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UNIFORM DISTRIBUTION OF SEQUENCES FROM DIRECT PRODUCTS OF GROUPS

L. KUIPERS and P. A. J. SCHEELBEEK

Introduction.

The present paper deals with some aspects of the theory of uniform distribution of sequences from compact topological (infinite) groups (Part I) and from finite groups as well (Part II). Part I is based upon the theory developed by B. ECKMANN [1], who proved the classical result: if $\mathcal{G}$ is a compact topological group, the sequence $\langle x_\nu \rangle$ $(\nu = 1, 2, \ldots)$ from $\mathcal{G}$ is uniformly distributed in $\mathcal{G}$ if and only if for each non-trivial irreducible representation $\mathcal{D}$ of $\mathcal{G}$

$$(1) \qquad \frac{1}{N} \sum_{\nu=1}^{N} \mathcal{D}(x_\nu) \rightarrow 0 \qquad \text{as } N \rightarrow \infty .$$

If in addition $\mathcal{G}$ is abelian, then $\mathcal{D}$ in (1) stands for a non-trivial character of $\mathcal{G}$.

In order to apply this theory to groups which are direct products of compact topological groups we make use of the Pontrjagin construction of the representations of such groups. See [2], p. 115. Let F be the direct product of two compact topological groups $\mathcal{G}$ and H. An element $z \in F$ is a pair (x, y) with $x \in \mathcal{G}$ and $y \in H$. Let g and h be irreducible representations of $\mathcal{G}$ and H resp. and of order m and n resp. Let $g(x) = (g_{ij}(x))$ and $h(y) = (h_{kl}(y))$. Form a matrix f of order mn by defining the elements

$$f_{(ik)(jl)}(z) = g_{ij}(x)\, h_{kl}(y) \; (i, j = 1, 2, \ldots, m \,;\; k, l = 1, 2, \ldots, n)$$

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and the array

$$(gh) = \begin{pmatrix} g_{11}\, h & \cdot & \cdot \\ g_{21}\, h & \cdot & \cdot \\ \cdot & \cdot & g_{mm}\, h \end{pmatrix},$$

where $g_{ij}\, h$ stands for the matrix

$$\begin{pmatrix} g_{ij}\, h_{11} & \cdot & \cdot \\ g_{ij}\, h_{12} & \cdot & \cdot \\ \cdot & \cdot & g_{ij}\, h_{nn} \end{pmatrix}.$$

Now $(f_{(ik)(jl)}(z))$ is a representation of F. One observes that the trace of F is equal to the product of the traces of G and H. Moreover F is irreducible. All irreducible representations of F (except for equivalence) can be found in this way. Theorems 1 and 2 are applications.

L. A. RUBEL [3] recently showed that Eckmann's theory also applies in the case of uniform distribution of sequences from $\mathcal{G}/\mathcal{G}_0$, where $\mathcal{G}$ is a locally compact group and $\mathcal{G}_0$ is a closed normal subgroup of $\mathcal{G}$ such that $\mathcal{G}/\mathcal{G}_0$ is compact. He applies his result to a problem posed by I. NIVEN [4], who introduced the concept of uniform distribution of a sequence of rational integers reduced modulo m (where m is a non-negative integer > 1) in the class of residues mod m. We remark however that in the case of Niven's theory and its subsequent developments sequences are considered containing elements which assume a finite number of distinct values only. In all these cases the criterion for uniform distribution reduces to a very simple useful lemma (theorem 3) which can be shown by applying properties of characters of finite additive abelian groups (Part II).

PART I

THEOREM 1. *Let $\mathcal{G}$ and H be two compact topological groups. Let $a \in \mathcal{G}$ be fixedpointfree under all non-trivial represenations of $\mathcal{G}$. Let $b \in H$. Then the sequence of elements (a, b), (a, b^2), (a^2, b), (a, b^3), (a^3, b), (a^2, b^2), (a^2, b^3), (a^3, b^2), (a, b^4), (a^4, b), ... is uniformly distributed in F, the direct product of $\mathcal{G}$ and H.*

PROOF. Since $a \in \mathcal{G}$ is fixedpointfree under all non-trivial representations of $\mathcal{G}$, the powers of a are dense in $\mathcal{G}$ (see [1], th. 1) and $\mathcal{G}$ is abelian. If $\mathcal{X}$ is a non-trivial character of $\mathcal{G}$, and h is an irreducible representation of H, then

$$\begin{pmatrix} \mathcal{X} h_{11} & \cdot & \cdot \\ \mathcal{X} h_{12} & \cdot & \cdot \\ \cdot & & \cdot\, \mathcal{X} h_{nn} \end{pmatrix}, \text{ or } \mathcal{X}(h_{ij})$$

is a non-trivial irreducible representation of F. Consider the first N elements of the sequence defined in the theorem. We want to show that

$$\frac{1}{N} \sum_{\lambda+\mu \leqq n} \mathcal{X}(h_{ij})(a^\lambda, b^\mu),$$

where the integer n is defined by $\frac{1}{2}(n-1)\, n \leqq N \leqq \frac{1}{2}\, n\,(n+1)$, goes to zero if $N \to \infty$. Now

$$\mathcal{X}(h_{ij})(a^\lambda, b^\mu) = \mathcal{X}(a^\lambda)(h_{ij}(b^\mu)).$$

Since all function values $h_{ij}(b^\mu)$ are uniformly bounded, it suffices to show that

$$\frac{1}{N}\{n\, \mathcal{X}(a) + (n-1)\, \mathcal{X}(a^2) + \dots + \mathcal{X}(a^n)\}$$

$$= \frac{1}{N}\{n\, \mathcal{X}(a) + (n-1)\, \mathcal{X}^2(a) + \dots + \mathcal{X}^n(a)\}$$

$$= \frac{1}{N}\left\{\frac{n\, \mathcal{X}(a)}{1-\mathcal{X}(a)} + \frac{\mathcal{X}^2(a)\, \{\mathcal{X}^n(a) - 1\}}{(1-\mathcal{X}(a))^2}\right\}.$$

Now replacing $\mathcal{X}^n(a)$ by $\mathcal{X}(a^n)$ and observing that $N = O(n^2)$ we see that the last expression $\to 0$ as $N \to \infty$ $(n \to \infty)$.

THEOREM 2. *Let $\mathcal{G}$ and H be two compact topological groups satisfying the second countability axiom and let $\langle x_n^{(1)} \rangle$, $\langle y_n^{(1)} \rangle$ $(n = 1, 2, \dots)$ be two sequences of elements from $\mathcal{G}$, and let $\langle x_n^{(2)} \rangle$, $\langle y_n^{(2)} \rangle$ $(n = 1, 2, \dots)$ be two sequences from H. Let $\langle x_n^{(1)} \rangle$, $\langle x_n^{(2)} \rangle$ be uniformly distributed in $\mathcal{G}$ and H, resp. Furthermore it is assumed that $\lim_{n\to\infty} (y_n^{(1)})^{-1} x_n^{(1)} = e^{(1)}$ and $\lim_{n\to\infty} (y_n^{(2)})^{-1} x_n^{(2)} = e^{(2)}$ where $e^{(1)}$ and $e^{(2)}$ are unit elements in $\mathcal{G}$ and H, resp. Then the sequence $\langle y_n^{(1)}, y_n^{(2)} \rangle$ is uniformly distributed in the direct product F of $\mathcal{G}$ and H.*

PROOF. First we notice that $\langle y_n^{(1)} \rangle$ is uniformly distributed in $\mathcal{G}$, as is $\langle y_n^{(2)} \rangle$ in H, according to D. HLAWKA [5], whose result is a generalization of a theorem of J. G. VAN DER CORPUT [6]. Now using Eckmann's criterion and the above notation we get

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m,\,n=1}^{N} f_{(i,k)\,(j,\,l)}(y_m^{(1)}, y_n^{(2)})$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{N} g_{ij}(y_m^{(1)}) \cdot \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} h_{kl}(y_n^{(2)}) = 0,$$

from which the theorem follows.

PART II

THEOREM 3. *If $\mathcal{G}$ is an abelian group, and H is a normal subgroup of finite index, then the sequence $\langle x_n \rangle$ $(n = 1, 2, \ldots)$ from $\mathcal{G}$ is uniformly distributed in the cosets of $\mathcal{G}$ mod H if and only if for every non-trivial character f of $\mathcal{G}$ which is trivial on H,*

$$\frac{1}{N} \sum_{n=1}^{N} f(x_n) \to 0 \text{ as } N \to \infty . \tag{2}$$

PROOF. Let $\langle x_n \rangle$ be uniformly distributed in the cosets X of $\mathcal{G}$ mod H, that is, let

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\substack{n=1 \\ x_n \in X}}^{N} 1 = \{\mathcal{G}:\, H\}^{-1} \text{ for each } X. \tag{3}$$

We always have

$$\sum_{n=1}^{N} f(x_n) = \sum_{X} f(X) \sum_{\substack{n=1 \\ x_n \in X}}^{N} 1 \tag{4}$$

for every character f of $\mathcal{G}$. For every non-trivial character f we have the property

$$\sum_{X} f(X) = 0. \tag{5}$$

Because of our assumption (3), (4) can be written in the from

$$\sum_{n=1}^{N} f(x_n) = \sum_{X} f(X) \left[\frac{N}{\{\mathcal{G}:\, H\}} + o(N) \right] \tag{6}$$

($o(N)$ depends on X). Now (5), (6) and (3) imply (2) for every non-trivial character.

Conversely, suppose that (2) is satisfied for every non-trivial character f (but trivial on H). Then we start from the identity

$$\sum_{\substack{n=1 \\ x_n \in X}}^{N} 1 = \frac{1}{\{\mathcal{G}:\, H\}} \cdot \sum_{f \in \widehat{\mathcal{G}/H}} \overline{f(X)} \sum_{n=1}^{N} f(x_n), \tag{7}$$

where X is any coset of $\mathcal{G}$ mod H, and where the summation on the right of (7) is performed over all characters involved. That (7) is true follows easily from the orthogonality properties of the f. Because of our assumption (2), (7) can be written in the form

$$\frac{1}{N} \sum_{\substack{n=1 \\ x_n \in X}}^{N} 1 = \frac{1}{\{\mathcal{G}:\, H\}} \cdot \left(1 + \sum_{f \text{ non tr}} \overline{f(X)}\, o(1)\right) \tag{8}$$

($o(1)$ depends on f). Letting $N \to \infty$, we find (3).

APPLICATION. Let m, s and $t \in \mathbf{Z}$ (the rational integers) and ≥ 1. Let $\mathcal{R} = \mathcal{R}[m, x, y]$ be the ring of polynomials in x and y over the ring $\mathbf{Z}/m\mathbf{Z}$ (or the class of residues mod m). Let $I = I(x^s, y^t)$ be the ideal generated by x^s and y^t, or let $I = \varphi x^s + \psi y^t$, where φ and ψ are arbitrary elements of $R[m, x, y]$.

Now consider the set R/I of residues of elements of R modulo I, that is the set of elements of R of the form

$$\sum_{k=1}^{t} (a_1^{(k)} x^{s-1} + \dots + a_s^{(k)})\, y^{t-k}.$$

Obviously the number of these residues is m^{st}. These elements form an additive group isomorphic with the direct product $\mathcal{G}$ of st additive abelian groups each of them consisting of the elements $0, 1, \dots, m-1$:

$$\mathcal{G} = \mathcal{G}_1^{(t)} * \dots * \mathcal{G}_s^{(t)} * \mathcal{G}_1^{(t-1)} * \dots * \mathcal{G}_s^{(t-1)} * \dots * \mathcal{G}_s^{(1)} .$$

It is well-known that each of the m^{st} characters of $\mathcal{G}$ is determined by an ordered set of st integers mod m, say the vector

$$C = (c_1^{(t)}, \dots, c_s^{(t)}, c_1^{(t-1)}, \dots, c_s^{(t-1)}, \dots, c_s^{(1)}),$$

and that its value assumed at the element

$$\mathcal{A} = (a_1^{(t)}, \dots, a_s^{(t)}, a_1^{(t-1)}, \dots, a_s^{(t-1)}, \dots, a_s^{(1)}) \in \mathcal{G}$$

is given by the expression

$$\exp 2\pi i \sum_{k=1}^{t} (a_1^{(k)} c_1^{(k)} + \dots + a_s^{(k)} c_s^{(k)})/m \tag{9}$$

or $e(AC, m)$, say.

Let $\langle A_n \rangle$ $(n = 1, 2, \dots)$ be a sequence of polynomials from $R[m, x, y]$. Then the question how these polynomials (reduced mod I) are distributed over the elements of R/I is equivalent to the question how the sequence of vectors of coefficients is distributed over the elements of $\mathbb{Z}^{st}/(m\mathbb{Z})^{st}$, where $\mathbb{Z}^{st}$ is the direct product of st rings $\mathbb{Z}$. Obiously $(m\mathbb{Z})^{st}$ is a subgroup of finite index (namely m^{st}) of $\mathbb{Z}^{st}$. Theorem 3 applies in this case. With the notations already used we have the following result.

THEOREM 4. *The sequence* $\langle A_n \rangle$ $(n=1, 2, \dots)$ *from* $R[m, x, y]$ *is uniformly distributed* mod I *if and only if for every element* $C \neq (0, 0, \dots, 0)$

$$\frac{1}{N} \sum_{n=1}^{N} e(A_n C, m) \to 0 \text{ as } N \to \infty .$$

REMARK 1. Let $s = 1$, $t = 1$. Then we have the case dealt with by I. NIVEN [4] and S. UCHIYAMA [7], and we find the criterion: a sequence $\langle a_k \rangle$ $(k = 1, 2, \dots)$ of integers is uniformly distributed mod m if and only if for every $j = 1, 2, \dots, m-1$

$$\frac{1}{N} \sum_{n=1}^{N} \exp (2\pi i\, a_n j/m) \to 0 \text{ as } N \to \infty.$$

REMARK 2. Let $\mathbb{Z}_p$ be the ring of p-adic integers (p being a prime number), let I be the ideal generated by p^k (k an integer ≥ 1). Then for an element ψ_k in $\mathbb{Z}_p/I$ we have the canonical representation

$$\psi = \psi_k = a_0 + a_1 p + \dots + a_{k-1} p^{k-1} \ (a_i = 0, 1, \dots, p-1).$$

Let $\langle x_n \rangle$ be a sequence in $\mathbb{Z}_p$. Since the non-trivial characters of $\mathbb{Z}_p/I$ are given by the expression

$$\exp (2\pi i\, h\psi(x_n)/p^k) \ (h = 1, 2, \dots, p^k - 1),$$

we find the criterion for the uniform distribution of $\langle x_n \rangle$ in $\mathbb{Z}_p/I$:

$$\frac{1}{N} \sum_{n=1}^{N} \exp\ (2\pi i\, h\psi\,(x_n)/p^k) \longrightarrow 0 \qquad \text{as } N \longrightarrow \infty$$

for every $h = 1, 2, \ldots, p^k - 1$. See J. CHAUVINEAU [8].

REMARK 3. Finally we comment on J. H. HODGES' paper [9]. Hodges considers a sequence $\langle A_k \rangle$ of polynomials from the ring $\mathcal{G}F[q, x]$, where $q = p^r$, p a prime and $r \in \mathbb{Z}$ with $r \geq 1$. If M denotes a monic polynomial of degree $m \geq 1$, then the problem is how $\langle A_k \rangle$ reduced mod M distributes. In fact, a residue of a polynomial A mod M is an expression of the form $\alpha_1 x^{m-1} + \ldots + \alpha_m$, where the α_i are of the form $a_1 \mu^{r-1} + \ldots + a_r$ with $a_i \in \mathcal{G}F(p)$, and μ is a zero of an irreducible polynomial of degree r from $\mathcal{G}F[p, x]$. Now define $e(A, M) = \exp\,(2\pi i a_1/p)$. Hodges shows that the condition

$$\frac{1}{N} \sum_{k=1}^{N} e\,(A_k, M) \longrightarrow 0 \qquad \text{as } N \longrightarrow \infty$$

is necessary for the uniform distribution of $\langle A_k \rangle$ mod M over the elements $\alpha_1 x^{m-1} + \ldots + \alpha_m$. We claim that the condition

$$\frac{1}{N} \sum_{k=1}^{N} e\ (A_k\, C, M) \longrightarrow 0 \qquad \text{as } N \longrightarrow \infty \tag{10}$$

(for every polynomial $C \not\equiv 0$ (mod M)) is necessary and sufficient for the uniform distribution of $\langle A_k \rangle$ and we state that the present case is an application of theorem 4. Take in theorem 4, $m = p$, $s = r$, $t = m$, and notice that $\sum_{k=1}^{N} e\,(A_k\, C, M)$ (see (10)) is an expression of the type (9). For, let

$$A_k \equiv \alpha_1^{(k)}\, x^{m-1} + \ldots + \alpha_m^{(k)} \ (\text{mod } M)$$

$$C \equiv \gamma_1\, x^{m-1} \quad + \ldots + \gamma_m \ \ (\text{mod } M),$$

then the coefficient of x^{m-1} in the product $A_k\, C$ is of the form $\alpha_1^{(k)}\, \gamma_m + \ldots + \alpha_m^{(k)}\, \gamma_1$, and by expressing each $\alpha_i^{(k)}$ and γ_j in terms of μ we find that the coefficient of μ^{r-1} in this sum is exactly of the form (9). Hence the criterion is the same as stated by theorem 4.

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