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THE FUNCTION CLASSES $\gamma_H(\beta, \delta, d)$ AND GLOBAL LINEAR GOURSAT PROBLEMS (*)

JAN PERSSON

Introduction.

The non-characteristic linear Cauchy problem when the coefficients are entire functions was treated in [4] and [5] as special cases of more general problems. It was proved that with entire data the Cauchy problem has an entire solution if the coefficients in the principal part of the operator are constants.

In [4] it was pointed out that the solution of

$$D_1 u = e^{x_1 + x_2} D_2 u, \quad u(0, x_2) = e^{x_2}, \tag{1.1}$$

is

$$u = (1 - e^{x_1} + e^{-x_2})^{-1}.$$

Since u is not an entire function we must put some restriction on the coefficients in the principal part. Recent studies on the characteristic Cauchy problem for the equation $D_t^m u = x^q D_x^n u$, $m < n$, $q \geq 0$, q, m, n integers, see [6], show that the dependence on the space variable has a remarkable impact on the solutions. See also Asadullin [1], and A. Friedman [2]. It was shown in [6] that the only analytic solutions around the origin of $D_t u = x^3 D_x^2 u$ are the trivial ones $Ax + B$, A and B being arbitrary constant. It was further shown in [6] that $D_t u = x^3 D_x^2 u + tx^3$ has no analytic solution in any neighbourhood of the origin.

It is then natural to modify (1.1) to

$$D_1 u = e^{x_2} D_2 u, \quad u(0, x_2) = e^{x_2}. \tag{1.2}$$

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The successive approximations give the solution of (1.2) explicitly as the series

$$u = \sum_{j=1}^{\infty} x_1^j e^{(j+1)x_2} + e^{x_2} .$$

Since

$$u(x_1, 0) = (1 - x_1)^{-1} ,$$

u cannot be an entire function. The solution of

$$D_1 u = e^{x_1} D_2 u, \quad u(0, x_2) = e^{x_2}, \tag{1.3}$$

is

$$u(x_1, x_2) = e^{x_2} \exp(e^{x_1} - 1).$$

So here u is entire.

We have taken the experience from (1.2) and (1.3) as a frame for the hypotheses in Theorem 1 and Theorem 2 in section 3 below. Theorem 1 is a slight modification of Theorem 1 in [5]. Here we use the function classes $\gamma_H(\beta, \delta, d)$ instead of the classes $\gamma(\beta, \delta, d)$ in [5]. See section 2. In section 2 we also give the notation. Otherwise we follow the proof of Theorem 1 in [5]. So we refer the reader to that paper for the main argument. Only the deviations from that proof is indicated in section 3.

Theorem 2 is a special case of Theorem 1. It generalizes Theorem 3 in [5]. It says that the coefficients in the principal part of the operator may be entire functions depending on the time variable only. The last mentioned theorems deal both with the non-characteristic Cauchy problem for entire functions.

So far the results in section 3. Still one might ask if it is possible to weaken the hypothesis on the coefficients in the principal part. Let $f(x_2)$ be an arbitrary entire function. The solution of

$$D_1 u = x_2 D_2 u, \quad u(0, x_2) = f(x_2), \tag{1.4}$$

is

$$u = f(x_2 e^{x_1}).$$

Thus u is an entire function. Now we look at

$$D_1 u = x_2^2 D_2 u, \quad u(0, x_2) = x_2 . \tag{1.5}$$

Its solution is

$$u = x_2 + x_1 x_2^2 (1 - x_1 x_2)^{-1} .$$

Here u is not an entire function. This is rather remarkable. But it is also in line with the results in [6]. We have not been able to prove a theorem that covers the situation in (1.4) and excludes the situation in (1.5). Our method of proof does not seem to apply to the situation in (1.4).

That a unique analytic solution of (1.1)-(1.5) exists in some neighbourhood of the origin follows from the Cauchy-Kovalevsky theorem. For a further background we refer the reader to [4] and [5] and the references given there.

In section 3 it is also indicated how Theorem 1 in [5] can be modified. The new version does not cover the old one and is not included in that theorem.

Added in proof: In a paper to appear in Norske Vid. Selsk. Forh. (Trondheim) we prove that $D_t u = x^2 D_x u + tx^2$ has no entire solution.

2. Preliminaries.

Let $x = (x_1, \dots, x_{n'}, x_{n'+1}, \dots, x_n) = (x', x'') \in R^{n'} \times R^{n-n'}$, where $0 \leq \leq n' \leq n$. A multi-index with non-negative integers as components is denoted by a Greek letter $\alpha = (\alpha_1, \dots, \alpha_n)$. We define $D_x = (D_{x_1}, \dots, D_{x_n}) = (\partial/\partial x_1, \dots, \partial/\partial x_n)$ and write $D_x^\alpha = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}$. We also write $|\alpha| = \alpha_1 + \dots + \alpha_n$. We define $\alpha \leq \delta \Longleftrightarrow \alpha_j \leq \delta_j$, $1 \leq j \leq n$.

Let $d \in R^n$. We define $\alpha d = \alpha_1 d_1 + \dots + \alpha_n d_n$. We shall also use variables $(y, x) \in C^s \times R^n$, $y \in C^s$, $x \in R^n$. If $y_j = y_j' + i y_j''$, $i = \sqrt{-1}$, y_j' and y_j'' real, we define $D_{y_j} = 2^{-1}(\partial/\partial y_j' - i\partial/\partial y_j'')$. Otherwise for these variables we use the natural extensions of the definitions above.

We repeat the definition of the class P as given in [5].

DEFINITION 1. *Let $p(t)$, $t \geq 0$, be a real valued continuously differentiable function. If it satisfies the following conditions*

(2.1) $$0 < p(t) < \log(t+4),$$

(2.2) $$0 < p'(t) < (t+1)^{-1},$$

(2.3) $p(t)$ *tends monotonically to* $+\infty$ *when* $t \to +\infty$,

(2.4) $p'(t)$ *tends monotonically to zero when* $t \to +\infty$,

and

(2.5) $p(t)/t$ *and* $p(p(t))/t$ *are decreasing,*

then p is said to belong to the class P.

We then define the function class $\gamma_H(\beta, \delta, d)$. It is defined in such a way that we can use most of the proof of Theorem 1 in [5].

DEFINITION 2. *The function* $g(y, x)$ *is complex-valued and defined in* $C^s \times R^n$, $\delta \in R^n$ *and* $\beta \in R^s$ *are multi-indices, and* $d \in R^n$, $d_j \geq 0$, $1 \leq j \leq n$. *The derivatives* $D_y^\gamma D_x^\xi g$, $\gamma \leq \beta$, *all* ξ, *exists and are continuous together with* g *itself. For every fixed* $x \in R^n$, $y \to g(y, x)$ *is a holomorphic function in* C^s. *If to every set* $K \subset C^s \times R^n$ *there exist a constant* $C > 0$ *and a function* $p \in P$ *such that*

$$(2.1) \qquad |D_x^\xi D_y^\gamma D_x^\alpha g(y, x)| \leq C(\xi d / p(\xi d))^{\xi d - 1}, \ (y, x) \in K, \ \alpha d + |\gamma| \leq |\beta| + \delta d, \quad \gamma \leq \beta, \quad \text{all } \xi,$$

then g *is said to belong to the function class* $\gamma_H(\beta, \delta, d)$. *Here* $0^{-1} = 1$ *in* (2.1).

We see immediately that $\gamma_H(\beta, \delta, d)$ restricted to $R^s \times R^n$ is a subspace of $\gamma(\beta, \delta, d)$. See Def. 2 in [5].

Let the function $g \in \gamma_H(\beta, \delta, d)$. If

$$D_{y_j}^k g(y, x) = 0, \quad y_j = 0, \quad 0 \leq k < \beta_j, \quad j = 1, \dots, s,$$

and

$$D_{x_j}^k g(y, x) = 0, \quad x_j = 0, \quad 0 \leq k < \delta_j, \quad j = 1, \dots, n,$$

then we write

$$g = O(y^\beta x^\delta).$$

3. Goursat problems with solutions in $\gamma_H(\beta, \delta, d)$.

We shall now prove the following theorem.

THEOREM 1. *The integer* n' *is restricted by* $0 \leq n' \leq n$. *Let* $d \in R^n$, $d_j = 1$, $1 \leq j \leq n'$, $d_j \geq 1$, $1 \leq j \leq n$. *The multi-indices* $\beta \in R^s$, $\delta \in R^n$, $\gamma^k \in R^s$, $\alpha^k \in R^n$, $1 \leq k \leq N$, *are restricted by*

$$(3.1) \qquad \delta_j = 0, \ n' < j \leq n, \ \gamma^k \leq \beta, \ \alpha^k d + |\gamma^k| \leq |\beta| + |\delta|, \ 1 \leq k \leq N,$$

and

$$(3.2) \qquad |\beta| - |\gamma^k| + |\delta| - \alpha^k d = 0 \Longrightarrow |\beta| > |\gamma^k|.$$

The functions $f(y, x)$, $a_k(y, x)$, $1 \leq k \leq N$, *belong to* $\gamma_H(\beta, \delta, d)$. *Thus to every* $r > 0$ *there exist a* $p \in P$ *and a constant* $C_1 > 0$ *such that*

$$(3.3) \qquad |D_x^\xi f(y, x)| \leq C_1 (\xi d / p(\xi d))^{\xi d - 1}, \ |y| + |x| \leq r, \quad \text{all } \xi,$$

and

(3.4) $\quad |D_x^{\xi} a_k(y,x)| \leq C_1 (\xi d/p(\xi d))^{\xi d-1}, \; |y|+|x| \leq r, \quad$ *all* $\xi,\ 1 \leq k \leq N.$

The functions a_k *are further restricted by*

(3.5) $$|\gamma^k| + \alpha^k d = |\beta| + |\delta| \Longrightarrow a_k \quad \textit{depends on } y \textit{ only.}$$

It follows that the Goursat problem

(3.6) $$D_y^{\beta} D_x^{\delta} u = \Sigma\, a_k D_y^{\gamma^k} D_x^{\alpha^k} u + f, \; u = O(y^{\beta} x^{\delta})$$

has one and only one solution u *in* $\gamma_H(\beta, \delta, d)$.

PROOF. We compare the theorem above with Theorem 1 in [5]. We see that (3.2) has no counterpart in [5]. In fact (3.2) is vital for the proof below. On the other hand there is no restriction of the type (4.5) in [5] above. Since the coefficients a_k in (3.5) are holomorphic in C^s it would make these coefficients constant. We shall follow the proof of Theorem 1 in [5] very closely. So we shall only write down the points of difference.

Let λ be an arbitrary fixed number in $0 < \lambda < 1$. We use the successive approximations in (4.7) in [5] connected with equation (3.6) above. We choose an arbitrary fixed $r > 1$. Then it follows from the continuity of the coefficients that the sum

$$\Sigma\, |a_k(y,x)|, \quad |y|+|x| \leq r,$$

is bounded.

We now make the following coordinate transformation.

$$y'_j = t^2 y_j, \quad 1 \leq j \leq s, \quad x'_j = t^{d_j} x_j, \quad 1 \leq j \leq n, \quad t \geq 1.$$

The new coefficients are of the form

$$a'_k(y', x') = t^{2(|\gamma^k| - |\beta|) - d(\delta - \alpha^k)}\, a_k(y,x), \quad 1 \leq k \leq N.$$

We also have

$$f'(y', x') = t^{-2|\beta| - |\delta|} f(y,x).$$

It now follows from (3.1) and (3.2) that we can choose t so great that in the compact set D in the primed space that corresponds to $|y|+|x| \leq r$ in the original space,

(3.7) $$\Sigma\, |a'_k(y', x')| < \lambda, \quad (y', x') \in D.$$

We note here that (3.2) is crucial. We now fix t so that (3.7) is true together with (4.2)′ and (4.3)′ in [5]. Then the proof goes through just as in [5].

It should be noted that lemma 5 in [5] must be modified. The function $u(y, x)$ in the lemma is now defined in $C^s \times R^n$ and for fixed x u is holomorphic in y. The integrations in the y_j-variable in the proof of the lemma are now made radially from 0 to y_j in the complex plane. It is also to be noted that the functions from the successive approximations are all holomorphic in the y-variables. This is verified if we use a suitable representation of the functions by the Cauchy integral formula in the y-variables and then perform the eventual necessary differentiations in the x-variables under the integral sign in the Cauchy integral formula. That the solution $u(x, y)$ then is holomorphic in $y \in C^s$ for fixed $x \in R^n$ is an immediate consequence of the uniform convergence of the successive approximations on every compact set in the primed space. Thus $u \in \gamma_H(\beta, \delta, d)$. Theorem 1 is proved.

We now look at Theorem 1 in [5]. We add condition (3.2) above and delete (4.5) in [5]. Then the conclusion of the theorem is still true. This is evident from the proof above.

What says Theorem 1 when we specialize to a Cauchy problem? We only look at the case when $d_j = 1$, $1 \leq j \leq n$. Then u can be extended to a function holomorphic in $x \in C^n$ for every fixed $y \in C^s$. Hartog's theorem then says that the extended function is holomorphic in $C^s \times C^n$. For a proof of Hartog's theorem see Hörmander [3], p. 28. Let now $s = 1$ and let $n' = 0$. Then (3.1), (3.2) and (3.5) says that (3.6) is a non-characteristic Cauchy problem with coefficients in the principal part that depend on y only. Thus we have also proved the following theorem that generalizes Theorem 3 in [5].

THEOREM 2. *The multi-indices* $\delta = (\delta_1, 0, \dots, 0) \in R^n$ *and* $\alpha^k \in R^n$, $1 \leq k \leq N$, *are restricted by*

$$\alpha^k \neq \delta, \quad |\alpha^k| \leq \delta, \quad 1 \leq k \leq N.$$

The entire functions f, a_k, $1 \leq k \leq N$, *in* C^n *are restricted by*

$$|\alpha^k| = |\delta| \Longrightarrow a_k \text{ depends on } x_1 \text{ only.}$$

If follows that the Cauchy problem

$$D_x^\delta u = \Sigma\, a_k D_x^{\alpha^k} u + f, \quad u \doteq o(x^\delta),$$

has one and only one entire solution. Here D_x *denotes complex differentiation.*

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