S in E_m is contained in no $(m - 1)$-dimensional hyperplane, then S has an interior point; and conversely.

For any point s_0 at positive distance from a convex set S in E_m, there exists a separating plane; that is, there exist $b \in E_m$, $\beta \in E_1$, such that $b \cdot s + \beta < 0$ for all $s \in S$, and $b \cdot s_0 + \beta > 0$.

For any point s_0 on the boundary of a convex set S in E_m, there exists a supporting plane; that is, there exist $b \in E_m$, $\beta \in E_1$, such that $b \neq 0$, $b \cdot s + \beta \leq 0$ for all $s \in S$, and $b \cdot s_0 + \beta = 0$.

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A real function $f(s)$ on a convex set S in E_m is called convex if, for any points $s, s' \in S$ and any real number α, $0 < \alpha < 1$,

$$f(\alpha s + (1 - \alpha) s') \leq \alpha f(s) + (1 - \alpha) f(s').$$

This is equivalent to convexity of the set

$$\{(s, y) : s \in S, \ y \geq f(s)\} \quad \text{in} \quad E_{m+1}.$$

For any point s_0 in the interior of the domain S of such a convex function, there exists a supporting function; that is, there exist $b \in E_m$, $\beta \in E_1$, such that $f(s) \geq b \cdot s + \beta$ for all $s \in S$, and $f(s_0) = b \cdot s_0 + \beta$. Even if S has no interior points, there exists a supporting function at some point.

2. Approximate supporting functions.

Although a convex function need not have a supporting function at the boundary of its domain (for example, $-(1 - x^2)^{\frac{1}{2}}$ on $[-1, 1]$), it does have approximate supporting functions in the following way.

THEOREM 1. Let $f(s)$ be a continuous convex function on a convex set S in E_m. Then, for any $s_0 \in S$ and any $\varepsilon > 0$, there exist $\delta > 0$, $b \in E_m$, and $\beta \in E_1$ such that

(*a*) $f(s) > b \cdot s + \beta$ for all $s \in S$; and

(*b*) $f(s) < b \cdot s + \beta + \varepsilon$ for $|s - s_0| < \delta$, $s \in S$.

PROOF. By continuity of f, the point $\left(s_0, f(s_0) - \frac{1}{2}\varepsilon\right)$ is at positive distance from the convex set $\{(s, y) : s \in S, \ y \geq f(s)\}$ in E_{m+1}. Thus there exists a separating plane; that is, there exist $b \in E_m$, $\gamma \in E_1$, and $\beta \in E_1$ such that

$$(b, \gamma) \cdot (s, y) + \beta < 0 \quad \text{for} \quad s \in S, \quad y \geq f(s);$$

and $(b, \gamma) \cdot \left(s_0, f(s_0) - \frac{1}{2}\varepsilon\right) + \beta > 0$.

If $\gamma \geq 0$, these inequalities would be contradictory for $s = s_0$, $y = f(s_0)$. Thus $\gamma < 0$, and we can take $\gamma = -1$. Then

$$b \cdot s - f(s) + \beta < 0 \quad \text{for all} \quad s \in S;$$

and $b \cdot s_0 - f(s_0) + \frac{1}{2}\varepsilon + \beta > 0$.

Condition (*b*) follows by continuity.

REMARK. The necessary condition above is easily seen to be sufficient for f to be continuous and convex on S.

3. Strong linear bounds.

THEOREM 2. Let $f(s)$ be a continuous convex function on a closed convex set S in E_m. If the graph of f contains no whole straight lines, then there exists a linear function $w(s)$ such that $f(s) \geq w(s)$ on S and $f(s) - w(s) \to \infty$ as $|s| \to \infty$ on S.

PROOF. We restrict consideration at first to f such that $f(s) \geq 0$ on S and $f(a) = 0$ for some $a \in S$. Let B be the set of points $b \in E_m$ such that $f(s) \geq b \cdot s + \beta$ on S for some $\beta \in E_1$. The set B is convex and contains 0.

Suppose that B is contained in some $(m-1)$-dimensional hyperplane. Then there exists $e \in E_m$, $e \neq 0$, such that $b \cdot e = 0$ for all $b \in B$. If $a + \lambda e \in E_m - S$ for some λ, then there exists a separating plane such that $c \cdot s + \gamma < 0$ for all $s \in S$ (so $c \in B$ and $c \cdot a + \gamma < 0$) and $c \cdot (a + \lambda e) + \gamma > 0$, so $c \cdot a + \gamma > 0$. Thus $a + \lambda e \in S$ for all λ. Since the graph of f contains no whole line, $f(a + \lambda e) > 0$ for some λ. Consider ε such that $0 < \varepsilon < f(a + \lambda e)$. For a corresponding approximate supporting function $b \cdot s + \beta$ at $a + \lambda e$,

$$f(s) > b \cdot s + \beta \quad \text{for all} \quad s \in S \quad (\text{so } b \in B),$$

and

$$f(s) < b \cdot s + \beta + \varepsilon \quad \text{for} \quad |s - a - \lambda e| < \delta.$$

Then $f(a) > b \cdot a + \beta = b \cdot (a + \lambda e) + \beta > f(a + \lambda e) - \varepsilon > 0$.

Thus B is contained in no $(m-1)$-dimensional hyperplane, and so has some interior point b_0, with corresponding β_0. Then $f(s) \geq w(s) = b_0 \cdot s + \beta_0$ for all $s \in S$.

Also, $f(s) - w(s) \to \infty$ as $|s| \to \infty$ on S, for suppose not. Then there exists a sequence of points $s_n \in S$ with $f(s_n) - w(s_n)$ bounded above and $|s_n| \to \infty$. For some subsequence, $s_n / |s_n| \to v \neq 0$. Now $b_0 + \varrho v \in B$ for ϱ sufficiently small, with corresponding β. Then

$$f(s) \geq (b_0 + \varrho v) \cdot s + \beta \quad \text{for all} \quad s \in S,$$

so

$$f(s_n) - w(s_n) \geq \varrho v \cdot (s_n / |s_n|) \, |s_n| + \beta - \beta_0 \to \infty.$$

For more general f, let a be some point for which f has a supporting function $b \cdot s + \beta$. Then $f(s) \geq b \cdot s + \beta$ for all $s \in S$, and $f(a) = b \cdot a + \beta$. Consequently, the convex function

$$f^{-}(s) = f(s) - f(a) - b \cdot (s - a)$$

has $f^-(s) \geq 0$ and $f^-(a) = 0$, while its graph contains no whole straight lines. Thus there exists a linear function $w^-(s)$ such that $f^-(s) \geq w^-(s)$ and $f^-(s) - w^-(s) \to \infty$ as $|s| \to \infty$ on S. Then

$$f(s) \geq w(s) = w^-(s) + f(a) + b \cdot (s - a)$$

and $f(s) - w(s) = f^-(s) - w^-(s) \to \infty$ as $|s| \to \infty$ on S.

REMARK. For S bounded, Theorem 2 is trivial.

4. Uniform approximate supporting functions.

THEOREM 3. Let D be a closed set in $E_n \times E_m$ such that $D_r = \{s : (r, s) \in D\}$ is convex for each $r \in E_n$. Suppose that D_r is « continuous » at $(r_0, s_0) \in D$ in the sense that $d(s_0, D_r) \to 0$ and $\sup \{d(s, D_{r_0}) : s \in D_r\} \to 0$ as $r \to r_0$ on $\{r : D_r \neq \Phi\}$. Let $f(r, s)$ be a continuous function on D, convex in s and such that the graph of $f(r_0, s)$ contains no whole straight lines. Then, for any $\varepsilon > 0$, there exist $b \in E_m$, $\beta \in E_1$, $\nu > 0$, and $\delta > 0$, such that

(a) $f(r, s) > b \cdot s + \beta + \nu |s - s_0|$ for $|r - r_0| < \delta$, and
(b) $f(r, s) < b \cdot s + \beta + \varepsilon$ for $|r - r_0| < \delta$, $|s - s_0| < \delta$.

PROOF. Let $w(s)$ be the linear function for $f(r_0, s)$ from Theorem 2: $f(r_0, s) \geq w(s)$, and $f(r_0, s) - w(s) \to \infty$ as $|s| \to \infty$ on D_{r_0}. For the given ε, let $v(s)$ be an approximate supporting function for $f(r_0, s)$ at s_0: $f(r_0, s) > v(s)$, and $f(r_0, s) < v(s) + \varepsilon$ for $|s - s_0| < \delta_0$. For $0 < \alpha < 1$, put

$$z(s) = \alpha w(s) + (1 - \alpha) v(s), \quad F(r, s) = f(r, s) - z(s).$$

Now $F(r_0, s_0) = f(r_0, s_0) - v(s_0) + \alpha [v(s_0) - w(s_0)] < 2\varepsilon$ for $\alpha < \varepsilon / |v(s_0) - w(s_0)|$.

By continuity of f at (r_0, s_0), there exists $\delta_1 > 0$ such that $F(r, s) < 3\varepsilon$ for $|r - r_0| < \delta_1$, $|s - s_0| < \delta_1$.

Also, $F(r_0, s) = \alpha [f(r_0, s) - w(s)] + (1 - \alpha) [f(r_0, s) - v(s)] \to \infty$ as $|s| \to \infty$ on D_{r_0}. Then there exists $m > 0$ such that $F(r_0, s) > 5\varepsilon$ for $|s - s_0| \geq \frac{1}{2} m$, $s \in D_{r_0}$. By uniform continuity of F on

$$D \cap \left\{(r, s) : |r - r_0| \leq \Delta, \ \frac{1}{2} m \leq |s - s_0| \leq 2m \right\}$$

and by continuity of D_r, there then exists $\delta_2 > 0$ such that $F(r, s) > 4\varepsilon$ for $|r - r_0| < \delta_2$, $|s - s_0| = m$, $(r, s) \in D$.

Also, $F(r_0, s) > 0$. By uniform continuity of F on $D \cap \{(r, s) : |r - r_0| \leq \Delta$, $|s - s_0| \leq 2m\}$ and by continuity of D_r, there exists $\delta_3 > 0$ such that $F(r, s) > -\varepsilon$ for $|r - r_0| < \delta_3$, $|s - s_0| \leq m$, $(r, s) \in D$. Then, under the same conditions, with $\nu = \varepsilon/2m$,

$$F(r, s) > -2\varepsilon + \nu |s - s_0|.$$

For any $\varepsilon' > 0$, there exists $\delta_4(\varepsilon') > 0$ such that $D_r = \Phi$ or $d(s_0, D_r) < \varepsilon'$ for $|r - r_0| < \delta_4(\varepsilon')$. Consider $|r - r_0| < \delta = \min\{\delta_1, \delta_2, \delta_3, \delta_4(\min[\delta_1, m])\}$. Then for $D_r \neq \Phi$, there exists $s' \in D_r$ such that $|s' - s_0| < \delta_1$ and m. For $|s - s_0| > m$, $s \in D_r$, take $s_1 = \gamma s + (1 - \gamma) s'$ for some γ, $0 < \gamma < 1$, such that $|s_1 - s_0| = m$. Then, by convexity,

$$F(r, s_1) \leq \gamma F(r, s) + (1 - \gamma) F(r, s'),$$

so

$$F(r, s) \geq F(r, s') + [F(r, s_1) - F(r, s')]/\gamma > -\varepsilon + \varepsilon/\gamma > -2\varepsilon + \nu |s - s_0|.$$

Thus $f(r, s) > z(s) - 2\varepsilon + \nu |s - s_0|$ for $|r - r_0| < \delta$, and $f(r, s) < z(s) - 2\varepsilon + 5\varepsilon$ for $|r - r_0| < \delta$, $|s - s_0| < \delta$. We obtain the required result by substituting $\varepsilon/5$ for ε in our work.

REMARKS 1. The straight line condition on the graph of f cannot be omitted, even for the weaker inequalities with $\nu = 0$; for example,

$$f(r, s) = rs \quad \text{on} \quad E_1 \times E_1, \quad \text{with} \quad r_0 = s_0 = 0.$$

2. The continuity condition on D_r cannot be omitted; for example,

$$f(r, s) = s^2 \quad \text{for} \quad r = 0, \quad -s^2 \quad \text{for} \quad rs = 1,$$

on $D = \{(r, s) : r = 0 \text{ or } rs = 1\}$ in $E_1 \times E_1$, with $r_0 = s_0 = 0$.

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