ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

ALFRED AEPPLI
YOSHIO AKIYAMA

Cohomology operations in Smith theory

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 24, nº 4 (1970), p. 741-833
<http://www.numdam.org/item?id=ASNSP_1970_3_24_4_741_0>

© Scuola Normale Superiore, Pisa, 1970, tous droits réservés.

L'accès aux archives de la revue « Annali della Scuola Normale Superiore di Pisa, Classe di Scienze » (http://www.sns.it/it/edizioni/riviste/annaliscienze/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme Numérisation de documents anciens mathématiques
http://www.numdam.org/

COHOMOLOGY OPERATIONS IN SMITH THEORY*

ALFRED AEPPLI and YOSHIO AKIYAMA

If the action of $Z_p = \{1, h, h^2, \dots, h^{p-1}\}$ on the p-fold cartesian product $X \times X \times \dots \times X$ (p a prime, X a cellular complex, $h(x_1, x_2, \dots, x_p) = (x_p, x_1, x_2, \dots, x_{p-1})$) is studied in Smith theory, the Steenrod operations appear in a natural way in some crucial formulas (which we call the Thom-Bott formulas). This has been noticed by R. Thom in [8]. Then R. Bott [1], Wu Wen-Tsün [9, 11], M. Nakaoka [4] and others developed the theory further, in particular M. Nakaoka used Smith theory to establish the axiomatic characterization of the Steenrod operations, and Wu Wen-Tsün gave applications to imbedding and immersion problems.

Here we develop the theory of the Steenrod operations again, completely inside Smith theory. For this purpose, we give a brief introduction into Smith theory in § 1 (based on the action of Z_p on a complex E), and we generalize it in § 2 : we consider all powers t^k, $k = 1, 2, \dots, p-1$, instead of confining our attention only to $t = 1 - h^\#$ and $s = 1 + h^\# + \dots + h^{\# p-1} = t^{p-1}$ (mod. p). We define the t^k-special cohomology groups ${}^kH$ (with coefficients in Z_p) and the generalized Smith operations μ_k and ν_k which are represented by cup products with the Wu classes $\mu_0 = \mu(1)$ and $\nu_0 = \nu(1)$ (up to a constant factor, cf. Proposition 2.13). This more general version of the Smith theory turns out to be the adequate instrument to play on. The notions and propositions in standard Smith theory find their natural counterparts in the generalized framework, and the generalized theory is used in an essential way in the following §' s.

§ 3 deals with the case of the p-cyclic action on the p-fold product $E = X \times X \times \dots \times X$ as described above. The δ_k-image ${}^kN^n = \delta_k^{p-k} H^{n-1}(E, \Delta)$ ($\delta_k = k-th$ Smith coboundary, Δ = diagonal in E) which is contained in ${}^kH^n(E, \Delta)$ is written as a direct sum $\Sigma\ {}^k\varphi_i H^{n-i} \Delta$ (Thom direct sum decomposition, Theorem 3.14 and Proposition 3.15; ${}^k\varphi_i$ is a suitable compo-

Pervenuto alla Redazione il 19 Giugno 1970.

(*) Work supported by NSF Grant GP-8691.

sition of coboundary operators). This leads in § 4 to a canonical representation of $\gamma_k^i a^p (a \in H^m X, a^p = a \otimes a \otimes \ldots \otimes a \in H^{mp} X \times \ldots \times X, \gamma_k^i$ induced by t^k) which is the content of the Thom-Bott formulas (Theorem 4.21). At the same time the existence and axiomatic characterization of the Steenrod operations is established. Finally, the Adem relations are proved in § 5, again inside Smith theory. For the completeness of the Adem relations, cf. the book [7] by Steenrod and Epstein. [7] contains the axiom system for the Steenrod operations which we adopt in the sequel, and [7] may be considered as a general reference source for cohomology operations over the coefficients Z_p.

There are various extensions of the theory that are not treated here, e. g. cohomology operations over any abelian group as coefficients or even more generally over a sheaf of abelian groups. Furthermore, we do not mention any geometric applications; we refer again to [7] for some theorems in homotopy theory, or especially to [9] for imbedding and non-imbedding theorems proved with the help of Smith theory and Steenrod operations.

§ 1. Introduction to Smith Theory.

In this § we state and prove a few basic results of standard Smith theory — called « standard » in contrast with « generalized » Smith theory which will be discussed in § 2. A more extensive exposition of standard Smith theory can be found in [2], [5, *i*] and [9, Chapter II].

Let E be a topological space with a deck transformation $h : E \longrightarrow E$ of period p, i. e., we assume that the maps $h^k : E \longrightarrow E$ are homeomorphisms without fixed point, for $k = 1, 2, \ldots, p-1$, and that h^p is the identity map 1 of E onto itself. Then $G = \{1, h, h^2, \ldots, h^{p-1}\}$ is said to act on E as a deck transformation group. In the sequel, we assume that (E, G) is simplicial, i. e. E is a simplicial complex and h is a simplicial map, so that G is a group of simplicial maps, and simplicial homology and cohomology theory applies. In more general cases, one can use cellular or singular theory. The coefficient domain for cochains and cohomology groups will always be $G \cong Z_p$ unless otherwise stated.

The map $h : E \longrightarrow E$ induces a cochain map $h^\# : C^\# E \longrightarrow C^\# E$ and a chain map $h_\# : C_\# E \longrightarrow C_\# E$. We introduce the following notation:

$$s = 1 + h^\# + (h^\#)^2 + \ldots + (h^\#)^{p-1} : C^\# E \longrightarrow C^\# E,$$

$$t = 1 - h^\# : C^\# E \longrightarrow C^\# E,$$

where $1 : C^{\#} E \longrightarrow C^{\#} E$ denotes the identity map. s and t are cochain maps of $C^{\#} E$ into itself. If $p = 2$, note that $t = s$. We denote these cochain maps by ϱ and $\bar{\varrho}$, agreeing that ϱ may stand for s, $\bar{\varrho}$ for t or vice versa, but that the meaning of ϱ and $\bar{\varrho}$ shall remain fixed in any given discussion. Note that $\varrho\bar{\varrho} = \bar{\varrho}\varrho = 1 - (h^{\#})^p = 0$. Hence we have:

LEMMA 1.1. $\varrho\bar{\varrho} = \bar{\varrho}\varrho = 0$.

We denote the orbit space of E under $G \cong Z_p$ by E_0, i. e., $E_0 = E/G$. A simplex $\tilde{\sigma}$ of E_0 is then viewed as the set $\{\sigma, h\sigma, h^2\sigma, \dots, h^{p-1}\sigma\}$. If for each $\tilde{\sigma}$ of E_0 a cell σ of E is chosen, then the set F of cells σ thus chosen will be called a fundamental domain of E under G.

LEMMA 1.2. $\ker \varrho = \operatorname{im} \bar{\varrho}$.

PROOF. By (1.1) it is clear that $\operatorname{im} \bar{\varrho} \subset \ker \varrho$. We will prove the other inclusion in each of the following two cases.

CASE 1. $\varrho = t$.
Suppose a cochain $u \in C^n E$ is in $\ker t$. Then $u = h^{\#} u$, so that $u(\sigma) = u(h_{\#}\sigma) = u(h_{\#}^2\sigma) = \dots = u(h_{\#}^{p-1}\sigma)$, for any n-simplex σ of E. Let F be a fundamental domain of E for G. Define an n-cochain v by

$$v(\sigma) = \begin{cases} u(\sigma), \text{ for } n\text{-simplex } \sigma \text{ in } F \\ 0, \text{ otherwise.} \end{cases}$$

Then we see that $u = sv$. Hence, $\ker t \subset \operatorname{im} s$.

CASE 2. $\varrho = s$.
Suppose that a cochain u of $C^n E$ is in $\ker s$, i. e., $su = (1 + h^{\#} + \dots + (h^{\#})^{p-1})u = 0$. Then $\sum_{j=0}^{p-1} u(h_{\#}^j\sigma) = 0$, for any n-simplex σ of E. Define an n-cochain v by the formula $v(h_{\#}^i\sigma) = \sum_{j=i}^{p-1} u(h_{\#}^j\sigma)$, for an n-simplex σ in the fundamental domain F. Then $(tv)\sigma = v(\sigma) - v(h_{\#}\sigma) = \sum_{j=0}^{p-1} u(h_{\#}^j\sigma) - \sum_{j=1}^{p-1} u(h_{\#}^j\sigma) = u(\sigma)$, for any n simplex of E. Hence $tv = u$, which shows that $\ker s \subset \operatorname{im} t$. Q. E. D.

DEFINITION. The ϱ-special cohomology group of E, denoted by ${}^{\varrho}H^* E$, is defined to be $H^*(\ker \varrho)$.
By (1.2), ${}^{\varrho}H^* E = H^*(\ker \varrho) = H^*(\operatorname{im} \bar{\varrho})$.

Let $\pi : E \to E_0$ be the natural projection of E onto its orbit space $E_0 = E/G$. The following lemma describes the relation between t-special cohomology and the ordinary cohomology groups.

Lemma 1.3_t. *The projection $\pi : E \to E_0$ induces an isomorphism $\pi^* : H^* E_0 \to {}^tH^* E$.*

Proof. First, we observe that $\pi^\# : C^\# E_0 \to C^\# E$ is a monomorphism. This is easily seen from the formula

$$\pi^\# (\sum_i g_i \tilde{\sigma}_i) = \sum_i (g_i s) \sigma_i ,$$

where $g_i \in G$, $\sum_i g_i \tilde{\sigma}_i \in C^\# E_0$, and $\sigma_i \in \tilde{\sigma}_i$ is a simplex in a fundamental domain F.

Secondly, we show that $\pi^\# C^\# E_0 = \ker t$. By the above formula it is immediate that $\pi^\# (C^\# E_0) \subset \operatorname{im} s = \ker t$. On the other hand if a cochain u is in $\ker t$, then by (1.2) u is written as $u = s(\sum_i g_i \sigma_i)$ where σ_i is a simplex in a fundamental domain F and $g_i \in G$. Let $\tilde{\sigma}_i$ be the simplex of E_0 such that $\tilde{\sigma}_i = \{\sigma_i, h\sigma_i, h^2 \sigma_i, \dots, h^{p-1} \sigma_i\}$. Then $\pi^\# (\sum_i g_i \tilde{\sigma}_i) = u$. Thus, $\pi^\# (C^\# E_0) \supset \supset \ker t$.

Therefore $\pi^\#$ defines an isomorphism $\pi^\# : C^\# E_0 \cong \ker t$. Since the coboundary operation δ commutes with $\pi^\#$, the map $\pi^\# : C^\# E_0 \cong \ker t$ induces an isomorphism $\pi^* : H^*(E_0) \cong {}^tH^*(E)$. Q. E. D.

In view of (1.3_t) we will identify the two groups $H^* E_0$ and ${}^tH^* E$.

The description of the relation between s-special cohomology groups and the ordinary cohomology groups is more complicated. Let $\oplus_p G = = G \oplus G \oplus \dots \oplus G$ be the p-fold direct sum of $G \cong Z_p$ with itself and let $\oplus_p^0 G$ denote its subgroup consisting of all elements $(g_1, g_2, \dots, g_p) \in \oplus_p G$ such that $\sum_{i=1}^{p} g_i = 0$ in G. Define a map $\bar{\pi}^\# : {}^sC^\# (E; G) \equiv \ker s \to C^\# (E_0; \oplus_p^0 G)$ as follows. Choose a fundamental domain F of E, then any element x of ${}^sC^\# (E; G)$ will be written as $x = \sum_i \sum_{j=1}^{p} g_{ij} h^j \sigma_i$ where the σ_i's are in F and $g_{ij} \in G$. Also $sx = \sum_i \sum_{j=1}^{p} \left(\sum_{k=1}^{p} g_{ik} \right) h^j \sigma_i = 0$ since x is an element of ${}^sC^\# (E; G)$, which implies $\sum_{k=1}^{p} g_{ik} = 0$ for all i. Set $\bar{\pi}^\# (x) = \sum_i (g_{i1}, g_{i2}, \dots, g_{ip}) \tilde{\sigma}_i$ where $\tilde{\sigma}_i$ are simplexes of E_0 such that $\pi \sigma_i = \tilde{\sigma}_i$. It is obvious that $\bar{\pi}^\# x$ is in $C^\# (E_0; \oplus_p^0 G)$. Moreover, it can be easily shown that $\bar{\pi}^\#$ is an isomorphism. Let $\delta^0 : C^\# (E_0; \oplus_p^0 G) \to C^\# (E_0; \oplus_p^0 G)$ be the coboundary operator given by the relation $\delta^0 = \bar{\pi}^{\#} \circ \delta \circ (\bar{\pi}^{\#})^{-1}$ where δ is the ordinary coboundary operator. Then $\delta^0 \bar{\pi}^{\#} = \bar{\pi}^{\#} \delta (\bar{\pi}^{\#})^{-1} \bar{\pi}^{\#} = \bar{\pi}^{\#} \delta$. Hence we have the following lemma:

LEMMA 1.3$_s$. Let $\oplus_p G$ be the p-fold direct sum of $G \cong Z_p$ and $\oplus_p^0 G$ be its subgroup consisting of all elements $(g_1, g_2, \dots, g_p)$ such that $\sum_{i=1}^{p} g_i = 0$. Let $\bar{\pi}^{\#} : {}^sC^{\#}(E; G) \longrightarrow C^{\#}(E_0; \oplus_p^0 G)$ be defined as above. Then $\bar{\pi}^{\#}$ induces an isomorphism

$$\bar{\pi}^* : {}^sH^*(E; G) \longrightarrow H^*(E_0; \oplus_p^0 G).$$

REMARK. Let $p = 2$. Then s and t are the same maps; and (1.3$_t$) and (1.3$_s$) imply that

$$H^*(E_0; Z_2) \cong {}^tH^*(E; Z_2) \cong {}^sH^*(E; Z_2).$$

Notice that $\oplus_2^0 Z_2 \cong Z_2$.

We have the following short exact sequence of cochain maps of cochain complexes:

$$0 \longrightarrow \ker \varrho \xrightarrow{\iota_\varrho} C^{\#} E \xrightarrow{\varrho} \operatorname{im} \varrho \longrightarrow 0$$

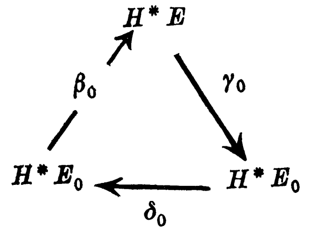
where ι_ϱ is the inclusion map. Hence, by the Kelley-Pitcher theorem it induces the following exact triangle:

$$\begin{array}{ccc} & H^* E & \\ {}^{\beta_\varrho}\nearrow & & \searrow^{\gamma_\varrho} \\ {}^{\varrho}H^* E = H^*(\ker \varrho) & \xleftarrow[\delta_\varrho]{} & H^*(\operatorname{im} \varrho) = {}^{\bar{\varrho}}H^* E, \end{array}$$

where γ_ϱ and β_ϱ are induced by ϱ and ι_ϱ, respectively, and $\delta_\varrho : {}^{\bar{\varrho}}H^n E \longrightarrow {}^{\varrho}H^{n+1} E$ is the Smith coboundary operator given by $\delta_\varrho [\varrho x] = [\delta x]$ for all $[\varrho x]$ in ${}^{\bar{\varrho}}H^n E$ and $n = 0, 1, 2, \dots$. δ denotes the ordinary coboundary operator. Notice that any element of ${}^{\bar{\varrho}}H^n E$ takes the form $[\varrho x]$, and that $[\delta x]$ is in ${}^{\varrho}H^{n+1} E$ because $[\varrho\, \delta x] = [\delta\, \varrho x] = 0$. This leads us to the following proposition.

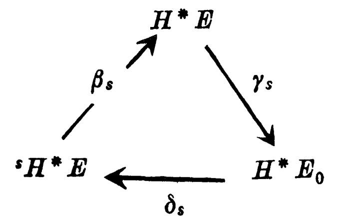
PROPOSITION 1.4 (*Richardson-Smith*). *For the fibration* $\{G \longrightarrow E \xrightarrow{\pi} E_0\}$ *described earlier, there are the following exact triangles*:

(1) *When* $p = 2$,

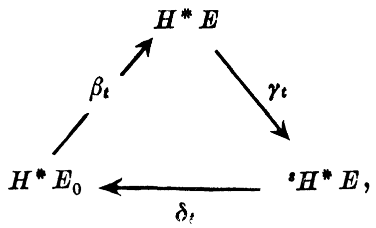


where γ_0 *and* β_0 *are induced by* $\varrho : C^{\#} E \longrightarrow \mathrm{im}\,\varrho$ *and the inclusion map* $\iota_\varrho : \ker \varrho \longrightarrow$ $\longrightarrow C^{\#} E$, *respectively, and* $\delta_0 : H^n E_0 \longrightarrow H^{n+1} E_0$ *is the Smith coboundary operation* $\delta_0 = \delta_\varrho = \delta_{\bar{\varrho}}$.

(2) *When* p *is an odd prime, we have two exact triangles*



and



where γ_ϱ *and* β_ϱ *are induced by* ϱ *and* ι_ϱ, *respectively, and* δ_ϱ *is the Smith coboundary operator defined before.*

Until now we have always assumed that the map h has no fixed points and we did not consider invariant sets in E. We shall relativize the preceding absolute situation in the following two cases.

Case 1. The group $G \cong Z_p$ acts on the space E leaving a subset L of E invariant under G (not necessarily pointwise), acting on $E - L$ as a deck transformation group. We assume that everything is simplicial. Let E_0 and L_0 denote the orbit spaces of E and L, respectively, under G; i. e. $E_0 = E/G$ and $L_0 = L/G$, where $L_0 \subset E_0$.

Lemma 1.5. In Case 1, consider the maps $\varrho, \bar{\varrho} : C^{\#}(E, L) \longrightarrow C^{\#}(E, L)$. Then $\ker \varrho = \mathrm{im}\,\bar{\varrho}$.

Remark. $t = 1 - h^{\#}$ and $s = 1 + h^{\#} + \dots + (h^{\#})^{p-1}$ make sense both in the absolute and the relativized cases. Hence we use ϱ and $\bar{\varrho}$ again for the relativized cases, as their meaning should be clear from the context.

The proof of (1.5) is similar to the proof of (1.2).

DEFINITION. The relativized ϱ-cochain group ${}^{\varrho}C^{\#}(E, L)$ is defined to be $\ker \varrho$, where $\varrho : C^{\#}(E, L) \to C^{\#}(E, L)$. In view of (1.4) ${}^{\varrho}C^{\#}(E, L) = \ker \varrho = \operatorname{im} \bar{\varrho}$.

DEFINITION. The relativized ϱ-special cohomology group ${}^{\varrho}H^{\#}(E, L)$ is defined to be $H^{\#}(\ker \varrho)$, where $\varrho : C^{\#}(E, L) \to C^{\#}(E, L)$ is the relativized map of Case 1.

As in the absolute case, the short exact sequence

$$0 \longrightarrow {}^{\varrho}C^{\#}(E, L) \xrightarrow{\iota_\varrho} C^{\#}(E, L) \xrightarrow{\varrho} {}^{\bar{\varrho}}C^{\#}(E, L) \longrightarrow 0$$

induces the exact triangle

$$\begin{array}{ccc} & H^{*}(E, L) & \\ {\scriptstyle \beta_\varrho} \nearrow & & \searrow {\scriptstyle \gamma_\varrho} \\ {}^{\varrho}H^{*}(E, L) & \xleftarrow[\delta_\varrho]{} & {}^{\bar{\varrho}}H^{*}(E, L). \end{array}$$

Also, the map $(\pi \mid_{E-L})^{\#} : C^{\#}(E_0, L_0) \to {}^{t}C^{\#}(E, L)$ is an isomorphism and yields an isomorphism $\pi^{*} : H^{*}(E_0, L_0) \to {}^{t}H^{*}(E, L)$. Hence (1.4) has the following counterpart.

PROPOSITION 1.6. *In the relativized Case* 1, *where* $(E, L)/G = (E_0, L_0)$, *we have the following exact triangles:*

(1) *When* $p = 2$,

$$\begin{array}{ccc} & H^{*}(E, L) & \\ {\scriptstyle \beta_0} \nearrow & & \searrow {\scriptstyle \gamma_0} \\ H^{*}(E_0, L_0) & \xleftarrow[\delta_0]{} & H^{*}(E_0, L_0) \end{array}$$

where γ_0 *and* β_0 *are induced by the maps* $\varrho = \bar{\varrho} = 1 + h^{\#} : C^{\#}(E, L) \to {}^{\varrho}C^{\#}(E, L)$ *and* $\iota_\varrho = \iota_{\bar{\varrho}} : {}^{\varrho}C^{\#}(E, L) \to C^{\#}(E, E)$, *respectively, and* $\delta_0 : H^{n}(E_0, L_0) \to H^{n+1}(E_0, L_0)$ *is the Smith coboundary operator,* $n = 0, 1, 2, \dots$.

(2) *When p is and odd prime, we have the two exact triangles*

$$\begin{array}{ccc} & H^*(E,L) & \\ {}^{\beta_s}\nearrow & & \searrow^{\gamma_s} \\ {}^{s}H^*(E,L) & \xleftarrow[\delta_s]{} & H^*(E_0,L_0) \end{array} \quad \text{and} \quad \begin{array}{ccc} & H^*(E,L) & \\ {}^{\beta_t}\nearrow & & \searrow^{\gamma_t} \\ H^*(E_0,L_0) & \xleftarrow[\delta_t]{} & {}^{s}H^*(E,L) \end{array}$$

where β_ϱ, γ_ϱ and δ_ϱ are similarly defined as in (1).

Case 2. In the second case of relativization, the group $G \cong Z_p$ acts on E leaving a subset L of E pointwise fixed and G is a deck transformation group on $E - L$. Let E_0 and L_0 be the orbit spaces of E and L, respectively, under G. Then, $L = L_0 \subset E_0$.

Lemma 1.7. In Case 2,

$$\mathrm{im}\,(s \mid C^\# E) = \ker\,(t \mid C^\# (E, L))$$

and

$$\mathrm{im}\,(t \mid C^\# E) = \ker\,(s \mid C^\# (E, L)).$$

Remark. In Case 2, where L is pointwise fixed under G we have the maps

$$s\,|_{C^\# E} : C^\# E \longrightarrow C^\# (E, L)$$

and

$$t\,|_{C^\# E} : C^\# E \longrightarrow C^\# (E, L)$$

as well as

$$s\,|_{C^\# (E, L)} : C^\# (E, L) \longrightarrow C^\# (E, L)$$

and

$$t\,|_{C^\# (E, L)} : C^\# (E, L) \longrightarrow C^\# (E, L).$$

Hence the statement of (1.7) makes sense, and its proof is similar to that of (1.2).

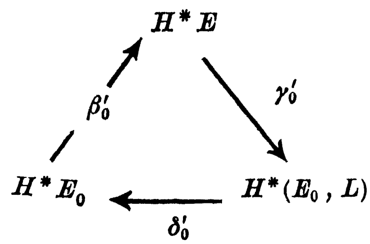
We have the following short exact sequence

$$0 \longrightarrow \ker\,(\varrho\,|_{C^\# E}) \xrightarrow{\ \iota_\varrho\ } C^\# E \xrightarrow{\ \varrho\ } \mathrm{im}\,(\varrho\,|_{C^\# E}) \longrightarrow 0$$

which leads to

Proposition 1.8. *In the relativized Case 2, where $(E, L)/G = (E_0, L)$, we have the following exact triangles*:

(1) *When* $p = 2$

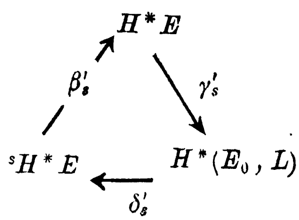


where γ_0' *and* β_0' *are induced by*

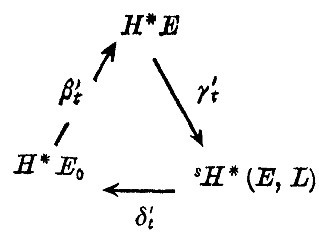
$$\varrho : C^\# E \longrightarrow \operatorname{im}(\varrho \mid C^\# E) = \ker(\varrho \mid C^\# (E, L)) \qquad \text{and}$$

$\iota_\varrho : \ker(\varrho \mid C^\# E) \longrightarrow C^\#$, *respectively, and* δ_0' *is the Smith coboundary operator.*

(2) *When* p *is an odd prime, we have the two exact triangles*:



and



where β_ϱ', γ_ϱ' *and* δ_ϱ' *are similarly defined as before in the obvious manner.*

Hereafter, unless we explicity mention otherwise, our statements will be concerned with Case 1, i. e. $(E_0, L_0) = (E, L)/G$. We can reduce a statement to the absolute situation by letting $L = \emptyset$. Moreover, since Case 2 is a special subcase of Case 1, it is enough to treat Case 1.

Definition. Let $\xi_\varrho : C^\#(E, L) \longrightarrow C^\#(E, L)$ be the map defined by $\xi_s = 1$ (identity) and $\xi_t = t^{p-2}$.

Lemma 1.9. $\xi_\varrho \varrho = s$.

Proof. If $\varrho = t$, then $\xi_t t = t^{p-2} t = t^{p-1} = \sum_{i=0}^{p-1} (-1)^i \binom{p-1}{i} (h^\#)^i \equiv s$ because $\binom{p-1}{1} \equiv (-1)^i \bmod p$. If $\varrho = s$, $\xi_s s = s$. Hence $\xi_\varrho \varrho = s$. Q.E.D.

Corollary 1.10.

$$\xi_\varrho \varrho = \varrho \xi_\varrho = \xi_{\bar\varrho} \bar\varrho = \bar\varrho \xi_{\bar\varrho} .$$

The proof is immediate from (1.9) as ϱ and ξ_ϱ commute.

Lemma 1.11. ξ_ϱ induces the switching homomorphism

$$\xi_\varrho^* : {}^{\bar\varrho}H^*(E, L) \to {}^{\varrho}H^*(E, L).$$

Proof. Let $\varrho = s$. Choose an arbitrary element $[u]$ of ${}^tH^*(E, L)$. Then $tu = u - h^\# u = 0$; hence $u = h^\# u$. Thus, $s(\xi_s u) = su = u + h^\# u + (h^\#)^2 u + \dots + (h^\#)^{p-1} u = pu \equiv 0$, which implies that $[\xi_s u] = \xi_s^* [u]$ is an element of ${}^sH^* E$, i. e. $\xi_s^* : {}^tH^*(E, L) \to {}^sH^*(E, L)$.

Next, let $\varrho = t$. Let $[v]$ be any element of ${}^sH^*(E, L)$, so that $sv = 0$. Since $0 = sv = t^{p-1} v = tt^{p-2} v = t\xi_t v$, we conclude that $\xi_t v$ is a t cocycle and hence $\xi_t^*[v]$ is an element of ${}^tH^*(E, L)$. Q. E. D.

Lemma 1.12.

(1) $$\xi_t s \equiv \begin{cases} 0, & \text{if } p \text{ is an odd prime} \\ s, & \text{if } p = 2. \end{cases}$$

(2) $$\xi_t^* \xi_s^* = \begin{cases} 0 : & {}^tH^*(E, L) \to {}^tH^*(E, L), \text{ if } p \text{ is an odd prime} \\ 1 : & {}^tH^*(E, L) \to {}^tH^*(E, L), \text{ if } p = 2. \end{cases}$$

Proof. (1) Suppose p is an odd prime. Then $\xi_t s = t^{p-2} t^{p-1} = t^{2p-3} = 0$, because $2p - 3 \geq p$. If $p = 2$, $\xi_t s = \xi_t t = s$ by (1.9).

(2) Let $[sx]$ be an element of ${}^tH^*(E, L)$. Then $\xi_t^* \xi_s^* [sx] = [t^{p-2} sx] = [t^{2p-3} x] = 0$ if p is an odd prime. When $p = 2$, $[\xi_t^* \xi_s^* sx] = [sx]$ is obvious. Q. E. D.

Lemma 1.13. The switching homomorphisms and the Smith coboundary operators commute, i. e.

$$\xi_{\bar\varrho}^* \delta_\varrho = \delta_{\bar\varrho} \xi_\varrho^* : {}^{\bar\varrho}H^*(E, L) \to {}^{\bar\varrho}H^*(E, L).$$

Proof. Let $[\varrho u]$ be an element of ${}^{\bar\varrho}H^*(E, L)$. Then

$$\begin{aligned} \xi_{\bar\varrho}^* \delta_\varrho [\varrho u] &= \xi_{\bar\varrho}^* [\delta u] \text{ by the definition of } \delta_\varrho \\ &= [\xi_{\bar\varrho} \delta u] \\ &= [\delta \xi_{\bar\varrho} u] \end{aligned}$$

$$= \delta_\varrho^- [\overline{\varrho} \; \xi_\varrho^- u]$$

$$= \delta_\varrho^- [\xi_\varrho^- \overline{\varrho u}]$$

$$= \delta_\varrho^- [\xi_\varrho \, \varrho u] \quad \text{by (1.10)}$$

$$= \delta_\varrho^- \xi_\varrho^* [\varrho u].$$

Hence $\xi_{\bar\varrho}^* \delta_\varrho = \delta_\varrho^- \xi_\varrho^*$. Q. E. D.

Now we are going to define two operators μ and ν, called Smith operations, which will play an important role in the later §′ s.

DEFINITION.

$$\mu = \delta_t \, \delta_s : H^k(E_0, L_0) \longrightarrow H^{k+2}(E_0, L_0).$$

$$\nu = \delta_t \, \xi_s^* : H^k(E_0, L_0) \longrightarrow H^{k+1}(E_0, L_0).$$

Since $H^*(E_0, L)$ and ${}^tH^*(E, L)$ are identified by π^*, we can also think of μ and ν as maps from ${}^tH^*(E, L)$ to ${}^tH^*(E, L)$ with degrees 2 and 1, respectively.

LEMMA 1.14. (1) μ and ν commute.

(2) $$\nu^2 = \begin{cases} 0 & \text{if } p \text{ is an odd prime} \\ \mu & \text{if } p = 2. \end{cases}$$

PROOF. (1) $\mu\nu = \delta_t \delta_s \delta_t \xi_s^*$ by definition

$$= \delta_t \delta_s \delta_t^* \delta_s \quad \text{by (1.13)}$$

$$= \delta_t \delta_s^* \delta_t \delta_s \quad \text{by (1.13)}$$

$$= \nu\mu \quad \text{by definition.}$$

(2) Let p be an odd prime, then

$$\nu^2 = \delta_t \xi_s^* \delta_t \xi_s^* \quad \text{by definition}$$

$$= \delta_t \delta_s \xi_t^* \xi_s^* \quad \text{by (1.13)}$$

$$= 0 \quad \text{by (1.12.2).}$$

If $p = 2$, $\nu^2 = \delta_t \delta_s \xi_t^* \xi_s^* = \delta_t \delta_s = \mu$ by (1.12.2). Q. E. D.

LEMMA 1.15. Given any element $[u]$ of $H^k(E_0, L_0)$, there exist cochains v^0, v^1, and v^2 such that $v^i \in C^{k+i}(E, L)$, $i = 0, 1, 2$; $u = sv^0$, $\delta v^0 = tv^1$, $\delta v^1 = sv^2$, $\nu[u] = [sv^1]$ and $\mu[u] = [sv^2]$.

PROOF. Since $[u]$ is in ${}^tH^k(E, L) \cong H^k(E_0, L_0)$, u is t-cocyle, i. e. $u = sv^0$ for some cochain v^0 in $C^k(E, L)$. Now $\delta_s[u] = \delta_s[sv^0] = [\delta v^0]$ is an element of ${}^sH^{k+1}(E, L)$ which implies that $\delta v^0 = tv^1$ for some $(k+1)$-cochain v^1 of $C^{k+1}(E, L)$. The class $\delta_t[tv^1] = [\delta v^1]$ belongs to ${}^tH^{k+2}(E, L)$, hence δv^1 is a t-cocycle. In particular, δv^1 is in im s, i. e., $\delta v^1 = sv^2$ with $v^2 \in C^{k+2}(E, L)$.

Finally, $\nu[u] = \delta_t \xi_s^*[sv^0] = \xi_t^* \delta_s[sv^0]$ by (1.13)

$$= \xi_t^*[\delta v^0]$$

$$= \xi_t^*[tv^1]$$

$$= [sv^1] \quad \text{by (1.9),}$$

and $\mu[u] = \delta_t \delta_s[sv^0] = \delta_t[\delta v^0] = \delta_t[tv^1] = [\delta v^1] = [sv^2]$. Q. E. D.

COROLLARY 1.16. Let 1 denote the class of $H^0(E_0 - L_0)$ ($\cong {}^tH^0(E - L)$) given by the 0-cocycle which is 1 on every vertex. Then there exist cochains c^i in $C^i(E - L)$, $i = 0, 1, 2$, such that $1 = sc^0$, $\delta c^0 = tc^1$, $\delta c^1 = sc^2$, $\nu_0 = \nu(1) = [sc^1]$ and $\mu_0 = \mu(1) = [sc^2]$.

The notation $\mu_0 = \mu(1)$ and $\nu_0 = \nu(1)$ will be frequently used in the sequel. They are called the Wu classes of the fibration $\{Z_p \to E - L \to \to E_0 - L_0\}$

LEMMA 1.17 (*Wu*). For any cochain x of $C^\#(E, L)$ and for any t-cochain y, we have $s(x \cup y) = (sx) \cup y$ and $t(x \cup y) = (tx) \cup y$.

PROOF. Since y is a t-cochain, $y = (h^\#)^j y$ for any j.

$$s(x \cup y) = \sum_{j=0}^{p-1} (h^\#)^j (x \cup y) = \sum_{j=0}^{p-1} (h^\#)^j x \cup (h^\#)^j y = \sum_{j=0}^{p-1} (h^\#)^j x \cup y = (sx) \cup y ;$$

and

$$t(x \cup y) = x \cup y - h^\#(x \cup y) = x \cup y - h^\# x \cup h^\# y = x \cup y - (h^\# x) \cup y = (tx) \cup y.$$

Q. E. D.

PROPOSITION 1.18. *For any element* $[x]$ *of* $H^*(E_0, L_0)$, *we have the following properties of* μ *and* ν :

$$(1) \qquad \mu[x] = \mu_0 \cup [x]$$

and

(2) $$\nu[x] = \nu_0 \cup [x].$$

Proof. By (1.16), there exist cochains c^i in $C^{\#}(E - L)$ such that $\mu_0 = [sc^2]$ and $\nu_0 = [sc^1]$ with the rest of the properties stated there. Therefore,

$$\begin{aligned}
\mu[x] &= \mu[1 \cup x] && \text{where } 1 \in C^0(E - L) \\
&= \mu[sc^0 \cup x] && \text{by (1.16)} \\
&= \mu[s(c^0 \cup x)] && \text{by (1.17)} \\
&= \delta_t[\delta(c^0 \cup x)] && \text{by the definition of } \mu = \delta_t \delta_s \\
&= \delta_t[(\delta c^0) \cup x] && \text{since } \delta x = 0 \\
&= \delta_t[(tc^1) \cup x] && \text{by (1.16)} \\
&= \delta_t[t(c^1 \cup x)] && \text{by (1.17)} \\
&= [\delta(c^1 \cup x)] && \\
&= [(\delta c^1) \cup x] && \text{because } \delta x = 0 \\
&= [(sc^2) \cup x] && \text{by (1.16)} \\
&= \mu_0 \cup [x].
\end{aligned}$$

Also

$$\nu[x] = \nu[1 \cup x] = \nu[(sc^0) \cup x] = \xi_t^* \delta_s[s(c^0 \cup x)] = \xi_t^*[\delta(c^0 \cup x)] =$$

$$= \xi_t^*[(\delta c^0) \cup x] = \xi_t^*[t(c^1 \cup x)] = [s(c^1) \cup x)] = [(sc^1) \cup x] = \nu_0 \cup [x]. \quad \text{Q.E.D.}$$

Remark. Although μ_0 is an element of $H^2(E_0 - L_0)$, the cup product with $[x] \in H^*(E_0, L_0) \cong {}^tH^*(E, L)$ yields an element $\mu_0 \cup [x]$ of $H^*(E_0, L_0)$. A similar statement applies to ν_0 and $\nu_0 \cup [x]$.

Suppose that $G \cong Z_p$ acts on spaces E' and E'' as a deck transformation group of prime period p, i. e., we assume there exist fixed point free homeomorphism $h' : E' \to E'$ and $h'' : E'' \to E''$ of prime period p.

Then we also have a homeomorphism $h' \times h'' : E' \times E'' \to E' \times E''$ on the product space. Clearly $h' \times h''$ has the period p. Denote the orbit spaces of E', E'' and $E' \times E''$ by E_0', E_0'' and $(E' \times E'')_0$, respectively. We have coverings $\{G \to E' \to E_0'\}, \{G \to E'' \to E_0''\}$ and $\{G \to E' \times E'' \xrightarrow{\pi_1} (E' \times E'')_0\}$. Define a map $h : (E' \times E'')_0 \to (E' \times E'')_0$ by $h(\pi_1(a' \times a'')) = \pi_1(h' a' \times a'')$.

h induces a covering $\{G \longrightarrow (E' \times E'')_0 \xrightarrow{\pi_2} E_0' \times E_0''\}$. Let ν_0', μ_0' be the Wu classes for E_0'; ν_0'', μ_0'' be the Wu classes for E_0''; and ν_0, μ_0 be the Wu classes for $E_0' \times E_0''$. The fallowing lemma describes the relation between ν_0, μ_0, and $\nu_0', \mu_0', \nu_0'', \mu_0''$.

Lemma 1.19.

(1) $\nu_0 = \nu_0' \otimes 1 + 1 \otimes \nu_0''$, and

(2) $\mu_0 = \mu_0' \otimes 1 - 1 \otimes \mu_0''$.

Proof. Let [1] be the class of $H^0(E_0' \times E_0'')$ given by the 0-cocycle 1 which is 1 et every vertex. Then, $\pi_1^\# \pi_2^\# (1) = 1_{E'} \times 1_{E''} \in C^0(E' \times E'')$, where $1_{E'}$ and $1_{E''}$ are the 0-cocycles which take the value 1 everywhere. By (1.16), there are cochains c^0, c^1 and c^2 such that $c^i \in C^i(E')$, $1_{E'} = s' c^0$, $\delta c^0 = t' c^1$, $\delta c^1 = s' c^2$, $\nu_0' = [s' c^1]$, and $\mu_0' = [s' c^2]$ where $t' = 1 - (h'^\#)$ and $s' = 1 + (h'^\#) + (h'^\#)^2 + \dots + (h'^\#)^{p-1}$. Similarly there are cochains v^0, v^1 and v^2 such that $v^i \in C^i(E'')$, $1_{E''} = s'' v^0$, $\delta v^0 = t'' v^1$, $\delta v^1 = s'' v^2$, $\nu_0'' = [s'' v^1]$ and $\mu_0'' = [s'' v^2]$ where t'' and s'' are defined by $t'' = 1 - (h''^\#)$ and $s'' = 1 + (h''^\#) + (h''^\#)^2 + \dots + (h''^\#)^{p-1}$. Let $t, s : C^\#((E' \times E'')_0) \longrightarrow C^\#((E' \times E'')_0)$ be defined as $t = 1 - h^\#$ and $s = 1 + h^\# + \dots + (h^\#)^{p-1}$. Then,

$$\begin{aligned}
\pi_2^\#(1) &= (\pi_1^\#)^{-1} \pi_1^\# \pi_2^\#(1) \\
&= (\pi_1^\#)^{-1} (1_{E'} \times 1_{E''}) \\
&= (\pi_1^\#)^{-1} (1_{E'} \times s'' v^0) \\
&= (\pi_1^\#)^{-1} \{1_{E'} \times (v^0 + h''^\# v^0 + \dots + (h''^\#)^{p-1} v^0)\} \\
&= (\pi_1^\#)^{-1} \{(1 + (h' \times h'')^\# + \dots + ((h' \times h'')^\#)^{p-1}) (1_{E'} \times v^0)\} \\
&= (\pi_1^\#)^{-1} \pi_1^\# (1_{E'} \times v^0)_B \\
&= (1_{E'} \times v^0)_B \\
&= (s' c^0 \times v^0)_B = \{((1 + h'^\# + \dots + (h'^\#)^{p-1}) c^0) \times v^0\}_B \\
&= (1 + h^\# + \dots + (h^\#)^{p-1}) (c^0 \times v^0)_B \\
&= s (c^0 \times v^0)_B,
\end{aligned}$$

where $(\dots)_B$ denotes the cochain $(\dots)$ in the orbit space $B = (E' \times E'')_0$.

We can assume that the simplexes appearing in $(1_{E'} \times v^0)$ are all in a fundamental domain F.

$$\nu_0 = \xi_t^* \delta_s[1] = \xi_t^* \delta_s[\pi_2^\# 1] = \xi_t^* \delta_s[s(c^0 \times v^0)_B] \tag{1}$$

$$= \xi_t^* [\delta (c^0 \times v^0)_B]$$

$$= \xi_t^* [(\delta c^0 \times v^0)_B + (c^0 \times \delta v^0)_B]$$

$$= \xi_t^* [(t' c^1 \times v^0)_B + (c^0 \times t'' v^1)_B]$$

$$= [\xi_t^\# (t' c^1 \times v^0)_B + (\xi_t^\# (c^0 \times t'' v^1))_B]$$

$$= [((\xi_{t'}^\# t' c^1) \times v^0)_B + ((\xi_{t'}^\# c^0) \times t'' v^1)_B]$$

$$= [(s' c^1 \times v^0)_B + ((1 - h'^\#)^{p-2} c^0 \times t'' v^1)_B]$$

$$= \Bigg[s' c^1 \times v^0)_B + \Bigg(\Bigg\{1 - \binom{p-2}{1} h'^\# + \\ + \binom{p-2}{2} (h'^\#)^2 \mp \dots + \binom{p-2}{p-2} (h'^\#)^{p-2} \Bigg\} c^0 \times t'' v^1 \Bigg)_B\Bigg]$$

$$= \Bigg[(s' c' \times v^0)_B + \Bigg(c^0 \times t'' \Bigg\{1 - \binom{p-2}{1} (h''^\#)^{p-1} + \\ + \binom{p-2}{2} (h''^\#)^{p-2} \mp \dots + \binom{p-2}{p-2} (h''^\#)^2 \Bigg\} v^1 \Bigg)_B\Bigg]$$

$$= \Bigg[(s' c^1 \times v^0)_B + \Bigg(c^0 \times (h''^\#)^2 t'' \Bigg\{1 - \binom{p-2}{1} (h''^\#) + \\ \pm \dots - \binom{p-2}{p-1} (h''^\#)^{p-3} + \binom{p-2}{p-2} (h''^\#)^{p-2} \Bigg\} v^1 \Bigg)_B\Bigg]$$

$$= [(s' c^1 \times v^0)_B + (c^0 \times (h''^\#)^2 (t'')^{p-1} v^1)_B]$$

$$= [(s' c^1 \times v^0)_B + (c^0 \times (h''^\#)^2 s'' v^1)_B]$$

$$= [(s' c^1 \times v^0)_B + (c^0 \times (s'' v^1))_B] = [(\nu_0' \times v^0)_B + (c^0 \times \nu_0'')_B],$$

which can be identified with $(\nu_0' \otimes 1 + 1 \otimes \nu_0'')$ of $H^1(E_0' \times E_0'')$.

(2) $\mu_0 = \delta_t \delta_s [1] =$

$= \delta_t[(t' c^1 \times v^0)_B + (c^0 \times t'' v^1)_B]$ by part (1)

$= \delta_t[t(c^1 \times v^0)_B + (c^0 \times v^1)_B - (c^0 \times h''^{\#} v^1)_B]$

$= \delta_t[t(c^1 \times v^0)_B + ((1-(h'^{\#})^{p-1}) c^0 \times v^1)_B]$

$= [\delta(c^1 \times v^0)_B + \delta((1 + h'^{\#} + \dots + (h'^{\#})^{p-2}) c^0 \times v^1)_B]$

$= [(s' c^2 \times v^0)_B - (c^1 \times t'' v^1)_B + ((1 + h'^{\#} + \dots + (h'^{\#})^{p-2}) t' c^1 \times v^1)_B +$

$+ ((1 + h'^{\#} + \dots + (h'^{\#})^{p-2}) c^0 \times s'' v^2)_B].$

Now

$-(c^1 \times t'' v^1)_B + ((1 + h'^{\#} + \dots + (h'^{\#})^{p-2}) t' c^1 \times v^1)_B$

$= -(c^1 \times t'' v^1)_B + (\{(1-(h'^{\#})^{p-1}) c^1\} \times v^1)_B$

$= -(c^1 \times t'' v^1)_B + (c^1 \times t'' v^1)_B = 0$

and

$((1 + h'^{\#} + \dots + (h'^{\#})^{p-2}) c^0 \times s'' v^2)_B$

$= (c^0 \times s''(1 + (h''^{\#})^{p-1} + \dots + (h''^{\#})^2) v^2)_B$

$= (c^0 \times (-s'' v^2))_B$

$= -(c^0 \times s'' v^2)_B.$

Hence,

$\mu_0 = [(s' c^2 \times v^0)_B - (c^0 \times s'' v^2)_B]$

$= [s(c^2 \times v^0)_B - (c^0 \times (1 + h''^{\#} + \dots + (h''^{\#})^{p-1}) v^2)_B]$

$= [s(c^2 \times v^0)_B - ((1 + h'^{\#} + \dots + (h'^{\#})^{p-1}) c^0 \times v^2)_B]$

$= [s(c^2 \times v^0)_B - s(c^0 \times v^2)_B]$

$= [\pi_2^{\#}(\tilde{c}^2 \times \tilde{v}^0 - \tilde{c}^0 \times \tilde{v}^2)]$ where $\tilde{c}^i \in C^i(E_0')$ and $\tilde{v}^i \in C^i(E_0'')$.

Hence under the isomorphism of $\pi_2^{\#}$, μ_0 can be identified with $[\tilde{c}^2 \times \tilde{v}^0 - \tilde{c}^0 \times \tilde{v}^2]$. Now, $\mu_0' = [s' c^2]$, $\mu_0'' = [s'' v^2]$, $1_{E'} = [s' c^0]$ and $1_{E''} = [s'' v^0]$ are identified with $[\tilde{c}^2]$, $[\tilde{v}^2]$, $[\tilde{c}^0]$ and $[\tilde{v}^0]$, respectively.

Hence $\mu_0 = \mu_0' \otimes 1 - 1 \otimes \mu_0''$ under the identification. Q.E.D.

Our final remark in this § is concerned with the naturality of the Smith operations. Suppose that $G \cong Z_p$ acts as a deck transformation group on E and $\bar{E}$ with generators h and $\bar{h}$. Let E_0 and $\bar{E}_0$ be the orbit spaces under G of E and $\bar{E}$, respectively, giving rise to two coverings $\{G \longrightarrow E \xrightarrow{\pi} E_0\}$ and $\{G \longrightarrow \bar{E} \xrightarrow{\bar{\pi}} \bar{E}_0\}$. Suppose further, that a map $f: E \longrightarrow \bar{E}$ is compatible with the action of G, i. e., assume that the map f induces $f_0: E_0 \longrightarrow \bar{E}_0$ satisfiying $f_0 \pi = \bar{\pi} f$.

PROPOSITION 1.20. *Let the Smith operations on E_0 be μ, ν and let them be $\bar{\mu}\, \bar{\nu}$ on $\bar{E}_0$. Then these operations commute with $f_0^*: H^* \bar{E}_0 \longrightarrow H^* E_0$*, i. e., *$f_0^* \bar{\mu} = \mu f_0^*$ and $f_0^* \bar{\nu} = \nu f_0^*$.*

PROOF. It is enough to prove the commutativity of the following diagrams at the center:

$$\begin{array}{ccccccc} H^* E_0 & \cong & {}^tH^* E & \xrightarrow{\delta_t \delta_s} & {}^tH^* E & \cong & H^* E_0 \\ \uparrow f_0^* & & \uparrow f^* & & \uparrow f^* & & \uparrow f_0^* \\ H^* \bar{E}_0 & \cong & {}^tH^* \bar{E} & \xrightarrow[\delta_t \delta_s]{} & {}^tH^* \bar{E} & \cong & H^* \bar{E}_0 \end{array}$$

and

$$\begin{array}{ccccccc} H^* E_0 & \cong & {}^tH^* E & \xrightarrow{\xi_t^* \delta_s} & {}^tH^* E & \cong & H^* E_0 \\ \uparrow f_0^* & & \uparrow f^* & & \uparrow f^* & & \uparrow f_0^* \\ H^* \bar{E}_0 & \cong & {}^tH^* \bar{E} & \xrightarrow[\xi_t^* \delta_s]{} & {}^tH^* \bar{E} & \cong & H^* \bar{E}_0 . \end{array}$$

Let $[su] \in {}^tH^* \bar{E}$, then by (1.15) there exist cochains v^1, v^2 in $C^\# \bar{E}$ such that $\delta u = tv^1$ and $\delta v^1 = sv^2$.

Hence,

$$\delta_t \delta_s f^* [su] = \delta_t \delta_s [f^\# su] = \delta_t \delta_s [sf^\# u] = \delta_t [\delta f^\# u] = \delta_t [f^\# \delta u] =$$

$$= \delta_t [f^\# tv^1] = \delta_t [tf^\# v^1] = [\delta f^\# v^1] = [f^\# \delta v^1] = f^* [sv^2] = f^* \delta_t \delta_s [su],$$

which implies the commutativity of the first diagram and the commutativity $f_0^* \bar{\mu} = \mu f_0^*$. The other part of the proof is similar. Q. E. D.

Corollary 1.22. Let ν_0, μ_0 be the Wu classes for $\{G \to E \to E_0\}$, and $\bar{\nu}_0, \bar{\mu}_0$ be the Wu class for $\{G \to \bar{E} \to \bar{E}_0\}$. Then $f_0^* \bar{\nu}_0 = \nu_0$ and $f_0^* \bar{\mu}_0 = \mu_0$.

Remark. It is easily seen that the naturality of the Smith operations holds also for the relativized cases.

§ 2. Generalized Smith Theory.

In this § we will generalize the standard Smith theory by considering $t, t^2, \dots, t^{p-1} = s$ and by taking every pair (t^k, t^{p-k}) into account instead of (t, s) or (s, t). We will see that almost all the statements in the standard Smith theory can be reformulated in the generalized framework.

Let $G = \{1, h, h^2, \dots, h^{p-1}\}$ act as a deck transformation group on E, and E_0 be the orbit space of E under G. Assume that E and E_0 are simplicial complexes and h is simplicial. Just as in § 1, we denote the induced cochain map of $C^\# E$ into $C^\# E$ by $h^\#$, i. e., $h^\# : C^\# E \to C^\# E$. As was stated before, the coefficient domain is always $G = Z_p$ where p is prime.

Let $t^k = (1 - h^\#)^k : C^\# E \to C^\# E$, for $k = 0, 1, 2, \dots, p$. Notice that $t^0 = 1$, $t^1 = t$, $t^{p-1} = s = 1 + h^\# + \dots + (h^\#)^{p-1}$ and $t^p = 0$. $t^{p-1} = s$ follows from the formula $\binom{p-1}{i} \equiv (-1)^i \bmod p$ when p is prime.

Lemma 2.1. For any $k = 0, 1, \dots, p$; $t^k t^{p-k} = 0$.

Proof. $t^k t^{p-k} = t^p = 1 - (h^\#)^p = 0$.

Lemma 2.2. For any $k = 0, 1, 2, \dots, p$, $\ker t^k = \operatorname{im} t^{p-k}$.

Proof. By (2.1) we have $\operatorname{im} t^{p-k} \subseteq \ker t^k$. Hence we will show the other inclusion, i. e., $\operatorname{im} t^{p-k} \supseteq \ker t^k$. The proof is by induction on k. By (1.2) the inclusion above is true for $k = 1$. Suppose that it holds for $k = 1, 2, \dots, r$. Let u be in $\ker t^{r+1}$, then $t^{r+1} u = t^r tu = 0$; hence by the induction hypothesis with $k = r$, we can write $tu = t^{p-r} v$ for some v in $C^\# E$. Therefore, $t(u - t^{p-r-1} v) = 0$. Again by the induction hypothesis with $k = 1$, $u - t^{p-r-1} w = t^{p-1} w$ for some cochain w of $C^\# E$. Hence, $u = t^{p-r-1} v + t^{p-1} w = t^{p-(r+1)}(v + t^r w)$ is a cochain in $\operatorname{im} t^{p-(r+1)}$, i. e., $\operatorname{im} t^{p-(r+1)} \supseteq \ker t^{r+1}$. This proves (2.2) for $k = 1, 2, \dots, p-1$. For $k = 0$ and $k = p$, the proof is immediate. Q. E. D.

Remark. (2.2) can also be shown as follows: Let $G \cong Z_p = \{1, h, h^2, \dots, h^{p-1}\} : E \to E$ act on E freely. Let A_G be the group ring of Z_p over Z_p

so that A_G acts on $C^\# E$, i. e. $A_G : C^\# E \to C^\# E$. $\{1, h^\#, (h^\#)^2, \dots, (h^\#)^{p-1}\}$ and $\{1, t, t^2, \dots, t^{p-1}\}$ are bases for A_G. Hence, every element a of A_G has a unique representation $a = \sum_{i=0}^{p-1} a_i^{(h)} (h^\#)^i = \sum_{i=0}^{p-1} a_i^{(t)} t^i$ where $a_i^{(h)}, a_i^{(t)} \in Z_p$. Let $E_0 = E/G$, and F be a fundamental domain. Then for any x of $C^\# E$ there exist unique x_i's in $C^\# F$ and unique y_i's in $C^\# F$ such that $x = \sum_{i=0}^{p-1} h^i x_i = \sum_{i=0}^{p-1} t^i y_i$. Hence im $t^{p-k} \subset \ker t^k$. On the other hand, if x is in ker t^k, then $t^k x = t^k \sum_{i=0}^{p-1} t^i y_i = \sum_{i=0}^{p-1} t^{i+k} y_i = 0$ with unique representation. Thus, $y_0 = y_1 = \dots = y_{p-k-1}$ and $x = t^{p-k} y_{p-k} + \dots + t^{p-1} y_{p-1} \in \text{im } t^{p-k}$.

DEFINITION. In a manner analogous to that of § 1, we define the t^k-special cohomology group, denoted by ${}^kH^* E$, to be $H^*(\ker t^k)$.

By virtue of (2.2), ${}^kH^* E = H^*(\text{im } t^{p-k})$.

We can describe the additive cohomology group ${}^kH^* E = \sum_m {}^kH^m E$ with the help of local coefficients over E_0. The p-sheeted covering $\{Z_p \to E \to E_0\}$ induces the locally trivial sheaf (or coefficient bundle) $\mathcal{B} = \{F \to B \to E_0\}$ with the stalk (or fiber) $F = Z_p \oplus \dots \oplus Z_p = (Z_p)^p$ where the topology of F is discrete. $G \cong Z_p$ acts on F by $h_F(x_1, \dots, x_p) = (x_p, x_1, \dots, x_{p-1})$ after convenient ordering of the coordinates in $F = (Z_p)^p$, and the map $t_F = 1 - h_F : F \to F$ is compatible with the action of G. Hence, t_F induces an action $t_B : B \to B$ on the total space B of $\mathcal{B}$. Let $\mathcal{B}_k = \ker t_B^k = \{F_k \to B_k \to E_0\}$ be the subsheaf of $\mathcal{B}$ given by $B_k = \ker t_B^k = \{x \mid x \in B, t_B^k x = 0\}$. Notice that $\mathcal{B}_0 = \{0 \to E_0 \to E_0\}$, $\mathcal{B}_1 = \{Z_p \to B_1 \to E_0\}$ and $\mathcal{B}_p = \mathcal{B}$.

LEMMA 2.3. (1) ${}^kH^* E \cong H^*(E_0; \mathcal{B}_k)$. In particular, ${}^0H^* \mathrm{E} = 0$, ${}^1H^* E \cong H^* E_0$, and ${}^pH^* E \cong H^* E$,

(2) $$\text{rank } F_k = k \quad (\text{i. e., } F_k \cong (Z_p)^k, \dim F_k = k).$$

PROOF. (1) This is immediate from ${}^kH^* E = H^*(\ker t^k) \cong H^*(E_0; \ker t_B^k) = H^*(E_0, \mathcal{B}_k)$.

(2) $\ker t_F^k = \text{im } t_F^{p-k}$ analogously to (2.2). Hence, rank $F_k = \dim(\ker t_F^k) = \dim(\text{im } t_F^{p-k})$. Clearly, (2) is true for $k = 0, 1, p-1, p$. Assume (2) for $k = 0, 1, \dots, r$ and for $k = p, p-1, \dots, p-r$. Then rank $t_F^{r+1} = \dim(\text{im } t_F^{r+1}) = \dim(\text{im } t_F(\text{im } t_F^r)) = \dim(\text{im } t_F (\ker t_F^{p-r})) = \dim(\ker t_F^{p-r}) - \dim(\ker t_F) = (p-r) - 1 = p - (r+1)$, by the induction hypothesis with $k = r$ and $k = 1$. Therefore, rank $F_{r+1} = p - \text{rank } t_F^{r+1} = r + 1$, and rank $F_{p-(r+1)} = \dim(\ker t_F^{p-(r+1)}) = \dim(\text{im } t_F^{r+1}) = p - (r+1)$. Q.E.D.

Remark. The cup product induces a ring structure in ${}^1H^*E \cong H^*E_0$ and in ${}^pH^*E \cong H^*E$; but, in general, not in ${}^kH^*E$ for $2 \leq k \leq p-1$. Cf. the formula for $t^k(a \cup b)$ used in the proof of Proposition 2.13.

We have the following short exact sequence :

$$0 \longrightarrow \ker t^k \xrightarrow{\iota_k} C^\# E \xrightarrow{t^k} \operatorname{im} t^k \longrightarrow 0.$$

Hence, by (2.2) and by the definition of the t^k-special cohomology ${}^kH^*$, the Richardson-Smith exact sequences (1.4) generalize for t^k as follows :

Proposition 2.4. *For the p-sheeted covering $\{Z_p \longrightarrow E \longrightarrow E_0\}$, there is the exact triangle*

$$\begin{array}{ccc} & H^*E & \\ {\scriptstyle\beta_k}\nearrow & & \searrow{\scriptstyle\gamma_k} \\ {}^kH^*E & \xleftarrow[\delta_k]{} & {}^{p-k}H^*E \end{array}$$

for each $k = 1, 2, \ldots, p-1$. Here γ_k is induced by t^k, β_k is induced by ι_k, and δ_k is the k-th Smith coboundary operation defined by $\delta_k[t^k x] = [\delta x]$ for any element $[t^k x]$ in ${}^{p-k}H^*E$.

(Notice that $[\delta x]$ is in ${}^kH^*E$, δ is the ordinary coboundary operation).

Remark. When $p = 2$, (2.4) reduces itself to the part (1) of (1.4).

We discuss two relativized cases as we did in § 1. They are :

Case 1. The group $G \cong Z_p$ acts on E leaving a subset L of E invariant (not necessarily pointwise) and its action on $E - L$ is free.

Case 2. The group $G \cong Z_p$ leaves a subset L of E pointwise fixed and acts on $E - L$ as a deck transformation group.

In either case, for simplicity we assume that L is a simplicial subcomplex of the complex E. The notation is the same as that of § 1.

Proposition 2.5. *In Case* 1 *where $(E, L)/G = (E_0, L_0)$, there is the exact triangle*

$$\begin{array}{ccc} & H^*(E, L) & \\ {\scriptstyle\beta_k}\nearrow & & \searrow{\scriptstyle\gamma_k} \\ {}^kH^*(E, L) & \xleftarrow[\delta_k]{} & {}^{p-k}H^*(E, L) \end{array}$$

for each $k = 1, 2, \ldots, p-1$.

REMARK. Relativized t^k-special cohomology groups ${}^kH^*(E, L)$ are defined to be $H^*(\ker t^k)$ where $t^k : C^\#(E, L) \longrightarrow C^\#(E, L)$ are the relativized cochain maps of Case 1. The proof of (2.5) is the same as in the standard Smith theory and is omitted here. Also notice that when $p = 2$, (2.5) is reduced to the part (1) of (1.6). The maps β_k, γ_k and δ_k are defined in the obvious manner as in (2.4).

LEMMA 2.6. In Case 2, we have

$$\operatorname{im}(t^{p-k} \mid C^\# E) = \ker(t^k \mid C^\#(E, L)).$$

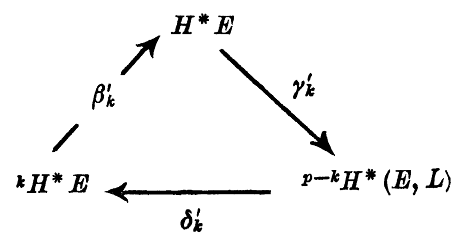
The proof is immediate.

The short exact sequences

$$0 \longrightarrow \ker(t^k \mid C^\# E) \xrightarrow{\iota_k} C^\# E \xrightarrow{t^k} \operatorname{im}(t^k \mid C^\# E) \longrightarrow 0$$

entail the following proposition, just as in § 1.

PROPOSITION 2.7. *In Case 2 where* $(E, L)/G = (E_0, L)$, *there is the exact triangle*



for each $k = 1, 2, \dots, p-1$; *where* β'_k, γ'_k *and* δ'_k *are defined in the obvious manner as before.*

DEFINITION. For each k, $k = 1, 2, \dots, p-1$, we define a map ξ_k : $C^\#(E, L) \longrightarrow C^\#(E, L)$ by $\xi_k = t^{p-k-1}$. The above definition holds also for the obsolute case where $L = \emptyset$. Note that $\xi_1 = \xi_t = t^{p-2}$ and $\xi_{p-1} = \xi_s = 1$ where ξ_t and ξ_s are defined in § 1.

LEMMA 2.8.

$$\xi_k t^k = t^k \xi_k = \xi_{p-k} t^{p-k} = t^{p-k} \xi_{p-k} = s$$

for all $k = 1, 2, \dots, p-1$.

Proof. This follows immediately from the definition of ξ_k.

Lemma 2.9. The map ξ_k induces the switching homomorphism $\xi_k^* : {}^{p-k}H^*(E, L) \to {}^kH^*(E, L)$, for each $k = 1, 2, \dots, p-1$.

Proof. If $[u]$ is an element of ${}^{p-k}H^*(E, L)$, then by definition $t^{p-k} u = 0$. Hence, $t^k \xi_k u = t^k t^{p-k-1} u = t^{k-1} (t^{p-k} u) = 0$, which implies that $\xi_k^*[u] = [\xi_k u]$ is an element of ${}^kH^*(E, L)$. Q. E. D.

Lemma 2.10.

(1) $\xi_k t^{p-k} = 0$, for $1 \leqq k \leqq p' = \dfrac{p-1}{2}$.

(2) $\xi_k^* \xi_{p-k}^* = 0$, for $1 \leqq k \leqq p-2$.

Proof. (1) $\xi_k t^{p-k} = t^{p-k-1} t^{p-k} = t^{2p-2k-1}$.

Since k varies from 1 to p', $2p - 2k - 1 \geqq 2p - 2p' - 1 = p$. Hence, $\xi_k t^{p-k} = 0$.

(2) $\xi_k^* \xi_{p-k}^* : {}^kH^*(E, L) \to {}^kH^*(E, L)$. Let $[t^{p-k} x]$ be an element of ${}^kH^*(E, L)$. Then $\xi_k^* \xi_{p-k}^* [t^{p-k} x] = [t^{p-k-1} t^{k-1} t^{p-k} x] = [t^{2p-k-2} x] = 0$ because $2p - k - 2 \geqq 2p - (p-2) - 2 = p$. Recall that any element of ${}^kH^*(E, L)$ is of the form $[t^{p-k} x]$ for some cochain x. Hence $\xi_k^* \xi_{p-k}^* = 0$ on ${}^kH^*(E, L)$, for $1 \leqq k \leqq p-2$. Q. E. D.

Lemma 2.11. $\xi_k^* \delta_{p-k} = \delta_k \xi_{p-k}^*$, for $k = 1, 2, \dots, p-1$.

Proof. As usual let $[t^{p-k} u]$ be an arbitrary element of ${}^kH^*(E, L)$. Then, $\xi_k^* \delta_{p-k} [t^{p-k} u] = \xi_k^* [\delta u] = [\xi_k \delta u] = [\delta \xi_k u] = \delta_k [t^k \xi_k u] = \delta_k [\xi_{p-k} t^{p-k} u] = \delta_k \xi_{p-k}^* [t^{p-k} u]$. Hence, $\xi_k^* \delta_{p-k} = \delta_k \xi_{p-k}^* : {}^kH^*(E, L) \to {}^kH^*(E, L)$, for $1 \leq k \leq p-1$. Q. E. D.

We now introduce the generalized Smith operations μ_k and ν_k, and state a few results about them.

Definition. The generalized Smith operations μ_k and ν_k are defined to be

$$\mu_k = \delta_k \delta_{p-k} : {}^kH^q(E, L) \to {}^kH^{q+2}(E, L)$$

and

$$\nu_k = \delta_k \xi_{p-k}^* : {}^kH^q(E, L) \to {}^kH^{q+1}(E, L),$$

where $k = 1, 2, \dots, p-1$ and q is a (non negative) integer.

When $p = 2$, they become

$$\mu = \nu^2 : {}^tH^q(E, L) \to {}^tH^{q+2}(E, L)$$

and

$$\nu = \delta_0 : {}^tH^q(E, L) \longrightarrow {}^tH^{q+1}(E, L)$$

where δ_0 is the operator introduced in the part (1) of (1.6).

As we remarked earlier in § 1, we identify $H^*(E_0, L_0)$ and ${}^tH^*(E, L)$ under π^*, and hence we can view μ_k and ν_k as maps from $H^*(E_0, L_0)$ into itself.

LEMMA 2.12. (1) $\mu_k \nu_k = \nu_k \mu_k$, for $1 \leqq k \leqq p-1$

(2) $$\nu_k^2 = \begin{cases} 0, & \text{for } 1 \leqq k \leqq p-2 \text{ and } p \geqq 3 \\ \mu, & \text{for } p = 2. \end{cases}$$

PROOF. (1) $\mu_k \nu_k = \delta_k \delta_{p-k} \delta_k \xi^*_{p-k}$ by definition

$= \delta_k \delta_{p-k} \xi^*_k \delta_{p-k}$ by (2.11)

$= \delta_p \xi^*_{p-k} \delta_k \delta_{p-k}$ by (2.11)

$= \nu_k \mu_k$.

(2) Let $p \geqq 3$. Then,

$\nu_k^2 = \delta_k \xi^*_{p-k} \delta_k \xi^*_{p-k}$ by definition

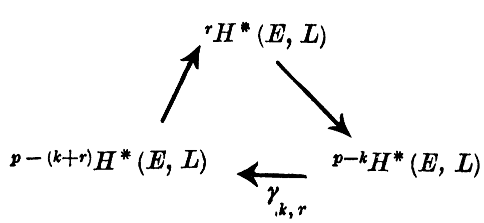
$= \delta_k \delta_{p-k} \xi^*_k \xi^*_{p-k}$ by (2.11)

$= 0$ by the part (2) of (2.10).

REMARK. The map $t^r : \text{im } t^k \longrightarrow \text{im } t^{k+r}$ induces a map $\gamma_{r,k} : H^*(\text{im } t^k) \equiv$ $\equiv {}^{p-k}H^*(E, L) \longrightarrow {}^{p-(k+r)}H^*(E, L) \equiv H^*(\text{im } t^{k+r})$. If $k = 0$, clearly $\gamma_{0,r} = \gamma_r$. When $k + r \leqq p$, the short exact sequence

$$0 \longrightarrow \ker(t^r \mid \text{im } t^k) \longrightarrow \text{im } t^k \xrightarrow{t^r} \text{im } t^{k+r} \longrightarrow 0$$

induces the exact cohomology triangle



This can be verified by showing that $\ker(t^r \mid \operatorname{im} t^k) = \ker(t^r \mid C^{\#}(E, L))$ when $k + r \leq p$, so that $H^*(\ker t^r \mid \operatorname{im} t^k) = H^*(\ker t^r \mid C^{\#}(E, L)) = {}^rH^*(E, L)$.

Recall that (1.18) in § 1 says that the action of the Smith operations μ and ν (i. e., of μ_1 and ν_1 in generalized Smith theory) is expressed by the cup product with the elements μ_0 and ν_0, the *Wu* classes. We want to prove analogous statements in generalized Smith theory for all μ_k and ν_k, $k = 1, 2, \ldots, p-1$. Let us begin with some preliminary considerations. For any cochains a and b in $C^{\#}(E, L)$,

$$\begin{aligned} t(a \cup b) &= (1 - h^{\#})(a \cup b) \\ &= a \cup b - h^{\#} a \cup h^{\#} b \\ &= (1 - h^{\#})\, a \cup b + a \cup (1 - h^{\#})\, b - (1 - h^{\#})\, a \cup (1 - h^{\#})\, b \\ &= ta \cup b + a \cup tb - ta \cup tb. \end{aligned}$$

We will write this equality as

$$t(a \cup b) = (t \cup 1 + 1 \cup t - t \cup t)(a \cup b),$$

where $(h' \cup h'')(a \cup b)$ means $h' a \cup h'' b$ for cochain maps $h', h'' : C^{\#}(E, L) \longrightarrow C^{\#}(E, L)$. Therefore, in general

$$\begin{aligned} t^k(a \cup b) &= (t \cup 1 + 1 \cup t - t \cup t)^k (a \cup b) \\ &= \left[\sum_{i=0}^{k} \binom{k}{i} (-1)^i (t \cup 1 + 1 \cup t)^{k-i} (t \cup t)^i\right] (a \cup b) \\ &= \left[\sum_{i=0}^{k} \binom{k}{i} (-1)^i \sum_{j=0}^{k-1} \binom{k-i}{j} (t^{k-i-j} \cup t^j)(t^i \cup t^i)\right] (a \cup b) \\ &= \left[\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^i \binom{k}{i} \binom{k-1}{j} (t^{k-j} \cup t^{i+j})\right] (a \cup b), \end{aligned}$$

i. e.

$$t^k(a \cup b) = \sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^i \binom{k}{i} \binom{k-j}{j} t^{k-j} a \cup t^{i+j} b,$$

for any integer k.

In (1.17), we observed that $t(x \cup y) = (tx) \cup y$ for any cochain x and t-cochain y, so that $t^k(x \cup y) = (t^k x) \cup y$, for $k = 1, 2, \ldots, p-1$. Hence, for any $[x]$ in ${}^kH^*(E, L)$, $\mu_0 \cup [x]$ and $\nu_0 \cup [x]$ are elements in ${}^kH^*(E, L)$.

PROPOSITION 2.13. *For any element* $[x]$ *of* ${}^kH^*(E, L)$ *and for* $k = 1, 2, \ldots, p-1$, *the following relations hold:*

(1) $\mu_k[x] = \mu_0 \cup [x]$,

and

(2) $k\nu_k[x] = \nu_0 \cup [x]$ (*i. e.,* $\nu_k[x] = k^{p-2}\nu_0 \cup [x]$).

Notice that by the remark preceding the proposition (2.13), the statement above makes sense.

PROOF. By (1.16), there are cochains c^0 and c^1 such that $1 = sc^0$, $\delta c^0 = tc^1$, $\mu_0 = \mu(1) = [\delta c^1]$ and $\nu_0 = \nu(1) = [sc^1]$.

(1) $\mu_0 \cup [x]$

$= [\delta c^1] \cup [x]$

$= [\delta(c^1 \cup x)]$ because $\delta x = 0$

$= \delta_k [t^k (c^1 \cup x)]$

$$= \delta_k \left[\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^i \binom{k}{i} \binom{k-i}{j} t^{k-j} c^1 \cup t^{i+j} x \right].$$

Since $[x]$ is an element of ${}^kH^*(E, L)$, $t^r x = 0$ for $r \geqq k$. Hence,

$$\mu_0 \cup [x] = \delta_k \left[\sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^i \binom{k}{i} \binom{k-i}{j} t^{k-j} c^1 \cup t^{i+j} x \right]$$

$$= \delta_k \left[\sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^i \binom{k}{i} \binom{k-i}{j} t^{k-j-1} \delta c^0 \cup t^{i+j} x \right]$$

$$= \delta_k \left[\delta \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^i \binom{k}{i} \binom{k-i}{j} t^{k-j-1} c^0 \cup t^{i+j} x \right]$$

$$= \delta_k \, \delta_{p-k} \left[t^{p-k} \sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1} (-1)^i \binom{k}{i} \binom{k-i}{j} t^{k-j-1} c^0 \cup t^{i+j} x \right] \qquad \text{(i)}$$

$$= \mu_k \left[\sum_{i=0}^{p-1} \sum_{j=0}^{p-i-1} (-1)^i \binom{p}{i} \binom{p-i}{j} t^{p-j-1} c^0 \cup t^{i+j} x \right]. \qquad \text{(ii)}$$

The last equality can be checked by applying the general formula for $t^k (a \cup b)$ repeatedly and summing, or by replacing $\sum_{i=0}^{k-1} \sum_{j=0}^{k-i-1}$ in (i) by $\sum_{i=0}^{k} \sum_{j=0}^{k-i}$ (since $t^r x = 0$ for $r \geqq k$) and by remarking that (i) looks like the sum for $t^k (c^0 \cup x)$ with $t^{k-j-1} c^0$ instead of $t^{k-j} c^0$. (ii) is the expansion of $t^p (c^0 \cup x)$ with $t^{p-j} c^0$ replaced by $t^{p-j-1} c^0$. Therefore

$$\mu_0 \cup [x]$$
$$= \mu_k \left[\sum_{i=0}^{p-1} \sum_{j=0}^{p-i-1} (-1)^i \binom{p}{i} \binom{p-i}{j} t^{p-j-1} c^0 \cup t^{i+j} x \right]$$

$$= \mu_k \left[\sum_{j=0}^{p-1} \binom{p}{j} t^{p-j-1} c^0 \cup t^j x \right] \text{ because } \binom{p}{i} = 0 \bmod p \text{ for } 1 \leqq i \leqq p-1$$

$$= \mu_k [t^{p-1} c^0 \cup x]$$

$$= \mu_k [sc^0 \cup x] = \mu_k [1 \cup x] = \mu_k [x].$$

(2) Let $[x]$ be an element of ${}^kH^* (E, L)$. Then $\nu_0 \cup [x] = \nu (1) \cup [x] = [sc^1 \cup x] = [t^{p-1} c^1 \cup x] = [t^{p-2} tc^1 \cup x] = [t^{p-2} \delta c^0 \cup x] = [\delta (t^{p-2} c^0 \cup x)] = \delta_k [t^k (t^{p-2} c^0 \cup x)] = \delta_k \left[\sum_{i=0}^{k} \sum_{j=0}^{k-i} (-1)^i \binom{k}{i} \binom{k-i}{j} t^{k-j+p-2} c^0 \cup t^{i+j} x \right]$, where $t^{k-j+p-2} c^0 \cup t^{i+j} x = 0$ if $k - j + p - 2 \geqq p$ or $i + j \geqq k$, i. e., $j \leqq k-2$ or $i + j \geqq k$. Hence, there is only one term left for $i = 0$, $j = k-1$. Therefore, $\nu_0 \cup [x] = \delta_k \left[\binom{k}{k-1} t^{p-1} c^0 \cup t^{k-1} x \right] = k\delta_k [sc^0 \cup t^{k-1} x] = k\delta_k [1 \cup t^{k-1} x] = k\delta_k [t^{k-1} x] = k\delta_k \, \xi^*_{p-k} [x] = k\nu_k [x]$. Since $k^{p-1} = 1 \bmod p$ by Fermat's theorem, we also have $\nu_k [x] = k^{p-2} \nu_0 \cup [x]$. Q. E. D.

Notice that (2.13) reduces to (1.18) when $k = 1$.

COROLLARY 2.14. For $k = 1, 2, \dots, p-1$, we have

(1) $\mu_k [x] = \mu_k (1) \cup [x]$,

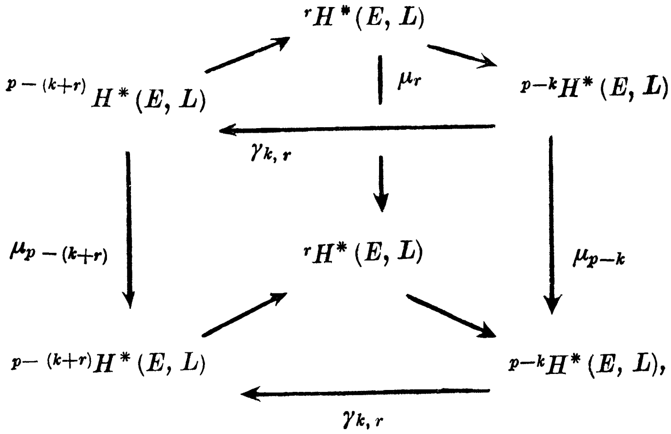
and

(2) $\nu_k[x] = \nu_k(1) \cup [x]$

where $\mu_k(1) = \mu_0$ and $\nu_k(1) = k^{p-2}\nu_0$ are elements of ${}^tH^2(E-L)$ and ${}^tH^1(E-L)$, respectively.

The corollary (2.14) can be proved directly starting with (2.12) and going through arguments as in [5, i, pp. 7-9]; in particular, (1.15) has analogues for a pair (t^k, t^{p-k}).

Proposition 2.13 (1) and the exact cohomology triangles which appeared in the Remark after Lemma 2.12 lead to a commutative diagram



for $k = 0, 1, \dots, p-1$ and $k + r \leqq p$, i. e. $\{\mu_k\}$ induces endomorphisms of degree 2 of the exact triangles above. An analogous statement holds for $\{\nu_k\}$, the induced endomorphisms being of degree 1.

§ 3. The Thom Direct Sum Decomposition for a p-cyclic Product.

In this §, we consider the case of the p-cyclic product of a simplicial or CW-complex and establish the Thom decomposition theorem (3.14) and (3.15) which says that the image ${}^kN^n = \delta_k\, {}^{p-k}H^{n-1}(E, \Delta)$ in ${}^kH^n(E, \Delta)$ is a direct sum of groups $(\delta_k\,\delta_{p-k})^j\theta_k\, H^{n-2j-1}\,\Delta$ and $(\delta_k\,\delta_{p-k})^i\delta_k\,\theta_k\, H^{n-2j-1}\,\Delta$, where i and j range over suitable sets of integers. The number λ_n which will be introduced in (3.5) will play an important role in the formulation of the Thom-Bott formulas and in the proof of the Cartan relations in § 4.

Many of the lemmas we introduce in this § to establish the direct sum representation as well as the Thom decomposition theorem itself are applied later in the development leading to the Thom-Bott formulas.

Let X be a finite simplicial or CW-complexe, and $E = X \times \dots \times X = X^p$ be the p-fold cartesian product of X (with the product topology). Define a homeomorphism $h : E \to E$ by $h(x_1, \dots, x_p) = (x_p, x_1, \dots, x_{p-1})$ for an element $(x_1, \dots, x_p)$ of E. Then the group $G = \{1, h, \dots, h^{p-1}\} \cong Z_p$ acts on E leaving the diagonal $\varDelta$ of E pointwise fixed. $\varDelta$ is homeomorphic to X. This is clearly a special case of the relativized Case 2 considered in the previous §. The map h induces a cochain map $h^{\#}: C^{\#} E \to C^{\#} E$ on the cochain group of E. In this § the coefficient domain will be always the cyclic p-group $G \cong Z_p$ unless otherwise stated. Whenever possible, we use the notations of § 2, e. g.

$$t = 1 - h^{\#}, \ t^k = (1 - h^{\#})^k, \ t^{p-1} = s = 1 + h^{\#} + \dots + (h^{\#})^{p-1}, \text{ etc.}$$

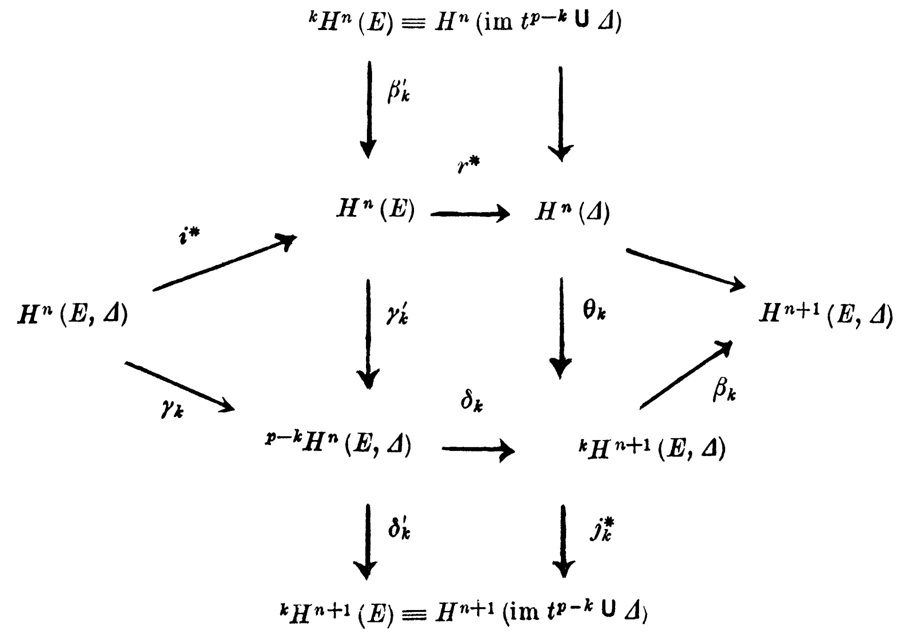
Recall that in the case where $(E, \varDelta)/G = (E_0, \varDelta)$, we have the following Richardson-Smith exact sequences by (2.5) and (2.7):

$$\dots \to H^n(E, \varDelta) \xrightarrow{\gamma_k} {}^{p-k}H^n(E, \varDelta) \xrightarrow{\delta_k} {}^{k}H^{n+1}(E, \varDelta) \xrightarrow{\beta_k} H^{n+1}(E, \varDelta) \to \dots$$

and

$$\dots \to H^n(E) \xrightarrow{\gamma'_k} {}^{p-k}H^n(E, \varDelta) \xrightarrow{\delta'_k} {}^{k}H^{n+1}(E) \xrightarrow{\beta'_k} H^{n+1}(E) \to \dots,$$

for $k = 1, 2, \dots, p-1$. γ_k and γ'_k are induced by t^k; β_k and β'_k are induced naturally; and δ_k and δ'_k are the generalized Smith coboundary operations. We also have the following diagram:



for $k = 1, 2, \dots, p-1$.

$$\cdots \longrightarrow H^n(E, \Delta) \xrightarrow{i^*} H^n(E) \xrightarrow{r^*} H^n(\Delta) \longrightarrow H^{n+1}(E, \Delta) \longrightarrow \cdots$$

is the exact sequence for the cohomology groups of the pair (E, Δ), and

$$\cdots \longrightarrow H^n(\operatorname{im} t^{p-k} \cup \Delta) \longrightarrow H^n(\Delta) \xrightarrow{\theta_k} {}^kH^{n+1}(E, \Delta) \longrightarrow H^{n+1}(\operatorname{im} t^{p-k} \cup \Delta) \longrightarrow \cdots$$

is the sequence obtained from the short exact sequence of cochain complexes

$$0 \longrightarrow {}^{(k)}C^{\#}(E, \Delta) \longrightarrow {}^{(k)}C^{\#}(E) \longrightarrow C^{\#}(\Delta) \longrightarrow 0$$

and hence it is exact. Here ${}^{(k)}C^{\#} E$ denotes the kernel of $t^k : C^{\#} E \longrightarrow C^{\#} E$ and ${}^{(k)}C^{\#}(E, \Delta)$ denotes the kernel of $t^k : C^{\#}(E, L) \longrightarrow C^{\#}(E, L)$. Hence rows and columns of the diagram are exact. θ_k is naturally induced by θ_t. When $p = 2$, we just write $\theta_t = \theta_s = \theta$.

LEMMA 3.1. The above diagram is anticommutative at the center and commutative everywhere else, for $k = 1, 2, \ldots, p-1$ if $p \geqq 3$. If $p = 2$, then the diagram is commutative everywhere.

PROOF. Commutativity outside the center of the diagram is immediate. We prove the anti-commutativity at the center when $p \geqq 3$. Let $[u]$ be an arbitrary element of $H^n E$; then u is written as $u = v + w$ where $v \in C^n(E, \Delta)$ and $w \in C^n \Delta$. Hence, $(\theta_k r^* + \delta_k \gamma'_k)[u] = [\theta_k w + \delta_k t^k v] = [\delta w + \delta v] = [\delta u] = 0$, which implies that $\theta_k r^* + \delta_k \gamma'_k = 0$.

It is clear that anticommutativity and commutativity are the same for Z_2. Q. E. D.

The map $t^k : C^{\#} E \longrightarrow C^{\#} E$ induces $(t^k)^* : H^*E \longrightarrow H^*E$ and $(\widetilde{t^k}) : H^*E \longrightarrow H^*(E, \Delta)$ on the cohomology groups. The induced maps are given by:

$$(t^k)^* = \beta'_{p-k} j^*_{p-k} \gamma'_k : H^*E \longrightarrow H^*E$$

and

$$(\widetilde{t^k}) = \beta_{p-k} \gamma'_k : H^*E \longrightarrow H^*(E, \Delta).$$

$(t^1)^*$ and $(\widetilde{t^1})$ will be denoted by t^* and $\tilde{t}$, respectively. Then $(t^k)^* = (t^*)^k$ and $(t^{p-1})^* = s^*$. Moreover, for any class $z_1 \otimes z_2 \otimes \cdots \otimes z_p$ in H^*E, we have $t^*(z_1 \otimes \cdots \otimes z_p) = z_1 \otimes \cdots \otimes z_p - z_2 \otimes z_3 \otimes \cdots \otimes z_p \otimes z_1$,

$$(t^k)^* = (t^{k-1})^*(z_1 \otimes \cdots \otimes z_p) - (t^{k-1})^*(z_2 \otimes z_3 \otimes \cdots \otimes z_p \otimes z_1)$$

and

$$s^*(z_1 \otimes \cdots \otimes z_p) = z_1 \otimes \cdots \otimes z_p + z_p \otimes z_1 \otimes \cdots \otimes z_{p-1} + z_{p-1} \otimes z_p \otimes z_1 \otimes \cdots \otimes z_{p-2} + \cdots + z_2 \otimes z_3 \otimes \cdots \otimes z_p \otimes z_1 .$$

Similar relations hold for the actions of $(\widetilde{t^k})$.

NOTATION. For $z \in H^n X$, let $z^p = z \otimes z \otimes \cdots \otimes z \in H^{np} E$.

LEMMA 3.2. Let n be a positive integer. For any z in $H^n X$, $\gamma'_k z^p = 0$ implies that $z^p = 0$ in $H^{np} E$, for any prime $p \geqq 2$.

PROOF. First, we consider the case where $n = \dim X$. Since $\gamma'_k z^p = 0$, by Richardson-Smith exactness, there is an element u in ${}^kH^{np} E$ such that $z^p = \beta'_k u$. By a dimension argument, both $\gamma'_{p-k} : H^{np} E \to {}^kH^{np}(E, \Delta)$ and $j^*_k : {}^kH^{np}(E, \Delta) \to {}^kH^{np} E$ are epimorphic. Hence, $u \in {}^kH^{np} E = j^*_k H^{np}(E, \Delta) = j^*_k \gamma'_{p-k} H^{np} E$, i. e., $z^p = \beta'_k u = \beta'_k j^*_k \gamma'_{p-k}\, y = (t^{p-k})^* y$, for some y in $H^{np} E$. Let y be written as $y = \Sigma z_{i_1} \otimes \cdots \otimes z_{i_p}$ (by the Künneth formula). Then $(t^{p-k})^* y$ does not contain any nonzero term of the form z^p. Since $z^p = (t^{p-k})^* y$, the only possible explanation is $z^p = 0$ in $H^{np} E$.

Second, suppose that $0 < n < \dim X$. Let $X^{(n)}$ be the n-skeleton of X, $E^{(n)} = X^{(n)} \times \cdots \times X^{(n)}$ be the p-fold cartesian product of $X^{(n)}$, $\Delta^{(n)}$ be the diagonal set of $E^{(n)}$ and $g : X^{(n)} \to X$ be the inclusion map. Notice that the superscript n does not tell the dimension of $E^{(n)}$ whereas it does for $X^{(n)}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} H^{np} E & \xrightarrow{\gamma'_k} & {}^{p-k}H^{np}(E, \Delta) \\ {\scriptstyle (g \times \cdots \times g)^*} \downarrow & & \downarrow {\scriptstyle (g \times \cdots \times g)^*_{p-k}} \\ H^{np} E^{(n)} & \xrightarrow{\gamma'_k} & {}^{p-k}H^{np}(E^{(n)}, \Delta^{(n)}) \end{array}$$

where $(g \times \cdots \times g)^* : H^* E \to H^* E^{(n)}$ is induced by the map $(g \times \cdots \times g) : E^{(n)} \to E$ and $(g \times \cdots \times g)^*_{p-k} : {}^{p-k}H^{np}(E, \Delta) \to {}^{p-k}H^{np}(E^{(n)}, \Delta^{(n)})$ is defined in the obvious way. Hence, $\gamma'_k z^p = 0$ implies that $0 = (g \times \cdots \times g)^*_{p-k} \gamma'_k z^p = \gamma'_k (g \times \cdots \times g)^* z^p = \gamma'_k (g^* z)^p$ by the commutativity of the above diagram. Thus, $(g^* z)^p = 0$ in $H^{np}(E^{(n)})$ because $\gamma'_k : H^{np}(E^{(n)}) \to {}^{p-k}H^{np}(E^{(n)}, \Delta^{(n)})$ is monomorphic on elements of the form z^p, by the argument in the preceding paragraph. The map $g^* : H^n X \to H^n X^{(n)}$ is a monomorphism. Therefore,

$(g \times \dots \times g)^* : (H^n X \otimes \dots \otimes H^n X) \to H^{np} E^{(n)}$ is a monomorphism by the Künneth isomorphism theorem. This implies that $z^p = 0$ in $H^{np} E$.

Q. E. D.

LEMMA 3.3 Let $n > 0$. Then

(1) $$(t^{p-k})^* H^n E = \beta'_k {}^k H^n E$$

and

(2) $$\tilde{t}^{p-k} H^n E = \beta_k {}^k H^n (E, \Delta),$$

for $k = 1, 2, \dots, p-1$.

PROOF. (1) By the definition of $(t^{p-k})^*$, we have $(t^{p-k})^* H^n E \subset \beta'_k {}^k H^n E$. Hence, we need only prove the other inclusion.

First suppose that n is divisible by p, say $n = pm$ for some m. Let x be an element of $\beta'_k {}^k H^n E$. Then $(t^k)^* x = \beta'_{p-k} j^*_{p-k} \gamma'_k x = 0$, because $\gamma'_k \beta'_k = 0$ by Richardson-Smith exactness. Hence

$$x = \sum_i z_i^p \bmod \operatorname{im} (t^{p-k})^*$$

$$= (\sum_i z_i)^p \bmod \operatorname{im} (t^{p-k}))^*.$$

Now $\gamma'_k x = 0$ implies that $\gamma'_k (\sum_i z_i)^p = 0$. By (3.2), $(\sum_i z_i)^p = 0$, i. e. x is in the image of the map $(t^{p-k})^* : H^n E \to H^n E$.

Secondly, suppose that n is not divisible by p. Then $(t^k)^* x = 0$ implies that x is in $(t^{p-k})^* H^n E$, because there is no term of the form z_i^p in the expression of x.

(2) Since $H^n E \xrightarrow{r^*} H^n \Delta \to H^{n+1} (E, \Delta) \xrightarrow{i^*} H^{n+1} E$ is exact (cohomology sequence for (E, Δ)) and r^* is epimorphic, we see that i^* is a monomorphism. The commutativity of the following diagram

$$\begin{array}{ccc} {}^k H^n (E, \Delta) & \xrightarrow{j_k^*} & {}^k H^n E \\ \downarrow{\scriptstyle \beta_k} & & \downarrow{\scriptstyle \beta'_k} \\ H^n (E, \Delta) & \xrightarrow[i^*]{} & H^n E \end{array}$$

and part (1) imply that $\operatorname{im} (i^* \beta_k) = \operatorname{im} (\beta'_k j_k^*) \subset \operatorname{im} \beta'_k = \operatorname{im} (t^{p-k})^* = \operatorname{im} (\beta'_k j_k^* \gamma_{p-k}) = \operatorname{im} (i^* \beta_k \gamma'_{p-k})$. Hence, $\operatorname{im} (i^* \beta_k) = \operatorname{im} (i^* \beta_k \gamma'_{p-k})$ so that $i^* (\operatorname{im} \beta_k) = i^* (\operatorname{im} \beta_k \gamma'_{p-k})$. Since i^* is a monomorphism the desired result follows.

Q. E. D.

The following lemma is a key step in proving the Thom direct sum decomposition.

DEFINITION. Let $n > 0$ and $1 \leqq k \leqq p-1$. Define ${}^kN^n = {}^kN^n(E, \varDelta)$ as $\delta_k\, {}^{p-k}H^{n-1}(E, \varDelta)$.

Clearly ${}^kN^n$ is a subgroup of ${}^kH^n(E, \varDelta)$.

LEMMA 3.4. Let n be a positive integer, then

$$ {}^kN^n = \delta_k\, {}^{p-k}N^{n-1} + \theta_k\, H^{n-1}\varDelta. $$

PROOF. Let u be an element of ${}^kN^n$. u is written as $u = \delta_k v$, for some v in ${}^{p-k}H^{n-1}(E, \varDelta)$. By (3.3.2), there is an element z in $H^{n-1}E$ such that $\beta_{p-k}\, v = (\widetilde{t^k})\, z$. Let $y = v - \gamma'_k z$. y is in ${}^{p-k}H^{n-1}(E, \varDelta)$ and moreover, $\beta_{p-k} y = = \beta_{p-k}\, v - \beta_{p-k}\, \gamma'_k z = \beta_{p-k}\, v - (\widetilde{t^k})\, z = 0$. Therefore, y is an element of ${}^{p-k}N^{n-1} = \delta_{p-k}\, {}^kH^{n-2}(E, \varDelta)$ by Richardson-Smith exactness. Write $y = = \delta_{p-k}\, w$, for some w in ${}^kH^{n-1}(E, \varDelta)$. Then $u = \delta_k\, v = \delta_k\,(y + \gamma'_k z) = = \delta_k\, y + \delta_k\, \gamma'_k z = \delta_k\,(\delta_{p-k}\, w) - \theta_k\, r^* z$, where the last equality follows from (3.1). Hence,

$$ {}^kN^n \subset \delta_k\, {}^{p-k}N^{n-1} + \theta_k\, H^{n-1}\varDelta. $$

Since r^* is an epimorphism,

$$ \begin{aligned} &\delta_k\, {}^{p-k}N^{n-1} + \theta_k\, H^{n-1}\varDelta \\ &\subset \delta_k\, {}^{p-k}N^{n-1} + \theta_k\, r^*\, H^{n-1} E \\ &\subset \delta_k\, {}^{p-k}N^{n-1} + \delta_k\, \gamma'_k\, H^{n-1} E \\ &\subset \delta_k\,({}^{p-k}N^{n-1} + {}^{p-k}H^{n-1}(E, \varDelta)) \subset {}^kN^n(E, \varDelta). \end{aligned} $$

Q. E. D.

Later in this chapter it will be observed that the decomposition of (3.4) is direct and that repeated use of (3.4) yields the Thom direct sum decomposition theorem (3.14).

LEMMA 3.5 Let $n > 0$ and $\alpha = np' - 1$ if $p \geqq 3$, $p' = \dfrac{p-1}{2}$. Let a be any class of $H^n S^n$. Then there is a nonzero element λ of $Z_p \cong G$ such that

$$ \lambda\, \gamma'_{p-k}\, a^p = \mu^\alpha_k\, \delta_k\, \theta_{p-k}\, a \text{ if } p \geqq 3. $$

Moreover, in case $p \geqq 3$, λ depends only on n and p, i. e. λ is independent of the choice of a in $H^n S^n$ and of the choice of k, $1 \leqq k \leqq p-1$.

If $p = 2$, we have

$$\gamma_0' \, a^2 = \delta_0^{n-1} \, \theta a.$$

PROOF. First we assume $p \geqq 3$. Let $X = S^n$, $E = X^p$ be the p-fold cartesian product of X, and Δ be the diagonal of E. We apply (3.4) repeatedly to the above situation to get

$$\begin{aligned}
{}^kN^{np} &= \delta_k \, {}^{p-k}N^{np-1} + \theta_k \, N^{np-1} \Delta \\
&= \delta_k \, {}^{p-k}N^{np-1} \qquad \text{because } H^{np-1} \Delta = 0 \\
&= \delta_k \, (\delta_{p-k} \, {}^kN^{np-2} + \theta_{p-k} \, H^{np-2} \Delta) \\
&= \delta_k \, \delta_{p-k} \, {}^kN^{np-2} \\
&\;\;\vdots \\
&= (\delta_k \, \delta_{p-k})^a \, \delta_k \, (\delta_{p-k} \, {}^kN^n + \theta_{p-k} \, H^n \Delta) \\
&= \mu_k^{a+1} \, {}^kN^n + \mu_k^a \, \delta_k \, \theta_{p-k} \, H^n S^n .
\end{aligned}$$

Next we will show that ${}^kN^n = 0$ as follows:

$$\begin{aligned}
{}^kN^n &= \delta_k \, {}^{p-k}N^{n-1} + \theta_k \, H^{n-1} S^n \qquad \text{by (3.4)} \\
&= \delta_k \, {}^{p+k}N^{n-1} \qquad \text{because } H^{n-1} S^n = 0 \\
&= \delta_k \, (\delta_{p-k} \, {}^kN^{n-2} + \theta_{p-k} \, H^{n-2} S^n) \qquad \text{by (3.4)} \\
&= \delta_k \, \delta_{p-k} \, {}^kN^{n-2} \\
&= \delta_k \, \delta_{p-k} \, \delta_k \, {}^{p-k}N^{n-3} \\
&= (\delta_k \, \delta_{p-k})^2 \, {}^kN^{n-4} \\
&\;\;\vdots \\
&= \begin{cases} (\delta_k \, \delta_{p-k})^{m-1} \, \delta_k \, {}^{p-k}N^1 & \text{if } n = 2m > 0 \\ (\delta_k \, \delta_{p-k})^m \, {}^kN^1 & \text{if } n = 2m + 1 \end{cases} \\
&= \begin{cases} (\delta_k \, \delta_{p-k})^m \, {}^kH^0 \, (E, \Delta) & \text{if } n = 2m \\ (\delta_k \, \delta_{p-k})^m \, \delta_k \, {}^{p-k}H^0 \, (E, \Delta) & \text{if } n = 2m + 1 \end{cases} \\
&= 0, \text{ because } {}^kH^0 \, (E, \Delta) = 0 \text{ for any } k = 1, 2, \dots, p-1.
\end{aligned}$$

Therefore, combining the two results we get

$$^{k}N^{np} = \mu_k^a \, \delta_k \, \theta_{p-k} \, H^n S^n \, .$$

On the other hand,

$$\begin{aligned} & i^* \beta_k \gamma'_{p-k} a^p \\ & = \beta'_k j_k^* \gamma'_{p-k} a^p \quad \text{(see the diagram in the proof of (3.3.2))} \\ & = (t^{p-k})^* a^p \qquad \text{by the definition of } (t^{p-k})^* \\ & = 0 \, ; \end{aligned}$$

hence $\beta_k \gamma'_{p-k} a^p = 0$ because i^* is a monomorphism as we saw in the proof of (3.3.2). By Richardson-Smith exactness, $\beta_k \gamma'_{p-k} a^p = 0$ implies that $\gamma'_{p-k} a^p$ is in $^{k}N^{np} = \delta_k \, ({}^{p-k}H^{np-1} (E, \Delta))$. Hence, there exists an element m in Z_p such that

$$\gamma'_{p-k} \, a^p = m \, \mu_k^a \, \delta_k \, \theta_{p-k} \, a$$

by what we proved in the preceding paragraph.

m is nonzero because $m = 0$ would mean $\gamma'_{p-k} a^p = 0$ and hence $a^p = 0$ by (3.2).

Let a_0 be the fundamental cohomology class of $H^n S^n$ and m_0 be the number corresponding to a_0 in the formula

$$\gamma'_{p-k} \, a_0^p = m_0 \, \mu_k^a \, \delta_k \, \theta_{p-k} \, a_0 \, .$$

Let a be any class of $H^n S^n$. Then $a = j \, a_0$ for some j in Z_p and

$$\begin{aligned} \gamma'_{p-k} \, a^p & = \gamma'_{p-k} (j \, a_0)^p \\ & = j^p \gamma'_{p-k} \, a_0^p \\ & = j \, \gamma'_{p-k} \, a_0^p \qquad \text{(by Fermat's theorem)} \\ & = j m_0 \, \mu_k^a \, \delta_k \, \theta_{p-k} \, a_0 \\ & = m_0 \, \mu_k^a \, \delta_k \, \theta_{p-k} \, a. \end{aligned}$$

This implies that $m = m_0$ and hence m is independent of the choice of a in $H^n S^n$.

To show the independence of m from k, $1 \leqq k \leqq p-1$, we write

$$\gamma_t a^p = m\, \mu^{\alpha}_{p-1}\, \delta_{p-1}\, \theta_1\, a$$

$$= m\, \mu^{\alpha}_{p-1}\, \delta_{p-1}\, \theta_1\, r^* \bar{a} \qquad \text{where } \bar{a} = a \otimes 1 \otimes \ldots \otimes 1$$

$$= -\, m\, \mu^{\alpha}_{p-1}\, \delta_{p-1}\, \delta_1\, \gamma'_1 \bar{a} \quad \text{by (3.1)}$$

$$= -\, m\, \mu^{\alpha+1}_{p-1}\, \gamma'_t \bar{a}$$

$$= -\, m\, (\cup\, \mu_0)^{\alpha+1} \cup \gamma'_t\, \bar{a} \qquad \text{by (2.13).}$$

Then, for $k = 2, 3, \ldots, p-1$, we have

$$\gamma'_k a^p = \gamma_{1,k-1}\, \gamma'_t\, a^p$$

$$= -\, m\, \gamma_{1,k-1}\, ((\cup\, \mu_0)^{\alpha+1} \cup \gamma'_t\, \bar{a})$$

$$= -\, m\, (\cup\, \mu_0)^{m+1} \cup \gamma_{1,k-1}\, \gamma'_t\, \bar{a} \qquad \text{by (1.17)}$$

$$= -\, m\, (\cup\, \mu_0)^{m+1} \cup \gamma'_k\, \bar{a}$$

$$= -\, m\, \mu^{\alpha+1}_k\, \gamma'_k \bar{a} \qquad \text{by (2.13.1)}$$

$$= m\, \mu^{\alpha}_k\, \delta_k\, \theta_{p-k}\, a\,;$$

and, thus, m is independent of k. For λ, we set $\lambda = m^{-1}$.

Finally, if $p = 2$ it is clear that a similar argument holds with $\lambda = 1$, and we have

$$\gamma'_0\, a^2 = \delta_0^{n-1}\, \theta a. \qquad \text{Q. E. D}$$

We shall write $\lambda = \lambda_n$ to indicate that λ depends exclusively on the dimension n of the cohomology class used in defining it (if $p \geqq 3$). λ_n will be determined later up to the sign (cf. Corollary 4.19 and the remark succeeding it).

Since $(t^k)^* (t^{p-k}) = 0$, we define the group ${}^k\mathcal{H}^*E$ by $\ker (t^k)^*/\mathrm{im}\,(t^{p-k})^*$, for $k = 1, 2, \ldots, p-1$. Then the following lemma characterizes this group.

LEMMA 3.6. Let $n > 0$ and let X be a finite simplicial or CW-complex.

(1) The map $\eta : H^n X \to {}^k\mathcal{H}^{np}E$, defined by $\eta(a) = [a^p]$ = the class of a^p mod $\mathrm{im}\,(t^{p-k})^*$, is an isomorphism, and

(2) ${}^k\mathcal{H}^rE = 0$ if r is not divisible by p.

Proof. It is obvious that η is well-defined.

(1) First we show that η is a monomorphism. Let $\eta(a)=0$. Then a^p is in the image of $(t^{p-k})^*$, i. e. $a^p=(t^{p-k})^* x$ for some class x in $H^{np}E$. Hence, $\gamma'_k a^p = \gamma'_k (t^{p-k})^* x = \gamma'_k \beta'_k j^*_k \gamma'_{p-k} x = 0$ by exactness. By (3.2), $a^p=0$.

Secondly, to prove that η is an epimorphism, let y be an arbitrary element of ${}^k\mathcal{H}^{np}E$. Then $(t^k)^* y = 0$ by construction, so $y \equiv \sum_i z_i^p$ mod im $(t^{p-k})^*$ by an argument in the proof of (3.3). Thus, $y=(\sum_i z_i)^p$ mod im $(t^{p-k})^*$, i. e., $y=\eta(\sum_i z_i)$.

(2) Let y be an element of ${}^kH^r E, p \nmid r$. $(t^k)^* y = 0$ by the definition of ${}^k\mathcal{H}^r E$. This implies that y is an element of im $(t^{p-k})^*$ because r is not divisible by p. Q. E. D.

Lemma 3.6 implies that $H^n X \cong {}^k\mathcal{H}^{np} E$ and ${}^k\mathcal{H}^r E = 0$, for all $k = 1, 2, \ldots, p-1$ and for any integer r which is not divisible by p, i. e. ${}^k\mathcal{H}^* E$ does not depend on k. So hereafter we shall omit k and just write $\mathcal{H}^* E$ for ${}^k\mathcal{H}^* E$.

Definition. We define a map $\gamma^*_{p-k}\colon \mathcal{H}^{np}E \to {}^kH^{np}(E,\Delta)$ by $\gamma^*_{p-k}[a^p] = \gamma'_{p-k} a^p$ where $[a^p]=\eta(a)$.

Then we get the following lemma:

Lemma 3.7. Let $n>0$ and $\alpha = np'-1$, if $p \geqq 3$. Then

(1) $\gamma^*_{p-k}\colon \mathcal{H}^{np}E \to {}^kH^{np}(E,\Delta)$ is a monomorphism for all $k=1,2,\ldots,p-1$, and

(2) if dim $X=n$, the following diagram is commutative:

$$\begin{array}{ccc} H^n\Delta \equiv H^n X & \xrightarrow{\quad\eta\quad} & \mathcal{H}^{np}(E) \\ & {\scriptstyle \mu_k^\alpha \delta_k \theta_{p-k}} \searrow \qquad \swarrow {\scriptstyle \lambda_n \gamma^*_{p-k}} & \\ & {}^kH^{np}(E,\Delta) & \end{array}$$

where λ_n is the number defined in (3.5).

(3) In particular, when $p=2$, the map $\gamma^*_0\colon \mathcal{H}^{2n}E \to {}^tH^{2n}(E,\Delta)$ is a

monomorphism; and the diagram

$$\begin{array}{ccc} H^n X & \xrightarrow{\eta} & \mathcal{H}^{2n}(E) \\ & \delta_0^{n-1}\theta \searrow \quad \swarrow \gamma_0^* & \\ & {}^t H^{2n}(E, \Delta) & \end{array}$$

is commutative if $\dim X = n$.

PROOF. (1) By (3.2) $\gamma_{p-k}^* [a^p] = \gamma'_{p-k} a^p = 0$ would imply that $a^p = 0$ and $[a^p] = \eta(a) = 0$.

(2) The proof is done by going into the sample spaces S^n and $(S^n)^p$. Let K be the p-fold cartesian product of the n-sphere S^n and let Δ_K be its diagonal set. The meaning of E and Δ are as usual, i. e. $E = X^p$ and $\Delta = \operatorname{diag} E$. Since $\dim X = n$, $H^i X = 0$ for $i \geqq n+1$. Hence, by Hopf's theorem, $H^n X \cong [X, S^n]/p^* [X, S^n]$ $(H^n(X; Z) \cong [X, S^n], p^* : H^n(X; Z) \longrightarrow H^n(X; Z)$ induced by multiplication by $p, Z \xrightarrow{p} Z)$, and this isomorphism is natural in the sense that for a given element x of $H^n X$, there exists a corresponding map $f : X \longrightarrow S^n$ such that $f^* a = x$ where a is the fundamental class of $H^n S^n$. Then we have the following commutative diagram:

$$\begin{array}{ccccc} H^{np}(K) & \xrightarrow{\gamma'_{p-k}} & {}^k H^{np}(K, \Delta_K) & \xleftarrow{\mu_k^a \delta_k \theta_{p-k}} & H^n S^n \\ {\scriptstyle (f \times \dots \times f)^*} \downarrow & & \downarrow {\scriptstyle (f \times \dots \times f)_k^*} & & \downarrow {\scriptstyle f^*} \\ H^{np}(E) & \xrightarrow[\gamma'_{p-k}]{} & {}^k H^{np}(E, \Delta) & \xleftarrow[\mu_k^a \delta_k \theta_{p-k}]{} & H^n X. \end{array}$$

Hence

$\mu_k^a \delta_k \theta_{p-k} x$

$= \mu_k^a \delta_k \theta_{p-k} f_a^*$

$= (f \times \dots \times f)_k^* \mu_k^a \delta_k \theta_{p-k} a$ by the commutativity of the above diagram

$= \lambda_n (f \times \dots \times f)_k^* \gamma'_{p-k} a^p$ by (3.5)

$= \lambda_n \gamma'_{p-k} (f \times \dots \times f)^* a^p$ by the commutativity of the above diagram

$$= \lambda_n \gamma'_{p-k} (f^* a)^p$$

$$= \lambda_n \gamma'_{p-k} x^p = \lambda_n \gamma^*_{p-k} [x^p]$$

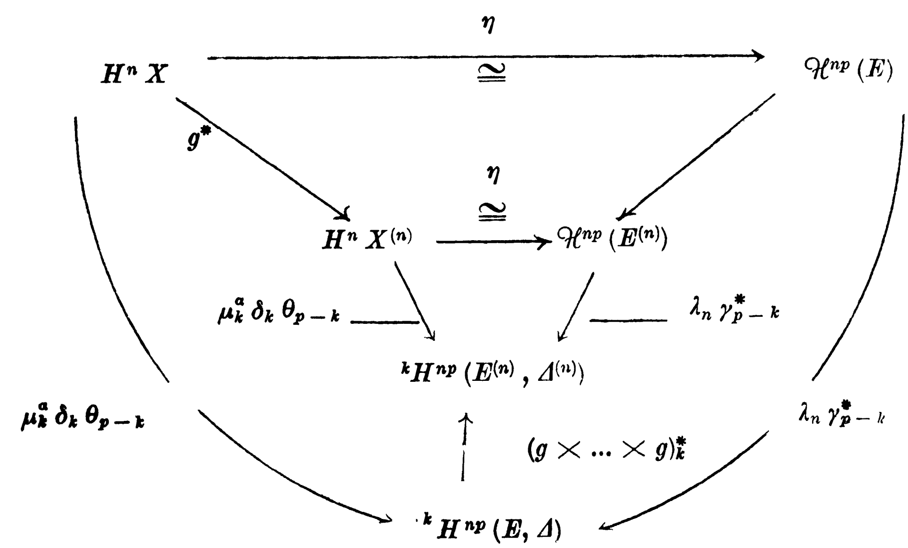
$$= \lambda_n \gamma^*_{p-k} \eta x, \qquad \text{which proves (2).}$$

(3) is proved by a similar argument. Q. E. D.

Corollary 3.8. $\mu^a_k \delta_k \theta_{p-k} : H^n X \to {}^k H^{np} (E, \Delta)$ and $\delta_0^{n-1} \theta : H^n X \to \to {}^t H^{2n} (E, \Delta)$ are monomorphisms, where $\dim X = n > 0$ and $\alpha = np' - 1$ (α is defined if p is an odd prime).

Proof. It is immediate from (3.6.1) and (3.7). Q. E. D.

Lemma (3.7) and its corollary (3.8) can be applied to the n skeleton $X^{(n)}$ of X when $\dim X > n$, as we did in the proof of (3.2). Then the diagram



is commutative everywhere except around the exterior triangle (see (3.2) for notations).

Let $x \in H^n X$. If $\mu^a_k \delta_k \theta_{p-k} x = 0$ then $\mu^a_k \delta_k \theta_{p-k} (g^* x) = (g \times \dots \times g)^*_k \mu^a_k \circ \circ \delta_k \theta_{p-k} x = 0$ so that $g^* x = 0$ by (3.8). Then $x = 0$ since g^* is a monomorphism. Also if $\gamma^*_{p-k} \eta x = 0$, then $\gamma^*_{p-k} \eta (g^* x) = (g \times \dots \times g)^*_k \gamma^*_{p-k} \eta x = 0$ which means that $g^* x = 0$ by (3.8) and hence $x = 0$. Analogous discussions hold for the case $p = 2$. Thus we have shown the following corollary:

COROLLARY 3.9. The maps

$$\gamma^*_{p-k}\eta : H^n X \longrightarrow {}^k H^{np}(E, \Delta)$$

and

$$\mu^a_k \delta_k \theta_{p-k} : H^n X \longrightarrow {}^k H^{np}(E, \Delta)$$

are monomorphic for all $k = 1, 2, \dots, p-1$, $n > 0$, $p \geqq 3$ and for any complex X. If $p = 2$, the maps

$$\gamma_0^* \eta : H^n X \longrightarrow {}^t H^{2n}(E, \Delta)$$

and

$$\delta_0^{n-1} \theta : H^n X \longrightarrow {}^t H^{2n}(E, \Delta)$$

are monomorphic, for $n > 0$ and for any complex X.

Note also that $\delta_0^{n-1}\theta = \nu^{n-1}\theta$ because $\nu = \delta_0$ when $p = 2$.

Let $(a_i) = (a_0, a_1, \dots, a_{n(p-1)-1})$ be a collection of elements $a_i \in H^{n+i} X$ for $0 \leqq i \leqq n(p-1)-1$. Such a $n(p-1)$-tuple (a_i) shall be called a system.

PROPOSITION 3.10. *Let $n > 0$, and $\alpha = np' - 1$ if $p \geqq 3$.*

(1) *In case $p \geqq 3$, define*

$$y = y(a_i) = \theta_k a_{n(p-1)-1} + \delta_k \theta_{p-k} a_{n(p-1)-2} +$$

$$+ (\delta_k \delta_{p-k}) \theta_k a_{n(p-1)-3} + \dots + (\delta_k \delta_{p-k})^\alpha \delta_k \theta_{p-k} a_0$$

where $(a_i) = (a_0, a_1, \dots, a_{n(p-1)-1})$ is a system. If y belongs to $\gamma'_{p-k} H^{np} E$, then y is of the form

$$y = \lambda_n \gamma'_{p-k} a_0^p$$

where λ_n is the number defined and characterized in (3.5).

(2) *Let $p = 2$.*

If $y = y(a_i) = \theta a_{n-1} + \delta_0 \theta a_{n-2} + \dots + \delta_0^{n-1} \theta a_0$, where $(a_i) = (a_0, a_1, \dots, a_{n-1})$, is in $\gamma'_0 H^{2n} E$, then y is of the form

$$y = \gamma'_0 a_0^2 .$$

PROOF. (1) Let $X^{(n)}$ be the n-skeleton of X, $E^{(n)} = (X^{(n)})^p = X^{(n)} \times \times \dots \times X^{(n)}$ be the p-fold cartesian product of $X^{(n)}$, $\Delta^{(n)}$ be the diagonal set of $E^{(n)}$ and $g : X^{(n)} \subset X$ be the inclusion map as in the proof of (3.2). $g^* : H^n X \longrightarrow H^n(X^{(n)})$ is monomorphic. Define a homomorphism $\widehat{g^*} : \mathcal{H}^{np}(E) \longrightarrow$

$\longrightarrow \mathcal{H}^{np}(E^{(n)})$ by $\widehat{g^*}[a^p] = [(g^* a)^p]$. Let $f = g \times \dots \times g : E^{(n)} \longrightarrow E$,

$$f^* : H^* E \longrightarrow H^* E^{(n)}$$

and

$$f^*_k = (g \times \dots \times g)^*_k : {}^kH^*(E, \Delta) \longrightarrow {}^kH^*(E^{(n)}, \Delta^{(n)})$$

be defined in the obvious manner. Then the following diagram is commutative:

$$\begin{array}{ccccccc}
H^n(X) & \underset{\cong}{\overset{\eta}{\longrightarrow}} & \mathcal{H}^{np}(E) & \overset{\gamma^*_{p-k}}{\longrightarrow} & {}^kH^{np}(E, \Delta) & \overset{\gamma'_{p-k}}{\longleftarrow} & H^{np}(E) \\
\downarrow g^* & & \downarrow \widehat{g^*} & & \downarrow f^*_k & & \downarrow f^* \\
H^n(X^{(n)}) & \underset{\eta}{\overset{\cong}{\longrightarrow}} & \mathcal{H}^{np}(E^{(n)}) & \underset{\gamma^*_{p-k}}{\longrightarrow} & {}^kH^{np}(E^{(n)}, \Delta^{(n)}) & \underset{\gamma'_{p-k}}{\longleftarrow} & H^{np}(E^{(n)}).
\end{array}$$

Here $g^*, \widehat{g^*}$ and γ^*_{p-k} are monomorphisms, g^* being a monomorphism by construction, $\widehat{g^*}$ being a monomorphism by the commutativity $\eta\, g^* = \widehat{g^*}\, \eta$ with η isomorphism, and γ^*_{p-k} being a monomorphism by (3.9); η is an isomorphism by (3.6.1); and $\gamma'_{p-k} : H^{np}(E^{(n)}) \longrightarrow {}^kH^{np}(E^{(n)}, \Delta^{(n)})$ is an epimorphism because the sequence $H^{np}(E^{(n)}) \overset{\gamma'_{p-k}}{\longrightarrow} {}^kH^{np}(E^{(n)}, \Delta^{(n)}) \overset{\delta'_{p-k}}{\longrightarrow} {}^{p-k}H^{np+1}(E^{(n)}) \equiv 0$ is exact. Hence, $f^*_k : {}^kH^{np}(E, \Delta) \longrightarrow {}^kH^{np}(E^{(n)}, \Delta^{(n)})$ is monomorphic on $\gamma^*_{p-k}\, \mathcal{H}^{np}(E)$. Now,

$$f^*_k y = f^*_k(\theta_k a_{n(p-1)-1} + \dots + (\delta_k \delta_{p-k})^a \delta_k \theta_{p-k} a_0)$$

$$= (\delta_k \delta_{p-k})^a \delta_k \theta_{p-k} g^* a_0 \qquad \text{by naturality}$$

$$= \lambda_n \gamma^*_{p-k}[(g^* a_0)^p] \qquad \text{by (3.7.2)}$$

$$= \lambda_n \gamma^*_{p-k} \widehat{g^*}[a_0^p] \qquad \text{by the definition of } \widehat{g^*}$$

$$= \lambda_n f^*_k \gamma^*_{p-k}[a_0^p] \qquad \text{by the commutativity of the above diagram}$$

$$= \lambda_n f^*_k \gamma'_{p-k} a_0^p,$$

i. e. $\quad f^*_k(y - \lambda_n \gamma'_{p-k} a_0^p) = 0.$

Since y is in $\gamma'_{p-k} H^{np} E$ by assumption, we can write y as

$$y = \lambda_n \gamma'_{p-k} a_0^p + \gamma'_{p-k} \Sigma (z_{i_1} \otimes \cdots \otimes z_{i_p})$$

where $\sum_{j=1}^{p} (\dim z_{i_j}) = np$.

It suffices to show that $\gamma'_{p-k}(\Sigma z_{i_1} \otimes \cdots \otimes z_{i_p}) = 0$. Since $j_k^* \theta_k = 0$ and $\beta'_k j_k^* \delta_k = \beta'_k \delta'_k = 0$ by the Richardson-Smith exact sequence, we have $\beta'_k j_k^* y = 0$. Hence,

$$\begin{aligned} 0 &= \beta'_k j_k^* \gamma'_{p-k} a_0^p + \beta'_k j_k^* \gamma'_{p-k} (\Sigma z_{i_1} \otimes \cdots \otimes z_{i_p}) \\ &= (t^{p-k})^* a_0^p + (t^{p-k})^* (\Sigma z_{i_1} \otimes \cdots \otimes z_{i_p}) \\ &= (t^{p-k})^* \Sigma z_{i_1} \otimes \cdots \otimes z_{i_p}, \end{aligned}$$

which implies that

$$\Sigma z_{i_1} \otimes \cdots \otimes z_{i_p} = z^p + \text{image of } (t^k)^*, \text{ and}$$

that $\gamma'_{p-k} \Sigma z_{i_1} \otimes \cdots \otimes z_{i_p} = \gamma'_{p-k} z^p$ by Richardson-Smith exactness. Thus, we have shown that

$$y = \lambda_n \gamma'_{p-k} a_0^p + \gamma'_{p-k} z^p.$$

Now, $f_k^* y = \lambda_n f_k^* \gamma'_{p-k} a_0^p$ implies that $f_k^* \gamma'_{p-k} z^p = f_k^* \gamma^*_{p-k} [z^p] = 0$. As we observed earlier, f_k^* is monomorphic on $\gamma^*_{p-k} (\mathcal{H}^{np} E)$. Hence, we get $\gamma^*_{p-k} [z^p] = \gamma'_{p-k} z^p = 0$, i. e., $y = \lambda_n \gamma'_{p-k} a_0^p$.

(2) is similar to (1). Q. E. D.

PROPOSITION 3. 11. *Let x be an element of ${}^k H^{np}(E, \Delta)$. If x belongs to $(\gamma'_{p-k} H^{np}(E, \Delta)) \cap {}^k N^{np}$, then x belongs to $\gamma^*_{p-k} (\mathcal{H}^{np} E)$, for any prime p.*

PROOF. We proceed just as in the proof of (3.10). Since x is in $\gamma'_{p-k} H^{np}(E, \Delta)$, we have $x = \gamma'_{p-k} z^p + \gamma'_{p-k} \Sigma z_{i_1} \otimes \cdots \otimes z_{i_p}$ ($z_{i_1} \otimes \cdots \otimes z_{i_p}$ mixed). x is also in ${}^k N^{np}$, so that $\beta'_k j_k^* x = 0$. This implies that $\Sigma z_{i_1} \otimes \otimes \cdots \otimes z_{i_p} \in$ image of $(t^k)^*$ and $\gamma'_{p-k}(\Sigma z_{i_1} \otimes \cdots \otimes z_{i_p}) = 0$. Therefore, $x = \gamma'_{p-k} z^p = \gamma^*_{p-k} [z^p]$. Q. E. D.

PROPOSITION 3.12. (*Uniqueness*). *Let n be a positive integer and k be an integer, $1 \leq k \leq p-1$. For any a in $H^n X$, there exists at most one system $(a_i) = (a_0, a_1, \ldots, a_{n(p-1)-1})$ such that $a_0 = a$, $a_i \in H^{n+i} X$ and $y = \lambda_n \gamma'_{p-k} a^p$ where $y = y(a_i)$ is the expression in Proposition* 3.10.

Proof. Suppose that we have distinct systems (a_i) and (b_i) such that $a_0 = b_0 = a$, a_i, $b_i \in H^{n+i} X$ and $y(a_i) = y(b_i) = \lambda_n \gamma'_{p-k} a^p$. Let $c_i = a_i - b_i$ and $m = \min\{i \mid c_i \neq 0\}$. Then clearly $1 \leqq m \leqq n(p-1)-1$, and

$$y(c_i) = \theta_k c_{n(p-1)-1} + \delta_k \theta_{p-k} c_{n(p-1)-2} + \dots + (\delta_k \delta_{p-k})^\alpha \delta_k \theta_{p-k} c_0 = 0,$$

where $\alpha = np' - 1$ as usual. Let

$$\overline{c_i} = \begin{cases} c_{m+i} & \text{if } 0 \leqq i \leqq n(p-1)-1-m \\ 0 & \text{if } n(p-1)-m \leqq i \leqq (n+m)(p-1)-1. \end{cases}$$

$(\overline{c_i}) = (c_m, c_{m+1}, \dots, c_{n(p-1)-1}, 0, \dots, 0)$ is a system such that $c_{m+i} \in H^{n+m+i} X$.
We shall consider the following two cases:

Case *A*. m is odd.
Let

$$y_1 = \theta_{p-k} \overline{c}_{(n+m)(p-1)-1} + \delta_{p-k} \theta_k \overline{c}_{(n+m)(p-1)-2} + \dots + (\delta_{p-k} \delta_k)^{(m+n)p'-1} \delta_{p-k} \theta_k \overline{c}_0 .$$

Then

$$y_1 = (\delta_{p-k} \delta_k)^{(mp-1)/2} \delta_{p-k} (\theta_k c_{n(p-1)} + \dots + (\delta_k \delta_{p-k})^\alpha \delta_k \theta_{p-k} c_0)$$
$$= (\delta_{p-k} \delta_k)^{(mp-1)/2} \delta_{p-k} y(c_i) = 0.$$

Hence $y_1 = 0 \in \operatorname{im} \gamma'_k$.
By (3.10) $0 = y_1 = \lambda_m \gamma'_k \overline{c}_0^p = \lambda_m \gamma'_k c_m^p$, which implies that $c_m^p = 0$ by (3.2) and that $c_m = 0$ contradicting our assumption $c_m \neq 0$.

Case *B*. m is even.
Let

$$y_2 = \theta_k \overline{c}_{(n+m)(p-1)-1} + \delta_k \theta_{p-k} \overline{c}_{(n+m)(p-1)-2} + \dots + (\delta_k \delta_{p-k})^{(n+m)p'-1} \delta_k \theta_{p-k} \overline{c}_0 .$$

Then

$$y_2 = (\delta_k \delta_{p-k})^{mp/2} y(c_i) = 0, \qquad \text{as in Case } A.$$

Hence, $0 = y_2 = \lambda_m \gamma'_{p-k} \overline{c}_0^p = \lambda_m \gamma'_{p-k} c_m^p$, which again implies that $c_m^p = 0$ by (3.2) and that $c_m = 0$ contradicting $c_m \neq 0$. Q. E. D.

(3.12) should be properly interpreted for $p = 2$.

Although the following result was stated and proved in the proof of (3.5) for $X = S^n$, we restate if for an arbitrary complex X.

LEMMA 3.13. Let n be a positive integer. For any class a in $H^n X$, $\gamma'_k a^p$ is in ${}^{p-k}N^{np}$.

PROOF. Since $i^* \beta_{p-k} \gamma'_k a^p = \beta'_{p-k} j^*_{p-k} \gamma'_k a^p = (t^k)^* a^p = 0$ and i^* is a monomorphism, $\beta_{p-k} \gamma'_k a^p = 0$. Thus, $\gamma'_k a^p$ belongs to $\ker \beta_{p-k} = \operatorname{im} \delta_{p-k} = {}^{p-k}N^{np}$. Q. E. D.

Now we state and prove the main theorem of this §.

THEOREM 3.14. (*Thom direct sum decomposition for $p \geqq 3$*). *Let $n > 0$ and $k = 1, 2, \ldots, p-1$. We have the following direct sum decomposition of ${}^kN^n$:*

(1) *If $n = rp - 2q$, for integers r and q such that $0 \leqq 2q < p$, then*

$$ {}^kN^n = \theta_k H^{n-1} \Delta + \delta_k \theta_{p-k} H^{n-2} \Delta + \delta_k \delta_{p-k} \theta_k H^{n-3} \Delta $$

$$ + \ldots + (\delta_k \delta_{p-k})^{rp'-q-1} \delta_k \theta_{p-k} H^r \Delta. $$

(2) *If $n = rp - (2q+1)$, for some r and q such that $0 < 2q+1 < p$, then*

$$ {}^kN^n = \theta_k H^{n-1} \Delta + \delta_k \theta_{p-k} H^{n-2} \Delta + \delta_k \delta_{p-k} \theta_k H^{n-3} \Delta $$

$$ + \ldots + (\delta_k \delta_{p-k})^{rp'-q-1} \theta_k H^r \Delta. $$

Notice that $r = \min \{ j \mid n/p \leqq j \}$.

PROOF. First, we prove the directness of the decomposition. Suppose that

$$ \theta_k x_{n-1} + \delta_k \theta_{p-k} x_{n-2} + \ldots + (\delta_k \delta_{p-k})^{rp'-q-1} \delta_k \theta_{p-k} x_r = 0 $$

where $x_i \in H^i \Delta$, for $r \leqq i \leqq n-1$. Denote the left hand side of the above equation by z. Apply $(\delta_k \delta_{p-k})^q$ to z and get $(\delta_k \delta_{p-k})^q z = 0$. Now, $(\delta_k \delta_{p-k})^q z$ is an element of dimension rp and is of the form given in Proposition 3.10, hence $x_{n-1} = x_{n-2} = \ldots = x_r = 0$ by (3.10) and (3.12). This proves the directness of the decomposition of part (1). For the directness of part (2), the argument is similar; we have to apply $(\delta_{p-k} \delta_k)^q \delta_{p-k}$ instead of $(\delta_k \delta_{p-k})^q$.

Now we prove the equalities of (1) and (2) by induction on n. Let $n = 1$. Then ${}^kN^1 = \delta_k {}^{p-k}H^0 (E, \Delta) = 0$ because ${}^{p-k}H^0 (E, \Delta) = 0$ for all k. Also, $\theta_k H^0 \Delta = \theta_k r^* H^0 E = \delta_k \gamma'_k H^0 E \subset \delta_k {}^{p-k}H^0 (E, \Delta) = 0$ by (3.1) and by the exactness of $H^0 E \xrightarrow{r^*} H^0 \Delta \to 0$. Hence, (1) and (2) hold and both sides are zero. Assume that they hold for $1, 2, \ldots, n$. We shall prove them for $n+1$ by considering the following two cases.

Case A. n is not divisible by p.

By (3.4), ${}^{k}N^{n+1} = \theta_k H^n \Delta + \delta_k {}^{p-k}N^n$. By the induction hypothesis, ${}^{p-k}N^n$ will be decomposed into the direct sum of groups by (1) or (2). Hence, (1) and (2) follow for $n+1$.

Case B. n is divisible by p.

Let $n = rp$. Again by (3.4) and by the induction hypothesis, we have

$$\begin{aligned}{}^{k}N^{n+1} &= \theta_k H^n \Delta + \delta_k {}^{p-k}N^n \\ &= [\theta_k H^n \Delta + \delta_k \theta_{p-k} H^{n-1} \Delta + \dots + (\delta_k \delta_{p-k})^{rp'-1} \delta_k \theta_{p-k} H^{r+1} \Delta] \\ &\quad + (\delta_k \delta_{p-k})^{rp'} \theta_k H^r \Delta.\end{aligned}$$

Since $\min \{j \in Z \mid n+1/p \leqq j\} = r+1$, we have to express the elements of $(\delta_k \delta_{p-k})^{rp'} \theta_k H^r \Delta$ as linear combinations of elements in the bracket. Let $a \in H^r \Delta$. By (3.13), $\gamma_k' a^p$ is in ${}^{p-k}N^{rp}$. By induction hypothesis and by (3.10) and (3.12), there exists uniquely determined elements $a_{n-1}, a_{n-2}, \dots, a_r$ such that $a_i \in H^i \Delta$, $a_r = a$ and

$$\begin{aligned}\lambda_r \gamma_k' a^p &= \theta_{p-k} a_{n-1} + \delta_{p-k} \theta_k a_{n-2} + \delta_{p-k} \delta_k \theta_{p-k} a_{n-3} + \dots + \\ &\quad + (\delta_{p-k} \delta_k)^{rp'-1} \delta_{p-k} \theta_k a_r .\end{aligned}$$

Hence,

$$\lambda_r \delta_k \gamma_k' a^p = \delta_k (\theta_{p-k} a_{n-1} + \dots + (\delta_{p-k} \delta_k)^{rp'-1} \theta_{p-k} a_{r+1}) + (\delta_k \delta_{p-k})^{rp'} \theta_k a,$$

which implies that

$$\begin{aligned}(\delta_k \delta_{p-k})^{rp'} \theta_k a &= -(\lambda_r \theta_k r^* a^p + \delta_k \theta_{p-k} a_{n-1} + \dots + \\ &\quad (\delta_k \delta_{p-k})^{rp'-1} \delta_k \theta_{p-k} a_{r+1}) .\end{aligned}$$

Q. E. D.

Similarly, for $p=2$ we have the following decomposition :

Proposition 3.15. (*Thom direct sum decomposition for $p=2$*). *Let $n>0$. If $n = 2r$, then*

$$\begin{aligned}{}^{t}N^n &= \theta H^{n-1} \Delta + \delta_0 \theta H^{n-2} \Delta + \dots + \delta_0^{r-1} \theta H^r \Delta \\ &= \theta H^{n-1} \Delta + \nu \theta H^{n-2} \Delta + \dots + \nu^{r-1} \theta H^r \Delta\end{aligned}$$

is a direct sum decomposition. If $n = 2r-1$, then

$${}^tN^n = \theta H^{n-1}\varDelta + \delta_0\,\theta H^{n-2}\varDelta + \dots + \delta_0^{r-2}\,\theta H^r\varDelta$$

$$= \theta H^{n-1}\varDelta + \nu\theta\, H^{n-2}\varDelta + \dots + \nu^{r-2}\,\theta H^r\varDelta$$

is direct.

PROOF is immediate by interpreting (3.14) in a proper way.

§ 4. Steenrod Powers and the Thom-Bott Formulas.

The Thom direct sum decomposition of ${}^kN^n\,(p \geq 3)$ and of ${}^tN^n\,(p=2)$ leads to representations of $\gamma_k' a^p$ and of $\gamma_0' a^2$ which give rise to the Steenrod cohomology operations $P_p^i\colon H^m \to H^{m+2i(p-1)}$ and $Sq^i\colon H^m \to H^{m+i}$. The Steenrod powers (for $p \geq 3$) or the Steenrod squares ($p=2$) appear in the direct summands of the Thom decomposition for $\gamma_k' a^p$ or $\gamma_0' a^2$. The existence of the Steenrod operations will be shown by proving the Steenrod-Epstein axioms for the appropriate parts in the direct summands of the Thom decomposition. The crucial step is the proof of the Cartan formula. Uniqueness (or axiomatic characterization) is proved again by making use of the Thom direct sum decomposition. This has been done already by Nakaoka [4]. The final theorem in this §, Theorem 4.21, sums up the connection between Smith theory and Steenrod operations by giving the Thom-Bott formulas in their general setting.

The following proposition is immediate from (3.12), (3.13), (3.14) and (3.15).

PROPOSITION 4.1. *Let $n > 0$. Let $\alpha = np' - 1$ and $1 \leqq k \leqq p-1$ if $p \geqq 3$. Let a be an element in $H^n X$. Then:*

(1) *For $p \geqq 3$, there is a unique system $(a_i^{(k)}) = (a_0^{(k)}, a_1^{(k)}, \dots, a_{n(p-1)-1}^{(k)})$ such that $a_i^{(k)} \in H^{n+i}X$, $a_0^{(k)} = a$ and*

$$\lambda_n\,\gamma_{p-k}'\,a^p = \theta_k\,a_{n(p-1)-1}^{(k)} + \delta_k\,\theta_{p-k}\,a_{n(p-1)-2}^{(k)} + \delta_k\,\delta_{p-k}\,\theta_k\,a_{n(p-1)-3}^{(k)} +$$

$$\dots + (\delta_k\,\delta_{p-k})^{\alpha}\,\delta_k\,\theta_{p-k}\,a_0^{(k)}\,.$$

(2) *For $p = 2$, there is a unique system $(a_i) = (a_0, a_1, \dots, a_{n-1})$ such that $a_i \in H^{n+i}X$, $a_0 = a$ and*

$$\gamma_0'\,a^2 = \theta\,a_{n-1} + \delta_0\,\theta\,a_{n-2} + \dots + \delta_0^{n-1}\,\theta a_0$$

$$= \theta\,a_{n-1} + \nu\theta\,a_{n-2} + \dots + \nu^{n-1}\,\theta\,a_0\,.$$

LEMMA 4.2. Let $(a_i^{(k)})$ be the system in (4.1.1). Then

$$(1) \qquad a^{(k)}_{n(p-1)-2i-1} = 0, \text{ for } 2 \leqq k \leqq p-1\,;$$

and

$$(2) \qquad a^{(1)}_{n(p-1)-2i} = a^{(2)}_{n(p-1)-2i} = \dots = a^{(p-1)}_{n(p-1)-2i}$$

where $1 \leqq i \leqq np'$.

PROOF. $\lambda_n \gamma'_k a^p = \lambda_n \gamma_{1,k-1} \gamma'_t a^p = \gamma_{1,k-1} (\theta_s a^{(1)}_{n(p-1)-1} + \delta_s \theta_t a^{(1)}_{n(p-1)-2} + \dots + (\delta_s \delta_t)^{\alpha} \delta_s \theta_t a^{(1)}_0)$ by (4.1.1). Now we have

$$\gamma_{1,k-1} (\delta_s \delta_t)^{i-1} \delta_s \theta_t a^{(1)}_{n(p-1)-2i} = \gamma_{1,k-1} \mu^{i-1}_{p-1} \delta_s \theta_t r^* x$$

$$(x = a^{(1)}_{n(p-1)-2i} \otimes 1 \otimes \dots \otimes 1)$$

$$= - \gamma_{1,k-1} \mu^{i-1}_{p-1} \delta_s \delta_t \gamma'_t x \qquad \text{by (3.1)}$$

$$= - \gamma_{1,k-1} \mu^{i}_{p-1} \gamma'_t x$$

$$= - \gamma_{1,k-1} (\cup \mu_0)^i \cup \gamma'_t x \qquad \text{by (2.13.1)}$$

$$= - (\cup \mu_0)^i \cup \gamma_{1,k-1} \gamma'_t x$$

$$= - (\cup \mu_0)^i \cup \gamma'_k x \qquad \text{by (1.17)}$$

$$= - \mu^{i-1}_{p-k} \delta_{p-k} \delta_k \gamma'_k x$$

$$= (\delta_{p-k} \delta_k)^{i-1} \delta_{p-k} \theta_k a^{(1)}_{n(p-1)-2i},$$

and for $k = 2, 3, \dots, p-1$ we have

$$\gamma_{1,k-1} (\delta_s \delta_t)^i \theta_s a^{(1)}_{n(p-1)-(2i-1)}$$

$$= \gamma_{1,k-1} \mu^{i}_{p-1} \theta_s r^* x' \qquad (x' = a^{(1)}_{n(p-1)-(2i-1)} \otimes 1 \otimes \dots \otimes 1)$$

$$= - \gamma_{1,k-1} \mu^{i}_{p-1} \delta_s \gamma'_s x'$$

$$= - \gamma_{1,k-1} \mu^{i}_{p-1} \delta_s \xi^*_t \gamma'_t x' \qquad \text{by (2.8)}$$

$$= - \gamma_{1,k-1} \mu^{i}_{p-1} \xi^*_s \delta_t \gamma'_t x' \qquad \text{by (2.11)}$$

$$= - (\cup \mu_0)^i \cup (\gamma_{1,k-1} \xi^*_s \delta_t \gamma'_t x') \qquad \text{by (2.13.1)}$$

$= 0$ because $\xi^*_s \delta_t \gamma'_t x'$ is written as $[t^{p-1} z]$ for some z and hence

$\gamma_{1,k-1}\, \xi_s^*\, \delta_t\, \gamma_t'\, x' = \gamma_{1,k-1}\, [t^{p-1}\, z] = [t^{p+k-2}\, z] = 0$. Therefore,

$$\lambda_n\, \gamma_k'\, a^p = \delta_{p-k}\, \theta_k\, a^{(1)}_{n(p-1)-2} + (\delta_{p-k}\, \delta_k)\, \delta_{p-k}\, \theta_k\, a^{(1)}_{n(p-1)-4} + \dots +$$

$$(\delta_{p-k}\, \delta_k)^a\, \delta_{p-k}\, \theta_k\, a_0^{(1)},$$

which implies (1) and (2) by (4.1.1.). Q. E. D.

Let β be the Bockstein homomorphism associated with the short exact sequence

$$0 \to Z_p \to Z_{p^2} \to Z_p \to 0$$

defined for absolute and relative cohomology.

LEMMA 4.3. For any prime $p \geqq 2$, we have

(1) $\beta\nu = \mu - \nu\beta : {}^tH^n(E, \Delta) \to {}^tH^{n+2}(E, \Delta)$.

(2) $\beta\mu = \mu\beta : {}^tH^n(E, \Delta) \to {}^tH^{n+3}(E, \Delta)$.

(3) $\beta\theta_t = -\theta_t\, \beta : H^n\, \Delta \to {}^tH^{n+2}(E, \Delta) = H^{n+2}(E_0, \Delta)$.

(4) $\delta_t\, \gamma_t'\, a^p = \beta\gamma_s'\, a^p$, for $a \in H^* X$.

PROOF. (1) By (1.15) and (1.16), there are cochains c^1, c^2 on $E - \Delta$ such that $\delta c^1 = sc^2$, $\mu_0 = \mu(1) = [sc^2]$ and $\nu_0 = \nu(1) = [sc^1]$. Since $\beta(\nu_0) = \beta[sc^1] = [\delta c^1] = [sc^2] = \mu_0$, we have $\beta\nu x = \beta(\nu_0 \cup x) = \beta\nu_0 \cup x - \nu_0 \cup \beta x = \mu_0 \cup x - \nu_0 \cup \beta x = \mu x - \nu\beta x$, for any $x \in H^* X$.

(2) $\beta\mu x = \beta(\mu_0 \cup x) = \beta\mu_0 \cup x + \mu_0 \cup \beta x = \beta^2\nu_0 \cup x + \mu_0 \cup \beta x = \mu_0 \cup \beta x = \mu\beta x$, for all $x \in H^* X$, because $\beta^2 = 0$.

(3) We have $0 \to Z_p \xrightarrow{\lambda} Z_{p^2} \xrightarrow{\mu} Z_p \to 0$ where $\lambda(n) = np$ and $\mu(m) = m$. This induces coefficient maps $\lambda_\#$ and $\mu_\#$ in the short exact sequences

$$0 \to C^\#(\Delta\,; Z_p) \xrightarrow{\lambda_\#} C^\#(\Delta\,; Z_{p^2}) \xrightarrow{\mu_\#} C^\#(\Delta\,; Z_p) \to 0,$$

$$0 \to C^\#(E_0, \Delta\,; Z_p) \xrightarrow{\lambda_\#} C^\#(E_0, \Delta\,; Z_p^2) \xrightarrow{\mu_\#} C^\#(E_0, \Delta\,; Z_p) \to 0.$$

Let $\tilde{\delta}$ be the ordinary coboundary operator on (E_0, Δ) over Z_p or Z_{p^2}. The map $\tilde{\delta}$ over Z_p induces $\theta_t : H^*\Delta \to H^*(E_0, \Delta)$. Let $\delta : C^\#(Y; Z_{p^2}) \to C^\#(Y; Z_{p^2})$ be the ordinary coboundary operator over Z_{p^2} for any space Y.

Let $[x]$ be a class in $H^n(\Delta; Z_p)$. x can also be viewed as an element of $C^n(\Delta; Z_{p^2})$ and $x \in C^n(\Delta; Z_{p^2})$ is a preimage of $x \in C^n(\Delta; Z_p)$ under $\mu_\#$. By definition $\beta[x] = [\lambda_\#^{-1} \delta x]$ where x is regarded as an element of $C^n(\Delta; Z_{p^2})$ so that we can apply δ. Hence, $\theta_t \beta[x] = \theta_t[\lambda_\#^{-1} \delta x] = [\tilde{\delta} \lambda_\#^{-1} \delta x] = [\lambda_\#^{-1} \tilde{\delta} \delta x]$ by naturality. On the other hand, $\beta \theta_t[x] = \beta[\tilde{\delta} x] = [\lambda_\#^{-1} \delta \tilde{\delta} x]$. Here again $\tilde{\delta} x$ is viewed as an element of $C^{n+1}(E_0, \Delta; Z_{p^2})$, so that $\delta(\tilde{\delta} x)$ makes sense.

Hence, it is enough to show $\delta \tilde{\delta} = - \tilde{\delta} \delta$. Let $s^p = [a_0 \dots a_p]$, $s^{p+1} = [a_0 \dots a_p \, a_{p+1}]$ be simplexes of Δ, and $s^{p+2} = [a_0 \dots a_p \, a_{p+1} \, a_{p+2}]$ be a simplex in $E_0 - \Delta$. Then

$$\begin{aligned} \delta \tilde{\delta} s^p &= \delta \{(-1)^{p+1} [a_0 \dots a_p \, a_{p+2}] + \text{others}\} \\ &= (-1)^{p+1} (-1)^{p+2} [a_0 \dots a_p \, a_{p+2} \, a_{p+1}] + \text{others} \\ &= - [a_0 \dots a_p \, a_{p+2} \, a_{p+1}] + \text{others} \\ &= s^{p+2} + \text{others}, \end{aligned}$$

whereas
$$\begin{aligned} \tilde{\delta} \delta s^p &= \tilde{\delta} \{(-1)^{p+1} s^{p+1} + \text{others}\} \\ &= (-1)^{p+1} (-1)^{p+2} s^{p+2} + \text{others} \\ &= - s^{p+2} + \text{others}. \end{aligned}$$

This is true for any $s^p < s^{p+1} \in \Delta$ and $s^{p+1} < s^{p+2} \in (E_0 - \Delta)$. If $s^p < s^{p+1} < s^{p+2}$ with $s^p \in \Delta$ and $s^{p+1}, s^{p+2} \in (E_0 - \Delta)$ or with $s^p, s^{p+1}, s^{p+2} \in \Delta$, then in both cases $\tilde{\delta} \delta (s^p)$ and $\delta \tilde{\delta} (s^p)$ do not have the term s^{p+2} obtained through s^{p+1}. This exhausts every case, and $\delta \tilde{\delta} = - \tilde{\delta} \delta$.

(4) Let a be in $H^* X$ and let c be a cocycle representing a. Then $a = [c]$ and $a^p = [c \times \dots \times c] \in H^* E$.

$\delta_t \gamma_t' \, a^p = \delta_t [t(c \times \dots \times c)] = [\delta u]$ and $\beta \gamma_s' \, a^p = \beta [s(c \times \dots \times c)] = [\delta u]$ imply (4) where $u \in C^\#(E, \Delta)$ with $tu = t(c \times c \times \dots \times c)$ (hence $su = s(c \times \times c \times \dots \times c)$).

REMARK. For $p = 2$, (4.3) becomes

(1)′ $\beta \nu = \nu^2 + \nu \beta$

(2)′ $\beta \nu^2 = \nu^2 \beta$

(3)′ $\beta \theta = \theta \beta$

(4)′ $\nu \gamma_0' \, a^2 = \beta \gamma_0' \, a^2$

and the proof is similar.

REMARK 2. (4.3.3) is a special case of the following remark. Let $0 \to C' \to C \to C'' \to 0$ be an exact sequence of free chain complexes, and $0 \to G' \to G \to G'' \to 0$ an exact sequence of modules (over a commutative ring R). Then the commutative diagram of R-modules

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C' \otimes G' & \to & C \otimes G' & \to & C'' \otimes G' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C' \otimes G & \to & C \otimes G & \to & C'' \otimes G & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & C' \otimes G'' & \to & C \otimes G'' & \to & C'' \otimes G'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

with exact rows and columns induces the anti-commutative diagram

$$\begin{array}{ccc} H_*(C'' \otimes G'') & \xrightarrow{\partial_*} & H_*(C' \otimes G'') \\ \downarrow \beta & & \downarrow \beta \\ H_*(C'' \otimes G') & \xrightarrow{\partial_*} & H_*(C' \otimes G') \end{array}$$

(cf. Proposition 2.1 of Cartan-Eilenberg's Homological Algebra, p. 56), $\partial_* \beta = -\beta \partial_*$, where ∂_* is the boundary homomorphism (belonging to $0 \to C' \to C \to C'' \to 0$) and β the Bockstein homomorphism for $0 \to G' \to \to G \to G'' \to 0$. A similar statement holds in cohomology.

By (4.2.2) we may write $a^{(k)}_{n(p-1)-2i}$ as $a_{n(p-1)-2i}$, for $1 \leqq k \leqq p-1$. Similarly, in view of (4.2.1), $a^{(1)}_{n(p-1)-(2i-1)}$ may be written as $a_{n(p-1)-(2i-1)}$.

LEMMA 4.4. Let $p \geq 3$. Then, $a_{n(p-1)-(2i-1)} = \beta a_{n(p-1)-2i}$, for all i, $1 \leqq i \leqq np'$.

PROOF. Let $\alpha = np' - 1$ and $a \in H^n X$. Then,

$\lambda_n \delta_t \gamma'_t a^p$

$= \delta_t (\theta_s a_{n(p-1)-1} + \dots + (\delta_s \delta_t)^\alpha \delta_s \theta_t a_0)$ by (4.1.1)

$= \nu \theta_t a_{n(p-1)-1} + \mu \theta_t a_{n(p-1)-2} + \dots + \mu^{\alpha+1} \theta_t a_0$.

On the other hand,

$$\lambda_n \delta_t \gamma_t' a_p$$

$$= \lambda_n \beta \gamma_s' a^p \qquad \text{by (4.3.4)}$$

$$= \beta (\delta_t \theta_s a_{n(p-1)-2} + \dots + (\delta_t \delta_s)^a \delta_t \theta_s a_0) \qquad \text{by (4.2.1)}$$

$$= \beta (\nu \theta_t a_{n(p-1)-2} + \dots + \mu^a \nu \theta_t a_0)$$

$$= (\mu - \nu\beta) \theta_t a_{n(p-1)-2} + \mu (\mu - \nu\beta) \theta_t a_{n(p-1)-4} + \dots + \mu^a (\mu - \nu\beta) \theta_t a_0$$

by (4.3.1) and (4.3.2)

$$= \nu\theta_t (\beta a_{n(p-1)-2}) + \mu \theta_t a_{n(p-1)-2} + \dots + \mu^a \nu \theta_t (\beta a_0) + \mu^{a+1} \theta_t a_0$$

by (4.3.3).

Hence, by comparison of these two expressions, we have

$$\nu \theta_t a_{n(p-1)-1} + \mu \nu \theta_t a_{n(p-1)-3} + \dots + \mu^a \nu \theta_t a_1$$

$$= \nu \theta_t (\beta a_{n(p-1)-2}) + \mu \nu \theta_t (\beta a_{n(p-1)-4}) + \dots + \mu^2 \nu \theta_t (\beta a_0).$$

By (3.14), we see that $a_{n(p-1)-(2i-1)} = \beta a_{n(p-1)-2i}$, for $1 \leqq i \leqq np'$. Q. E. D.

LEMMA 4.5. Let $p = 2$, and $(a_i) = (a_0, a_1, \dots, a_{n-1})$ be the system characterized by (4.1.2). Then, $a_{2i+1} = \beta a_{2i}$, for $0 \leqq 2i \leqq n-2$. Hence, in particular, $\beta a_{2i+1} = 0$.

PROOF. Suppose that n is even > 0.

$$\nu \gamma_0' (a \otimes a) = \nu (\theta a_{n-1} + \nu\theta a_{n-2} + \dots + \nu^{n-1} \theta a_0)$$

$$= \sum_{i=1}^{n} \nu^i \theta a_{n-i}.$$

Also

$$\nu \gamma_0' (a \otimes a) = \beta \gamma_0' (a \otimes a)$$

$$= \beta (\theta a_{n-1} + \dots + \nu^{n-1} \theta a_0)$$

$$= \theta \beta a_{n-1} + (\nu^2 + \nu\beta) \theta a_{n-2} + \nu^2 \theta \beta a_{n-3}$$

$$+ \nu^2 (\nu^2 + \nu\beta) \theta a_{n-4} + \dots + \nu^{n-2} (\nu^2 + \nu\beta) \theta a_0$$

$$= \{\nu\theta(\beta a_{n-2}) + \nu^2 \theta a_{n-2} + \nu^3 \theta(\beta a_{n-4}) + \nu^4 \theta a_{n-4} + \dots + \nu^{n-1}\theta(\beta a_0) +$$

$$+ \nu^n \theta a_0\} + \{\theta(\beta a_{n-1}) + \nu^2 \theta(\beta a_{n-3}) + \nu^4 \theta(\beta a_{n-5}) + \dots + \nu^{n-2}\theta(\beta a_1)\}.$$

Now let us look at the second term of the sum in the last equality. We want to show that $\beta a_{n-1} = \beta a_{n-3} = \dots = \beta a_1 = 0$. Apply ν to the sum to get

$$\nu\theta(\beta a_{n-1}) + \nu^3 \theta(\beta a_{n-3}) + \dots + \nu^{n-1}\theta(\beta a_1),$$

which is the expansion of the system

$$(\widetilde{a_i}) = (\widetilde{a_0}, \dots, \widetilde{a_n}) = (0, \beta a_1, 0, \beta a_3, \dots, 0, \beta a_{n-1}, 0).$$

Therefore,

$$\nu\theta(\beta a_{n-1}) + \nu^3 \theta(\beta a_{n-3}) + \dots + \nu^{n-1}\theta(\beta a_1) = \gamma_0'(0 \otimes 0) = 0.$$

By uniqueness, $\beta a_1 = \beta a_3 = .. = \beta a_{n-1} = 0$. Hence, we see that

$$\nu\gamma_0'(a \otimes a)$$

$$= \nu\theta a_{n-1} + \nu^2 \theta a_{n-2} + \dots + \nu^n \theta a_0$$

$$= \nu\theta(\beta a_{n-2}) + \nu^2 \theta a^{n-2} + \dots + \nu^n \theta a_0$$

which entails $a_{n-1} = \beta a_{n-2}$, $a_{n-3} = \beta a_{n-4}$, ... and $a_1 = \beta a_0$. Thus the statement is true for even n. For odd n, a similar discussion will show the same result. Q. E. D.

Combining what we have shown we state the following proposition.

PROPOSITION 4.6. *Let* $n > 0$.

(1) *Let* $\alpha = np' - 1$ *and* $p \geqq 3$. *For any given class* a *of* $H^n(X)$, *there exists a unique system* $(a_{2i}) = (a_0, a_2, a_4, \dots, a_{2\alpha})$ *such that* $a_0 = a$ *and* $a_{2i} \in H^{n+2i}(X)$, *which satisfies the following relations*:

$$\lambda_n \gamma_t' a^p = \sum_{i=0}^{\alpha} \mu_{p-1}^{\alpha-i}(\theta_s \beta + \delta_s \theta_t) a_{2i}$$

$$= \sum_{i=0}^{\alpha} (\cup \mu_0)^{\alpha-i} \cup (\theta_s \beta a_{2i} - \mu_0 \cup \gamma_t' \overline{a_{2i}}),$$

and if $k = 2, 3, \dots, p-1$, *then*

$$\lambda_n \gamma_k' a^p = \sum_{i=0}^{\alpha} \mu_{p-k}^{\alpha-i} \delta_{p-k} \theta_k a_{2i}$$

$$= -\sum_{i=0}^{a} (\cup \mu_0)^{a-i+1} \cup \gamma'_k \overline{a_{2i}}$$

where $\overline{a_{2i}} = (a_{2i} \otimes 1 \otimes \dots \otimes 1) \in H^{n+2i}(E)$ *and* λ_n *is the number introduced and characterized in* (3.5).

(2) *Let* $p = 2$. *Then there exists a unique system* $(a_i) = (a_0, a_1, \dots, a_{n-1})$ *such that* $a_0 = a$, $a_i \in H^{n+i} X$, $a_{2i+1} = \beta a_{2i}$, *and*

$$\gamma'_0 (a \otimes a) = \sum_{i=0}^{n-1} \nu^{n-i-1} \theta a_i$$

$$= \sum_{i=0}^{n-1} (\cup \nu_0)^{n-i} \cup \gamma'_0 (a_i \otimes 1).$$

PROOF. (1) The first equality for $\lambda_n \gamma'_t a^p$ follows from (4.1.1) and (4.4). Since $\delta_s \theta_t a_{2i} = \delta_s \theta_t r^* \overline{a_{2i}} = -\delta_s \delta_t \gamma'_t \overline{a_{2i}} = -\mu_{p-1} \gamma'_t \overline{a_{2i}}$, the second equality for $\lambda_n \gamma'_t a^p$ follows from (2.13.1). Also the first equality for $\lambda_n \gamma'_k a^p$, $2 \leqq k \leqq p-1$, is immediate from (4.1.1) and (4.2). The second equality for $\lambda_n \gamma'_k a^p$ follows directly from (2.13.1).

(2) The first equality is directly coming from (4.1.2), and the rest is a routine manipulation due to (2.13.2). Q. E. D.

REMARK. The relations in (4.6.1) are not in final form. Their revised forms which will be given later are known as the Thom-Bott formulas.

Our next aim is to show the Cartan formula for the system defined in (4.6). For this purpose, we make a few preliminary computations.

In (4.6), let $\beta a_{2i} = a_{2i+1}$. Then $\beta \overline{a_{2i}} = \overline{a_{2i+1}}$ by the naturality of β, where $\overline{a_j} = a_j \otimes 1 \otimes \dots \otimes 1$. We have

$$\begin{aligned}
&\theta_s \beta a_{2i} \\
&= \theta_s a_{2i+1} \\
&= \theta_s r^* \overline{a_{2i+1}} \\
&= -\delta_s \gamma'_s \overline{a_{2i+1}} && \text{by (3.1)} \\
&= -\delta_s \xi_t^* \gamma'_t \overline{a_{2i+1}} && \text{by (2.8)} \\
&= -\nu_{p-1} \gamma'_t \overline{a_{2i+1}} \\
&= \nu_0 \cup \gamma'_t \overline{a_{2i+1}} && \text{by (2.13.2)} \\
&= \nu_0 \cup \gamma'_t \beta \overline{a_{2i}}.
\end{aligned}$$

Therefore, for any $a \in H^m X$ where $m > 0$, we see that

$$\lambda_m \gamma_t' a^p = \sum_{i=0}^{\sigma} (\cup \mu_0)^{\sigma-i} \cup (\theta_* \beta a_{2i} - \mu_0 \cup \gamma_t' \overline{a_{2i}}) \qquad \text{by (4.6)}$$

$$= \sum_{i=0}^{\sigma} (\cup \mu_0)^{\sigma-i} \cup (\nu_0 \cup \gamma_t' \beta - \mu_0 \cup \gamma_t') \overline{a_{2i}}$$

$$= - \sum_{i=0}^{\sigma} (\cup \mu_0)^{\sigma+1-i} \cup \gamma_t' \overline{a_{2i}} + \sum_{i=0}^{\sigma} (\cup \mu_0)^{\sigma-i} \cup \nu_0 \cup \gamma_t' \beta \overline{a_{2i}}$$

when $p \geq 3$ and $\sigma = mp' - 1$. And if $p = 2$, then

$$\gamma_0' (a \otimes a) = \sum_{i=0}^{n-1} (\cup \nu_0)^{n-i} \cup \gamma_0' (a_i \otimes 1) \qquad \text{by (4.6.2)}$$

$$= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (\cup \nu_0)^{n-2i} \cup \gamma_0' (a_{2i} \otimes 1)$$

$$= \sum_{i=0}^{\left[\frac{n-1}{2}\right]} (\cup \nu_0)^{n-2i-1} \cup \gamma_0' \beta (a_{2i} \otimes 1)$$

with $\beta (a_{n-1} \otimes 1) = 0$.

The preceding formulas will be needed later in the proof of the Cartan formula.

Let m and n be positive integers. Let $a \in H^m X$, $b \in H^n X$. If $p \geqq 3$, let $\sigma = mp' - 1$, $\tau = np' - 1$ and $\alpha = (m + n) p' - 1$. Finally, let $c = a \cup b$.

DEFINITION. (1) Let i and j be integers. Let $p \geqq 3$. Define a_{2i} by

$$a_{2i} = \begin{cases} 0 & \text{if } i < 0 \text{ or if } i > \sigma + 1 \\ (\cup a)^p & \text{if } i = \sigma + 1 \\ \text{the uniquely determined element in (4.6) if } 0 \leqq i \leqq \sigma . \end{cases}$$

(2) When $p = 2$, we define

$$a_i = \begin{cases} 0 & \text{if } i < 0 \text{ or if } i > n \\ (a \cup a) & \text{if } i = n \\ \text{the uniquely determined element in (4.6), if } 0 \leqq i \leqq n - 1. \end{cases}$$

Make a similar definition for b_{2j} and c_{2r} when $p \geqq 3$, and for b_j, c_r when $p = 2$.

PROPOSITION 4.7. *Let $p \geqq 3$, then*

(1) $c_{2r} = (a \cup b)_{2r} = \sum_{i+j=r} a_{2i} \cup b_{2j}$ (*Cartan formula*),

and

(2) $\lambda_{m+n} = (-1)^{mnp'} \lambda_m \lambda_n$.

If $p = 2$, then

(3) $c_r = (a \cup b)_r = \sum_{i+j=r} a_i \cup b_j$.

PROOF. Let $p \geqq 3$, and let

$$A = \sum_{r=0}^{\alpha} \sum_{i+j=r} (\cup \mu_0)^{\alpha+1-r} \cup \gamma_t' (\overline{a_{2i}} \cup \overline{b_{2j}})$$

$$- \sum_{r=0}^{\alpha} \sum_{i+j=r} (\cup \mu_0)^{\alpha-r} \cup \nu_0 \cup \gamma_t' \beta (\overline{a_{2i}} \cup \overline{b_{2j}}).$$

We will show that $A = -\lambda_m \lambda_n \gamma_t' (a^p \cup b^p)$.

$$A = \sum_{i=0}^{\sigma} \sum_{j=0}^{\tau} (\cup \mu_0)^{\alpha+1-i-j} \cup \gamma_t' (\overline{a_{2i}} \cup \overline{b_{2j}})$$

$$+ \lambda_n \sum_{i=0}^{\sigma} (\cup \mu_0)^{\sigma+1-i} \cup \gamma_t' (\overline{a_{2i}} \cup b^p)$$

$$+ \lambda_m \sum_{i=0}^{\tau} (\cup \mu_0)^{\tau+1-j} \cup \gamma_t' (a^p \cup \overline{b_{2j}})$$

$$- \sum_{i=0}^{\sigma} \sum_{j=0}^{\tau} (\cup \mu_0)^{\alpha-i-j} \cup \nu_0 \cup \gamma_t' \beta (\overline{a_{2i}} \cup \overline{b_{2j}})$$

$$- \lambda_n \sum_{i=0}^{\sigma} (\cup \mu_0)^{\sigma-i} \cup \nu_0 \cup \gamma_t' \beta (\overline{a_{2i}} \cup b^p)$$

$$- \lambda_m \sum_{j=0}^{\tau} (\cup \mu_0)^{\tau-j} \cup \nu_0 \cup \gamma_t' \beta (a^p \cup \overline{b_{2j}}).$$

Let

$$I = \lambda_n \sum_{i=0}^{\sigma} (\cup\, \mu_0)^{\sigma+1-i} \cup \gamma_t' (\overline{a_{2i}} \cup b^p)$$

$$- \lambda_n \sum_{i=0}^{\sigma} (\cup\, \mu_0)^{\sigma-1} \cup \nu_0 \cup \gamma_t' (\overline{a_{2i}} \cup b^p)$$

$$= \lambda_n \sum_{i=0}^{\sigma} (\cup\, \mu_0)^{\sigma+1-i} \cup \gamma_t' (\overline{a_{2i}} \cup b^p)$$

$$- \lambda_n \sum_{i=0}^{\sigma} (\cup\, \mu_0)^{\sigma-i} \cup \nu_0 \cup \gamma_t' (\beta\overline{a_{2i}} \cup b^p)$$

$$= \lambda_n \sum_{i=0}^{\sigma} (\cup\, \mu_0)^{\sigma+1-i} \cup (\gamma_t' \overline{a_{2i}} \cup b^p + \overline{a_{2i}} \cup \gamma_t' b^p - \gamma_t' \overline{a_{2i}} \cup \gamma_t' b^p)$$

$$- \lambda_n \sum_{i=0}^{\sigma} (\cup\, \mu_0)^{\sigma-i} \cup \nu_0 \cup (\gamma_t' \beta\overline{a_{2i}} \cup b^p + \beta\overline{a_{2i}} \cup \gamma_t' b^p - \gamma_t' \beta\overline{a_{2i}} \cup \gamma_t' b^p)$$

$$= - \lambda_m \lambda_n (\gamma_t' a^p) \cup b^p - \lambda_m \lambda_n a^p \cup (\gamma_t' b^p) + \lambda_m \lambda_n (\gamma_t' a^p) \cup (\gamma_t' b^p)$$

$$+ \lambda_m \lambda_n a^p \cup (\gamma_t' b^p) + \lambda_n \sum_{i=0}^{\sigma} ((\cup\, \mu_0)^{\sigma+1-i} \cup \overline{a_{2i}} - (\cup\, \mu_0)^{\sigma-i} \cup \nu_0 \cup \beta\overline{a_{2i}}) \cup (\gamma_t' b^p)$$

$$= - \lambda_m \lambda_n \gamma_t' (a^p \cup b^p) + \lambda_m \lambda_n a^p \cup (\gamma_t' b^p)$$

$$+ \lambda_n \sum_{i=0}^{\sigma} ((\cup\, \mu_0)^{\sigma+1-i} \cup \overline{a_{2i}} - (\cup\, \mu_0)^{\sigma-i} \cup \nu_0 \cup \beta\overline{a_{2i}}) \cup (\gamma_t' b^p).$$

Set also

$$II = \lambda_m \sum_{j=0}^{\tau} (\cup\, \mu_0)^{\tau+1-j} \cup \gamma_t' (a^p \cup \overline{b_{2j}})$$

$$- \lambda_m \sum_{j=0}^{\tau} (\cup\, \mu_0)^{\tau-j} \cup \nu_0 \cup \gamma_t' \beta (a^p \cup \overline{b_{2j}}).$$

Then, just as in the calculation for I, we get

$$II = - \lambda_m \lambda_n \gamma_t' (a^p \cup b^p) + \lambda_m \lambda_n (\gamma_t' a^p) \cup b^p$$

$$+ \lambda_m (\gamma_t' a^p) \cup \sum_{j=0}^{\tau} ((\cup\, \mu_0)^{\tau+1-j} \cup \overline{b_{2j}} - (\cup\, \mu_0)^{\tau-j} \cup \nu_0 \cup \beta\, \overline{b_{2j}}).$$

Hence,

$$A = I + II + \sum_{i=0}^{\sigma} \sum_{j=0}^{\tau} (\cup \mu_0)^{\alpha+1-i-j} \cup \gamma_t' (\overline{a_{2i}} \cup \overline{b_{2j}})$$

$$- \sum_{i=0}^{\sigma} \sum_{j=0}^{\tau} (\cup \mu_0)^{\alpha-i-j} \cup \nu_0 \cup \gamma_t' \beta (\overline{a_{2i}} \cup \overline{b_{2j}})$$

$$= - 2\lambda_m \lambda_n \gamma_t' (a^p \cup b^p) + \lambda_m \lambda_n (\gamma_t' a^p) \cup b^p + \lambda_m \lambda_n a^p \cup (\gamma_t' b^p)$$

$$+ \sum_{i=0}^{\sigma} \sum_{j=0}^{\tau} (\cup \mu_0)^{\alpha+1-i-j} \cup (\gamma_t' (\overline{a_{2i}} \cup \overline{b_{2j}}) - (\gamma_t' \overline{a_{2i}}) \cup (\overline{b_{2j}}) - \overline{a_{2i}} \cup \gamma' \overline{b_{2j}})$$

$$- \sum_{i=0}^{\sigma} \sum_{j=0}^{\tau} (\cup \mu_0)^{\alpha-i-j} \cup \nu_0 \cup (\gamma_t' \beta (\overline{a_{2i}} \cup \overline{b_{2j}}) - (\gamma_t' \beta \overline{a_{2i}}) \cup \overline{b_{2j}} - (\beta \overline{a_{2i}}) \cup (\gamma_t' \overline{b_{2j}})$$

$$- (-1)^m (\gamma_t' \overline{a_{2i}}) \cup (\beta \overline{b_{2j}}) - (-1)^m \overline{a_{2i}} \cup (\gamma_t' \beta \overline{b_{2j}}))$$

$$= - 2\lambda_m \lambda_n \gamma_t' (a^p \cup b^p) + \lambda_m \lambda_n (\gamma_t' a^p) \cup b^p + \lambda_m \lambda_n a^p \cup (\gamma_t' b^p)$$

$$- \sum_{i=0}^{\sigma} \sum_{j=0}^{\tau} (\cup \mu_0)^{\alpha+1-i-j} \cup (\gamma_t' \overline{a_{2i}}) \cup (\gamma_t' \overline{b_{2j}})$$

$$+ \sum_{i=0}^{\sigma} \sum_{j=0}^{\tau} (\cup \mu_0)^{\alpha-i-j} \cup \nu_0 \cup (\gamma_t' \beta \overline{a_{2i}}) \cup (\gamma_t' \overline{b_{2j}})$$

$$+ (-1)^m \sum_{i=0}^{\sigma} \sum_{j=0}^{\tau} (\cup \mu_0)^{\alpha-i-j} \cup \nu_0 \cup (\gamma_t' \overline{a_{2i}}) \cup (\gamma_t' \beta \overline{b_{2j}})$$

$$= - 2\lambda_m \lambda_n \gamma_t' (a^p \cup b^p) + \lambda_m \lambda_n (\gamma_t' a^p) \cup b^p + \lambda_m \lambda_n a^p \cup (\gamma_t' b^p)$$

$$- \left(\sum_{i=0}^{\sigma} (\cup \mu_0)^{\sigma+1-i} \cup (\gamma' \overline{a_{2i}}) - \sum_{i=0}^{\sigma} (\cup \mu_0)^{\sigma-i} \cup \nu_0 \cup (\gamma_t' \beta \overline{a_{2i}}) \right)$$

$$\cup \left(\sum_{j=0}^{\tau} (\cup \mu_0)^{\tau+1-j} \cup (\gamma_t' \overline{b_{2j}}) - \sum_{j=0}^{\tau} (\cup \mu_0)^{\tau-j} \cup \nu_0 \cup (\gamma_t' \beta \overline{b_{2j}}) \right)$$

because $(\cup \nu_0)^2 = 0$

$$= - 2\lambda_m \lambda_n \gamma_t' (a^p \cup b^p) + \lambda_m \lambda_n (\gamma_t' a^p \cup b^p + a^p \cup \gamma_t' b^p - \gamma_t' a^p \cup \gamma_t' b^p)$$

$$= - 2\lambda_m \lambda_n \gamma_t' (a^p \cup b^p) + \lambda_m \lambda_n \gamma_t' (a^p \cup b^p) = - \lambda_m \lambda_n \gamma_t' (a^p \cup b^p).$$

Hence, $A = -\lambda_m \lambda_n \gamma'_t (a^p \cup b^p)$ and $\lambda_m \lambda_n \gamma'_t (a^p \cup b^p) = -\sum_{r=0}^{a} \sum_{i+j=r} (\cup\, \mu_0)^{a+1-r} \cup \gamma'_t (\overline{a_{2i}} \cup \overline{b_{2j}}) + \sum_{r=0}^{a} \sum_{i+j=r} (\cup\, \mu_0)^{a-r} \cup \nu_0 \cup \gamma'_t (\overline{a_{2i}} \cup \overline{b_{2j}})$. By (3.10) and by the existence and uniqueness of the representation of (4.6.1), the right hand side in the last formula is $\lambda_{m+n} \gamma'_t (a \cup b)^p$. Hence

$$\lambda_{m+n} \gamma_t (a \cup b)^p = \lambda_m \lambda_n \gamma'_t (a^p \cup b^p)$$

and $(a \cup b)_{2r} = \sum_{i+j=r} a_{2i} \cup b_{2j}$. Since $(a \cup b)^p = (-1)^{mnp'p} (a^p \cup b^p) = (-1)^{mnp'} (a^p \cup b^p)$, we see that

$$\lambda_{m+n} = (-1)^{mnp'} \lambda_m \lambda_n .$$

This proves (1) and (2).

For (3), let $p = 2$.

Let $A = \sum_{r=0}^{m+n-1} \sum_{i+j=r} (\cup\, \nu_0)^{n+m-r} \cup \gamma'_0 (\overline{a_i} \cup \overline{b_j})$. Then

$$A = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (\cup\, \nu_0)^{n+m-i-j} \cup \gamma'_0 (\overline{a_i} \cup \overline{b_j})$$

$$+ \sum_{i=0}^{m-1} (\cup\, \nu_0)^{m-i} \cup \gamma'_0 (\overline{a_i} \cup b^2)$$

$$+ \sum_{j=0}^{n-1} (\cup\, \nu_0)^{n-j} \cup \gamma'_0 (a^2 \cup \overline{b_j}).$$

Now

$$I = \sum_{i=0}^{m-1} (\cup\, \nu_0)^{m-i} \cup \gamma'_0 (\overline{a_i} \cup b^2)$$

$$= \sum_{i=0}^{m-1} (\cup\, \nu_0)^{m-i} \cup (\gamma'_0\, \overline{a_i} \cup b^2 + \overline{a_i} \cup \gamma'_0\, b^2 + \gamma'_0\, \overline{a_i} \cup \gamma'_0\, b^2)$$

$$= \gamma'_0\, a^2 \cup b^2 + \left(\sum_{i=0}^{m-1} (\cup\, \nu_0)^{m-i} \cup \overline{a_i} \right) \cup \gamma'_0\, b^2 + \gamma'_0\, a^2 \cup \gamma'_0\, b^2$$

and similarly

$$II = a^2 \cup \gamma'_0\, b^2 + \gamma'_0\, a^2 \cup \gamma'_0\, b^2 + \gamma'_0\, a^2 \cup \left(\sum_{j=0}^{n-1} (\cup\, \nu_0)^{n-j} \cup \overline{b_j} \right).$$

Hence

$$A = \mathrm{I} + \mathrm{II} + \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (\cup \nu_0)^{m+n-i-j} \cup \gamma_0' \, (\overline{a_i} \cup \overline{b_j})$$

$$= a^2 \cup \gamma_0' \, b^2 + \gamma_0' \, a^2 \cup b^2$$

$$+ \gamma_0' \, a^2 \cup \left(\sum_{j=0}^{n-1} (\cup \nu_0)^{n-j} \cup \overline{b_j} \right) + \left(\sum_{i=0}^{m-1} (\cup \nu_0)^{m-i} \cup \overline{a_i} \right) \cup \gamma_0' \, b^2$$

$$+ \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} (\cup \nu_0)^{m+n-i-j} \cup (\overline{a_i} \cup \gamma_0' \, \overline{b_j} + \gamma_0' \, \overline{a_i} \cup \overline{b_j} + \gamma_0' \, \overline{a_i} \cup \gamma_0' \, \overline{b_j})$$

$$= a^2 \cup \gamma_0' \, b^2 + \gamma_0' \, a^2 \cup b^2 + \gamma_0' \, a^2 \cup \gamma_0' \, b^2$$

$$= \gamma_0' \, (a^2 \cup b^2)$$

$$= \gamma_0' \, (a \cup b)^2$$

Hence, $(a \cup b)_r = \sum_{i+j=r} a_i \cup b_j$ when $p = 2$. Q. E. D.

Definition. Let i be an integer.

(1) When $p \geqq 3$, $D_p^i : H^m(X; Z_p) \to H^{m+2i}(X; Z_p)$, $m \geqq 0$, is defined by

$$D_p^i \, a = \begin{cases} a_{2i} & \text{if } 0 \leqq i \leqq \alpha \\ \lambda_m (\cup a)^p & \text{if } i = mp' = \alpha + 1 \\ 0 & \text{otherwise.} \end{cases}$$

When $m = 0$, $D^0 a = a$ and $D^i a = 0$, for $i \neq 0$.

(2) When $p = 2$, $D^i : H^m(X; Z_2) \to H^{m+i}(X; Z_2)$, $m \geqq 0$, is defined by

$$D_i \, a = \begin{cases} a_i & \text{if } 0 \leqq i \leqq m-1 \\ a \cup a & \text{if } i = m \\ 0 & \text{otherwise.} \end{cases}$$

For $a \in H^0(X; Z_2)$, $D^0 a = a$ and $D^i a = 0$ if $i \neq 0$.

Proposition 4.8. *The operators D_p^i and D^i defined above have the following properties:*

(1) D_p^0 *and* D^0 *are both identity maps.*

(2) D_p^i *and* D^i *are additive homomorphisms.*

(3) D_p^i *and* D^i *are natural.*

(4) *The Cartan formulas hold, i. e.,*

$$D_p^i(a \cup b) = \sum_{j=0}^{i} D_p^{i-j} a \cup D_p^j b, \quad \text{when } p \geqq 3;$$

and

$$D^i(a \cup b) = \sum_{j=0}^{i} D^{i-j} a \cup D^j b, \quad \text{when } p = 2.$$

(5) *When* $p = 2$, $D^1 = \beta$, *where* β *is the Bockstein homomorphism associated with the exact sequence*

$$0 \to Z_2 \to Z_4 \to Z_2 \to 0.$$

PROOF. (1) is immediate from the definition.

(2) Let a and b be arbitrary elements in $H^m X$, whith $m > 0$. Then, for $p \geqq 3$, $\gamma_t'(a+b)^p = \gamma_t'((a+b) \otimes \dots \otimes (a+b)) = \gamma_t' a^p + \gamma_t' b^p + \gamma_t' z$, for some $z \in \beta_t'({}^tH^{mp} E)$. Hence $\gamma_t' \beta_t' = 0$ implies that $\gamma_t'(a+b)^p = \gamma_t' a^p + \gamma_t' b^p$, which in turn implies that D_p^i is a homorphism for each $i = 0, 1, 2, \dots$ (and hence for any integer i). When $m = 0$, (2) is trivial.

When $p = 2$, the proof is exactly the same as above.

(3) Naturality of D_p^i and D^i follows from the naturality of γ_s', μ, ν, θ_t, γ_0', δ_0 and θ. Hence $f^* D_p^i = D_p^i f^*$ and $f^* D^i = D^i f^*$, for any map $f: X \to Y$. (Of course, for $f^*: H^* Y \to H^* X$ the proper coefficient domain (Z_p or Z_2) has to be taken).

(4) We have already proved the Cartan formulas in (4.7).

(5) follows from (4.5) which says $a_{2j+1} = \beta a_{2i}$. In particular, $D^1 a = a_1 = \beta a$. Q. E. D.

We extend the cohomology operations

$$D_p^i H^m(X; Z_p) \to H^{m+2i}(X; Z_p) \text{ and } D^i: H^m(X; Z_2) \to H^{m+i}(X; Z_2)$$

canonically to the relative cohomology groups for any pair (X, Y) via reduced cohomology groups and $H(X, Y) \cong \tilde{H}(X/Y)$. See [7, pp. 122-124]. Properties (1), (2), (3), (4) and (5) of Proposition 4.8 hold in the relative cases,

$$D_p^i: H^m(X, Y; Z_p) \to H^{m+2i}(X, Y; Z_p) \text{ and } D^i: H^m(X, Y; Z_2) \to H^{m+i}(X, Y; Z_2).$$

COROLLARY 4.9 (*Existence of Steenrod Squares*). Let $p = 2$. Then

(1) For all integers i, there is a natural transformation of functors which is a homomorphism

$$D^i : H^n(X, Y) \to H^{n+i}(X, Y), \ n \geqq 0.$$

(2) $D^0 = 1$.

(3) If dim $a = n$, $D^n a = (\cup a)^2 = a \cup a$.

(4) If $i > \dim a$, $D^i a = 0$. $D^i = 0$ for $i < 0$.

(5) Cartan formula:

$$D^i(a \cup b) = \sum_{j=0}^{i} D^{i-j} a \cup D^j b.$$

(6) D^1 is the Bockstein homomorphism β of the coefficient sequence

$$0 \to Z_2 \to Z_4 \to Z_2 \to 0.$$

(6) is known to follow from (1), (2), (3), (4) and (5). The properties (1) through (6) are taken from [11; pp. 1-2]. Hence (4.9) implies the existence of a cohomology operator satisfying the Steenrod-Epstein axioms when $p = 2$. We set $Sq^i = D^i$. To show the existence of « Steenrod powers » for Z_p, $p \geqq 3$, we need a further investigation of the D_p^i's.

LEMMA 4.10. Let (X, Y) be a finite pair and let $\delta : H^m Y \to H^{m+1}(X, Y)$ be the coboundary operator. Then

$$D_p^i \delta = \delta D_p^i \text{ and } \delta Sq^i = Sq^i \delta.$$

PROOF. This can be shown with the help of Proposition 4.8 and Corollary 4.9 in exactly the same way as in [7; pp. 2,3].

Or present aim is to prove divisibility, i. e. $D_p^i \equiv 0$ unless $i \equiv 0 \mod (p-1), p \geqq 3$. For this purpose, we first introduce the universal example. Write the $(2n+1)$-spere S^{2n+1} as $\left\{(z_0, z_1, \dots, z_n) \mid z_i \in C, \sum_{i=0}^{n} z_i \bar{z}_i = 1\right\}$ Define an action of $G = \{1, h, h^2, \dots, h^{p-1}\} \cong Z_p$, $p \geqq 3$, on S^{2n+1} by $h(z_0, z_1, \dots, z_p) = e^{\frac{2\pi i}{p}}(z_0, z_1, \dots, z_p)$. The action of G on S^{2n+1} induces the orbit space $L_p^{2n+1} = S^{2n+1}/G$, called the $(2n+1)$-dimensional lens space for Z_p, $p \geqq 3$. This induces a bundle (a p-sheeted covering) $\{Z_p \to S^{2n+1} \to L_p^{2n+1}\}$. Define $S^\infty = \bigcup_{n=0}^{\infty} S^{2n+1}$ and $L_p^\infty = \bigcup_{n=0}^{\infty} L_p^{2n+1}$ with weak topologies induced by $\{S^{2n+1}\}$ and $\{L_p^{2n+1}\}$, respectively.

The Z_p-bundle $\{Z_p \to S^\infty \to L_p^\infty\}$ is universal for the fiber Z_p.

Similarly, when $p = 2$, we can define the antipodal map $h: S^n \to S^n$ for $n \geqq 0$, so the group $G = \{1, h\} \cong Z_2$ acts on S^n. It is well known that $\{Z_2 \to S^\infty \to P^\infty\}$, where $P^\infty = L_2^\infty$ is the infinite-dimensional real projective space, is the universal object for Z_2 fibrations.

Lemma 4.11.

(1) Let μ_0 and ν_0 be the Wu classes for the covering $\{Z_p \to S^\infty \to \to L_p^\infty\}$, $p \geqq 3$. Then

$$H^{2i}(L_p^\infty) \cong Z_p \text{ is generated by } (\cup \mu_0)^i,$$

and

$$H^{2i+1}(L_p^\infty) \cong Z_p \text{ is generated by } (\cup \mu_0)^i \cup \nu_0, \text{ for all } i = 0, 1, 2, \ldots$$

(2) $H^*(P^\infty) \cong Z_2[\nu_0]$.

Proof. (1) Since $H^0(L_p^\infty) \cong Z_p$, we restrict ourselves to a positive integer n and show first that $H^n(L_p^\infty) \cong Z_p$. By Richardson-Smith exactness (1.4), we have the following two exact sequences:

$$\ldots \to 0 \xrightarrow{\gamma_s} H^n(L_p^\infty) \xrightarrow{\delta_s} {}^sH^{n+1}(S^\infty) \xrightarrow{\beta_s} 0 \to \ldots$$

and

$$\ldots \to 0 \xrightarrow{\gamma_t} {}^sH^n(S^\infty) \xrightarrow{\delta_t} H^{n+1}(L_p^\infty) \xrightarrow{\beta_t} 0 \to \ldots,$$

because $H^n(S^\infty) = 0$, for $n \geqq 1$. Hence,

$$H^n(L_p^\infty) \cong {}^sH^{n+1}(S^\infty) \text{ and } {}^sH^n(S^\infty) = H^{n+1}(L_p^\infty)$$

which imply that $H^1(L_p^\infty) \cong H^{2j-1}(L_p^\infty)$ and ${}^sH^1(S^\infty) \cong H^{2j}(L_p^\infty)$ for $j \geqq 1$. Therefore, it is enough to show that $H^1(L_p^\infty) \cong Z_p$ and ${}^sH^1(S^\infty) \cong Z_p$. Again by the Richardson-Smith exact sequence, we have

$$H^0(S^\infty) \xrightarrow{\gamma_s} H^0(L_p^\infty) \xrightarrow{\delta_s} {}^sH^1(S^\infty) \to 0$$

and

$$H^0(S^\infty) \xrightarrow{\gamma_t} {}^sH^0(S^\infty) \xrightarrow{\delta_t} H^1(L_p^\infty) \to 0.$$

Clearly $\gamma_s: H^0(S^\infty) \to H^0(L_p^\infty)$ and $\gamma_t: H^0(S^\infty) \to {}^sH^0(S^\infty)$ are zero maps. By exactness, it follows that $\delta_s: H^0(L_p^\infty) \to {}^sH^1(S^\infty)$ and $\delta_t: {}^sH^0(S^\infty) \to H^1(L_p^\infty)$ are isomorphisms. Since $H^0(L_p^\infty) \cong Z_p$ and ${}^sH^0(S^\infty) \cong Z_p$, we see that ${}^sH^1(S^\infty) \cong H^1(L_p^\infty) \cong Z_p$.

By (1.14.2), $(\cup \nu)^2 = \nu^2(1) = 0$. In the proof of (4.3.1) we showed that $\mu_0 = \beta\nu_0$. Also we observed that the δ_ϱ's are isomorphisms. Hence $(\cup \mu_0)^j = (\delta_t \delta_s)^j(1) \neq 0$ for all $j = 0, 1, 2, \ldots$. Similarly $(\cup \mu_0)^j \cup \nu_0 \neq 0$, for all j. Thus we have (1), i. e. $H^*(L_p^\infty; Z_p)$ is the tensor product of the exterior algebra in ν_0 and the polynomial algebra in $\mu_0 = \beta\nu_0$.

(2) is well-known, and is proved in Z_2 Smith theory similarly to (1). Q.E.D.

REMARK. In [7; p. 68], (4.11.1) is proved in a different manner using a cell decomposition of L_p^∞. See also [9; chapter II, § 2] and « Seminar on Transformation Groups» (by A. Borel), Annals of Mathematics Studies Number 46, where more general cases are treated.

With the help of (1.19) and (4.11) we prove the following proposition due to Wu [10]:

PROPOSITION 4.12 (Wu).

(1) *When* $p \geqq 3$,

$$D_p^i \nu_0 = \begin{cases} \nu_0 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0, \end{cases}$$

and

$$D_p^i \mu_0 = \begin{cases} \mu_0 & \text{if } i = 0 \\ \lambda_2 (\cup \mu_0)^p & \text{if } i = p-1 \\ 0 & \text{otherwise,} \end{cases}$$

where $\nu_0 = \nu(1)$ *and* $\mu_0 = \mu(1)$ *are the Wu classes in* $H^*(E_0 - \Delta)$.

(2) *When* $p = 2$,

$$Sq^i \nu_0 = \begin{cases} \nu_0 & \text{if } i = 0 \\ \nu_0 \cup \nu_0 = \mu_0 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

where ν_0 *and* μ_0 *are similarly defined as in* (1) *with* Z_2 *coefficients.*

PROOF. (1) We use the universal example $\{Z_p \longrightarrow S^\infty \longrightarrow L_p^\infty\}$. Because of (4.11), we have

$$D_p^i \nu_0 = n_i (\cup \mu_0)^i \cup \nu_0$$

and

$$D_p^i \mu_0 = m_i (\cup \mu_0)^{i+1},$$

for some n_i and m_i in Z_p, $p \geqq 3$. By the naturality and universality of L_p^∞, the above relationships hold in general for the same n_i and m_i and with μ_0 and ν_0 in appropriate Z_p-coverings. Now by (1.19), the Wu classes for $L_p^\infty \times L_p^\infty$ are

$$\nu_0 \otimes 1 + 1 \otimes \nu_0 \text{ and } \mu_0 \otimes 1 - 1 \otimes \mu_0 .$$

Let $D_p = \sum_{i=0}^{\infty} D_p^i = 1 + D_p^1 + D_p^2 + \dots$, then the Cartan farmula $D(a \cup b) = Da \cup Db$ is known to be equivalent to $D(a \otimes b) = Da \otimes Db$. See [7 ; p. 2] for its proof. Hence

$$\begin{aligned} & D_p^i(\nu_0 \otimes 1 + 1 \otimes \nu_0) \\ & = (D_p^i \nu_0) \otimes 1 + 1 \otimes (D_p^i \nu_0) \\ & = n_i((\cup \mu_0)^i \cup \nu_0) \otimes 1 + n_i 1 \otimes ((\cup \mu_0)^i \cup \nu_0). \end{aligned}$$

On the other hand

$$\begin{aligned} & D_p^i(\nu_0 \otimes 1 + 1 \otimes \nu_0) = n_i(\mu_0 \otimes 1 - 1 \otimes \mu_0)^i \cup (\nu_0 \otimes 1 + 1 \otimes \nu_0) \\ & = n_i \{(\cup \mu_0)^i \otimes 1 + (-1)^i 1 \otimes (\cup \mu_0)^i + \dots\} \cup (\nu_0 \otimes 1 + 1 \otimes \nu_0) \\ & = n_i \{((\cup \mu_0)^i \cup \nu_0) \otimes 1 + (-1)^i 1 \otimes ((\cup \mu_0)^i \cup \nu_0) + \text{other terms}\}. \end{aligned}$$

Hence n_i must be zero so that the two expressions for $D_p^i(\nu_0 \otimes 1 + 1 \otimes \nu_0)$ become equal when $i > 0$. It is clear that $n_0 = 1$.

Similarly, we have

$$\begin{aligned} & D_p^i(\mu_0 \otimes 1 - 1 \oplus \mu_0) \\ & = (D_p^i \mu_0) \otimes 1 - 1 \otimes D_p^i \mu_0 \\ & = m_i(\cup \mu_0)^{i+1} \otimes 1 - m_i 1 \otimes (\cup \mu_0)^{i+1}, \end{aligned}$$

and also

$$\begin{aligned} & D_p^i(\mu_0 \otimes 1 - 1 \otimes \mu_0) = m_i(\mu_0 \otimes 1 - 1 \otimes \mu_0)^{i+1} \\ & = m_i \{(\cup \mu_0)^{i+1} \otimes 1 + (-1)^i 1 \otimes (\cup \mu_0)^{i+1} + \text{other terms}\} \end{aligned}$$

which equals $m_i \{(\cup \mu_0)^{i+1} \otimes 1 - 1 \otimes (\cup \mu_0)^{i+1}\}$ if and only if $i = 0$ or $i = p-1$. Hence $m_i = 0$ for $i \neq 0, p-1$. By definition, $m_0 = 1$ and $m_{p-1} = \lambda_2$.

(2) When $p=2$, $Sq^0 \nu_0 = \nu$ and $Sq^1 \nu_0 = \beta\nu_0 = \mu_0 = \nu_0 \cup \nu$ by (4.9). $Sq^i \nu_0 = 0$ unless $i = 0, 1$, by definition. Q. E. D.

Lemma 4.13. *$D_p^i : H^1 X \to H^{1+2i} X$ is zero for $i > 0$ and $p \geqq 3$.*

Proof. Every 1-cohomology class in X is induced by a map $f: X \to K(Z_p; 1) = L_p^\infty$ where $K(Z_p; 1)$ denotes the Eilemberg-McLane space of type $(Z_p; 1)$. By writing the homotopy exact sequence for the covering $\{Z_p \to S^\infty \to L_p^\infty\}$, we easily see that $L_p^\infty = K(Z_p; 1)$. By (4.12) and naturality of D_p^i, we have $D_p^i x = D_p^i (f^* \nu_0) = f^* D_p^i \nu_0 = 0$, for all $x \in H^1(X)$.
Q. E. D.

Remark. The proof of (4.13) does not work for $p = 2$ and (4.13) is false for $p = 2$.

As a consequence of (4.12) and (4.8.4), the following proposition holds for the operations $\nu D_p^i : H^m(E_0 - L_0) \to H^{m+2i+1}(E_0 - L_0)$, $\mu D_p^i : H^m(E_0 - L_0) \to H^{m+2i+2}(E_0 - L_0)$ for $p \geqq 3$, and $\nu Sq^i : H^m(E_0 - L_0) \to H^{m+i+1}(E_0 - L_0)$ when $p = 2$.

Proposition 4.14. *Let $p \geqq 3$, then*

(1) $\nu D_p^i = D_p^i \nu$ and

(2) $\mu D_p^i = \sum_{j=0}^{\langle i \rangle} (-\lambda_2)^j \mu^{j(p-1)} D_p^{i-j(p-1)} \mu$

where $\langle i \rangle = \left[\frac{i}{p-1}\right] = max\left\{ n \mid n \leqq \frac{i}{p-1} \right\}$.

If $p = 2$, then

(3) $\nu Sq^i = \sum_{j=0}^{i} \nu^j Sq^{i-j} \nu$.

Proof. (1) Let $\nu_0 = \nu(1)$ as usual. Then

$$
\begin{aligned}
D_p^i \nu x &= D_p^i (\nu_0 \cup x) && \text{by (2.13.2)} \\
&= \sum_{j=0}^{i} D_p^{i-j} \nu_0 \cup D_p^j x && \text{by the Cartan formula} \\
&= \nu_0 \cup D_p^i x && \text{by (4.17)} \\
&= \nu D_p^i x .
\end{aligned}
$$

(2) For any $x \in H^m(E_0 - L_0)$ we have

$$D_p^r \mu x = D_p^r (\mu_0 \cup x) \qquad \text{by (2.13.1)}$$

$$= \sum_{j=0}^{r} D_p^{r-j} \mu_0 \cup x \qquad \text{by the Cartan formula}$$

$$= \mu_0 \cup D_p^r x + \lambda_2 (\cup \mu_0)^p \cup D_p^{r-(p-1)} x \quad \text{by (4.12)}$$

$$= (\mu D_p^r + \lambda_2 \mu^p D_p^{r-(p-1)}) x,$$

i. e. $D_p^r \mu = \mu D_p^r + \lambda_2 \mu^p D_p^{r-(p-1)}$. Hence

$$\mu D_p^i = D_p^i \mu - \lambda_2 \mu^p D_p^{i-(p-1)}$$

$$= D_p^i \mu - \lambda_2 \mu^{p-1} (\mu D_p^{i-(p-1)})$$

$$= D_p^i \mu - \lambda_2 \mu^{p-1} (D_p^{i-(p-1)} \mu - \lambda_2 \mu^p D_p^{i-2(p-1)})$$

$$= D_p^i \mu + (-\lambda_2) \mu^{p-1} D_p^{i-(p-1)} \mu + (-\lambda_2)^2 \mu^{2p-1} D_p^{i-2(p-1)}$$

$$= \dots$$

$$= \sum_{j=0}^{\langle i \rangle} (-\lambda_2)^j \mu^{j(p-1)} D_p^{i-j(p-1)} \mu.$$

(3) Now, let $p = 2$. Then

$$Sq^r \nu x = Sq^r (\nu_0 \cup x) = \nu_0 \cup Sq^r + (\cup \nu_0)^2 \cup Sq^{r-1} x$$

$$= (\nu Sq^r + \nu^2 Sq^{r-1}) x, \qquad \text{i. e.}$$

$$Sq^r \nu = \nu Sq^r + \nu^2 Sq^{r-1}.$$

Hence

$$\nu Sq^i = Sq^i \nu + \nu^2 Sq^{i-1}$$

$$= Sq^i \nu + \nu (Sq^{i-1} \nu + \nu^2 Sq^{i-2})$$

$$= Sq^i \nu + \nu Sq^{i-1} \nu + \nu^3 Sq^{i-2}$$

$$= \dots$$

$$= \sum_{j=0}^{i} \nu^j Sq^{i-j} \nu. \qquad \text{Q. E. D.}$$

Proposition 4.15 (*Divisibility by* $p-1$). *Let* $p \geqq 3$. *Then* $D_p^i = 0$ *if* $i \not\equiv 0$ *mod* $(p-1)$.

Proof. Let $D_p^i : H^m \longrightarrow H^{m+2i}$ for $m \geqq 0$. Suppose that $i = \alpha + 1 = mp'$. Then for $a \in H^m X$, we have $D_p^i a = D_p^{\alpha+1} a = \lambda_m (\cup a)^p$ by definition. If $D_p^i a \neq 0$, then m must be even because if dim $a = m$ is odd, we would have $(\cup a)^p = -(\cup a)^p$ and $(\cup a)^p = 0$. Let $m = 2m'$. Then $i = mp' = m'(p-1) \equiv 0$ mod $(p-1)$. So $D_p^{\alpha+1} = 0$ if $i = \alpha + 1 \not\equiv 0$ mod $(p-1)$. Suppose that $1 \leqq i \leqq \alpha$. In this case, we apply induction from above on $m = \dim a$, where $a \in H^m X$. Assume that X is a finite-dimensional CW-complex. The statement is trivial for $m > \dim X$. By (4.6), we can write

$$\lambda_m \gamma_s' a^p = \sum_{i=0}^{\alpha} \mu^{\alpha-1} \nu\theta_t D_p^i a = -\sum_{i=0}^{\alpha} \mu^{\alpha+1-i} \gamma_s' D_p^i \bar{a}$$

because $r^* \overline{a_{2i}} = a_{2i} = D_p^i a$, $r^* \bar{a} = a$ and the naturality of D_p^i imply that $r^* D_p^i \bar{a} = D_p^i a$. Since D_p^i is natural, $\gamma_s' D_p^i = D_p^i \gamma_s'$. Hence $\lambda_m \gamma_s' a^p = -\sum_{i=0}^{\alpha} \mu^{\alpha+1-i} D_p^i \gamma_s' \bar{a}$. Assume that (4.15) is true for all x with $\dim x > \dim a = m$. If $i \not\equiv 0$ mod $(p-1)$, then $i - j(p-1) \not\equiv 0$ mod $(p-1)$ for $j = 0, 1, \dots, \langle i \rangle$ and hence $D_p^{i-j(p-1)}(\mu \gamma_s' \bar{a}) = 0$ by induction hypothesis. By (4.14), $\mu D_p^{i-j(p-1)} \gamma_s' \bar{a} = 0$. Consequently, $\mu^{\alpha+1-i} D_p^i \gamma_s' \bar{a} = 0$ if $i \not\equiv 0$ mod $(p-1)$ where $\alpha = mp' - 1$, i. e. $\mu^{\alpha-i} \nu\theta_t D_p^i a = 0$ if $i \not\equiv 0$ mod $(p-1)$. The uniqueness of the representation implies that $D_p^i a = 0$, for $i \not\equiv 0$ mod $(p-1)$.

We have, therefore, proved (4.15) for the category of finite-dimensional CW-complexes; but by the limit procedure of J. Milnor [3], the proof extends to the category of all CW-complexes. Q. E. D.

Now we are ready to show the existence and uniqueness of the Steenrod cohomology operations. Proposition 4.15 suggests that we define operations $P_p^j = \lambda_{2j}^{-1} D_p^{j(p-1)} : H^m \longrightarrow H^{m+2j(p-1)}$, for $m \geqq 0$.

Theorem 4.16. (*Existence of the Steenrod powers for* Z_p). *The operations* $P_p = \Sigma P_p^i = 1 + P_p^1 + P_p^2 + \dots$ *are the Steenrod powers over* Z_p, $p \geqq 3$, i.e. *they fulfill the Steenrod axioms* [7, p. 76]:

(1) $P_p^i : H^m \longrightarrow H^{m+2i(p-1)}$ *is an additive homomorphism which is natural for all* i *and* $m \geqq 0$.

(2) $P_p^0 = 1$.

(3) *If* $\dim a = 2m$, *then* $P_p^m a = (\cup a)^p$.

(4) *If* $2j > \dim a$ *or* $j < 0$, *then* $P_p^j a = 0$.

(5) *The Cartan formula holds, i.e. for any two cohomology classes a and b,*

$$P_p^i(a \cup b) = \sum_{j=0}^{i} P_p^{i-j} a \cup P_p^j b.$$

REMARK. For $p = 2$, we already proved the existence of Steenrod squares Sq^i in (4.9).

PROOF. (1) and (2) are immediate from (4.8).

(3) If $\dim a = 2m$, then $P_p^m a = \lambda_{2m}^{-1} D_p^{m(p-1)} a = \lambda_{2m}^{-1} \lambda_{2m} (\cup a)^p = (\cup a)^p$.

(4) is immediate from the definition of P_p^j.

(5) $P_p^i(a \cup b)$

$$= \lambda_{2i}^{-1} D_p^{i(p-1)}(a \cup b)$$

$$= \lambda_{2i}^{-1} \sum_{j=0}^{i} D_p^{(i-j)(p-1)} a \cup D_p^{j(p-1)} b \qquad \text{by (4.8)}$$

$$= \lambda_{2i}^{-1} \sum_{j=0}^{i} \lambda_{2(i-j)} P_p^{i-j} a \cup \lambda_{2j} P_p^j b$$

$$= \sum_{j=0}^{i} \lambda_{2i}^{-1} \lambda_{2(i-j)} \lambda_{2j} P_p^{i-j} a \cup P_p^j b$$

$$= \sum_{j=0}^{i} P_p^{i-j} a \cup P_p^j b,$$

because $\lambda_{2(i-j)} \lambda_{2j} = (-1)^{4(i-j)jp'} \lambda_{2(i-j)+2j} = \lambda_{2i}$ by (4.7.2). Q. E. D.

Next we want to prove the uniqueness of the Steenrod powers and squares. Before we proceed, the following lemma is needed.

LEMMA 4.17. Let ${}^{p-k}N^m = \delta_{p-k}\, {}^kH^{m-1}(E, \Delta)$ as usual. If m is not divisible by p, δ_k is monomorphic on ${}^{p-k}N^m$.

PROOF. Let v be an arbitrary element in ${}^{p-k}N^m$. Assume that $\delta_k v = 0$. Then by exactness and by the relation $\gamma_k = \gamma'_k i^*$, we see that $v = \gamma_k y = \gamma'_k i^* y$, for some y in $H^m(E, \Delta)$. Since $\beta_{p-k}\delta_{p-k} = 0$, $0 = \beta_{p-k} v = \beta_{p-k}\gamma'_k i^* y = (\widetilde{t^k})\, y$. Hence $i^* y = (t^{p-k})^* z$ for some $z \in H^m E$ because m is not divisible by p. Since $\gamma'_k \beta'_k = 0$, we finally see that $v = \gamma'_k i^* y = \gamma'_k (t^{p-k})^* z = \gamma'_k \beta'_k j_k^* \gamma'_{p-k} z = 0$. Q. E. D.

THEOREM 4.18. (1) *The operation* $P_p = \sum_{i=0}^{\infty} P_p^i$ *is uniquely characterized by the properties* (1), (2), (3), (4) *and* (5) *in Theorem* 4.16.

(2) *The operation* $Sq = \sum_{i=0}^{\infty} Sq^i$ *is uniquely characterized by the properties* (1), (2), (3) (4) *and* (5) *in Corollary* 4.9.

PROOF. (1) By (2), (3) and (4) of Theorem 4.16, we see that

$$P_p^i \mu_0 = \begin{cases} \mu_0 & \text{if } i = 0 \\ (\cup \mu_0)^p & \text{if } i = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$P_p^i \nu_0 = \begin{cases} \nu_0 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

By (5), the Cartan formula, the relations $P_p^i \nu = \nu P_p^i$ and $P_p^i \mu = \mu P_p^i + \mu^p P_p^{i-1}$ follow just as in the proof of (4.14). Repeated use of $P_p^i \mu = \mu P_p^i + \mu^p P_p^{i-1}$ gives

$$\mu P^i = \sum_{j=0}^{i} (-1)^j \mu^{j(p-1)} P_p^{i-j} \mu$$

as in (4.14.2).

Let z be a class in $H^* E$ where $E = X^p$, and let $(E, \Delta)/Z_p = (E_0, \Delta)$ as usual. For any $q \geqq 2$, we see that

$$\begin{aligned} \delta_s(\cup \gamma_s' z)^q &= \delta_s(\gamma_s' z \cup (\cup \gamma_s' z)^{q-1}) \\ &= \delta_s \gamma_s (z \cup (\cup \gamma_s' z)^{q-1}) && \text{by (1.17)} \\ &= 0 && \text{by exactness.} \end{aligned}$$

Hence $\mu(\cup \gamma_s' z)^q = 0$ if $q \geqq 2$. In particular, if we let $z = \bar{a} = a \otimes 1 \otimes \cdots \otimes 1 \in H^m E$, then $\mu(\cup \gamma_s' \bar{a})^q = 0$, for $q \geqq 2$. Define

$$m' = \begin{cases} m/2 & \text{if } m \text{ is even} \\ \dfrac{m+1}{2} & \text{if } m \text{ is odd.} \end{cases}$$

By the preceding argument, $\mu P_p^{m'} \gamma_s' \bar{a} = 0$, because if $\dim a = m = 2m'$ then $\mu P_p^{m'} \gamma_s' \bar{a} = \mu (\cup \gamma_s' \bar{a})^p = 0$, and if $\dim a = m = 2m' - 1$ then $m < 2m'$

implies that $\mu P_p^{m'}(\gamma_s' \overline{a}) = 0$, by (4) of (4.16). Hence,

$$\sum_{j=0}^{m} (-1)^j \mu^{j(p-1)} P_p^{m'-j} \mu (\gamma_s' \overline{a}) = \mu P_p^{m'} (\gamma_s' \overline{a}) = 0, \text{ i. e.}$$

by setting $i = m' - j$ we get

$$\sum_{i=0}^{m'} (-1)^i \mu^{(m'-i)(p-1)} \nu \theta_t P_p^i a = 0$$

by (4.10) and $P_p^i \nu = \nu P_p^i$. Now we consider the following two cases.

CASE *A*. m is even.

$m = 2m'$. Then $\alpha + 1 = m'(p-1)$ and $\langle \alpha \rangle = m' - 1$. Also,

$$\sum_{i=0}^{m'} (-1)^i \mu^{(m'-i)(p-1)} \nu \theta_t P_p^i a$$

$$= \mu \sum_{i=0}^{\langle \alpha \rangle} (-1)^i \mu^{\alpha - i(p-1)} \nu \theta_t P_p^i a + (-1)^{m'} \nu \theta^t P_p^{m'} a = 0.$$

We have $\nu \theta_t P_p^{m'} a = \nu \theta_t (\cup a)^p = 0$. Hence $\mu x = 0$, where $x = \sum_{i=0}^{\langle \alpha \rangle} (-1)^i \mu^{\alpha - i(p-1)} \circ \nu \theta_t P_p^i a$. By (4.17), therefore $\delta_s x = 0$. Hence, by Richardson-Smith exactness, x is in the image $\gamma_s(H^{mp}(E, \Delta)) \subset \gamma_s'(H^{mp} E)$. This implies that $x = \lambda_m \gamma_s' a^p$ by (3.10), i. e. $\lambda_m \gamma_s' a^p = \sum_{i=0}^{\langle \alpha \rangle} (-1)^i \mu^{\alpha - i(p-1)} \nu \theta_t P_p^i a$. The uniqueness of the representation of $\gamma_s' a^p$ implies that the P_p^i's are uniquely determined.

CASE B. m is odd.

We have $m = 2m' - 1$, $\alpha + 1 = mp' = m'(p-1) - p'$ and $\langle \alpha \rangle = m' - 1$. We start from

$$\sum_{i=0}^{m'} (-1)^i \mu^{(m'-i)(p-1)} \nu \theta_t P_p^i a = 0$$

which we proved earlier. The above equation is rewritten as

$$\mu^{p'+1} \sum_{i=0}^{\langle \alpha \rangle} (-1)^i \mu^{\alpha - i(p-1)} \nu \theta_t P_p^i a = 0$$

because $p' + 1 + \alpha - i(p-1) = p' + 1 + m'(p-1) - p' - 1 - i(p-1) = m'(p-1) - i(p-1) = (m'-i)(p-1)$ and because $P_p^{m'} a = 0$ as $2m' > \dim a = m = 2m' - 1$. Hence,

(i) $$\mu^{p'+1} \sum_{i=1}^{\langle \alpha \rangle} (-1)^i \mu^{\alpha - i(p-1)} \nu \theta_t P_p^i a + \mu^{p'+1+\alpha} \nu \theta_t a = 0.$$

By (4.6) and (4.15)

$$\lambda_m \gamma'_s a^p = \sum_{i=0}^{\langle a \rangle} \mu^{a-i(p-1)} \nu\theta_t D_p^{i(p-1)} a,$$

i. e.

$$\mu^a \nu\theta_t a = \lambda_m \gamma'_s a^p - \sum_{i=1}^{\langle a \rangle} \mu^{a-i(p-1)} \nu\theta_t D_p^{i(p-1)} a.$$

Therefore,

$$\begin{aligned} &\mu^{p'+1+a} \nu\theta_t a \\ &= \lambda_m \mu^{p'+1} \gamma'_s a^p - \mu^{p'+1} \sum_{i=1}^{\langle a \rangle} \mu^{a-i(p-1)} \nu\theta_t D_p^{i(p-1)} a \\ &= - \mu^{p'+1} \sum_{i=1}^{\langle a \rangle} \mu^{a-i(p-1)} \nu\theta_t D_p^{i(p-1)} a \end{aligned}$$

because $\mu^{p'+1} \gamma'_s a^p = \mu^{p'} \delta_t \delta_s \gamma'_s a^p = - \mu^{p'} \delta_t \theta_s r^* a^p = - \mu^{p'} \nu\theta_t (\cup a)^p = 0$.

Substituting the above expression for $\mu^{p'+1+a} \nu\theta_t a$ in (i), we see that

$$\begin{aligned} &\mu^{p'+1} \sum_{i=1}^{\langle a \rangle} \mu^{a-i(p-1)} \nu\theta_t ((-1)^i P_p^i a - D_p^{i(p-1)} a) \\ &= \sum_{i=1}^{\langle a \rangle} \mu^{(m-i)(p-1)} \nu\theta_t ((-1)^i P_p^i a - D_p^{i(p-1)} a) = 0. \end{aligned}$$

Notice that $l = \dim ((-1)^1 P_p^1 a - D_p^{p-1} a) = m + 2(p-1)$. Apply $\mu^{p(p-2)-p'}$ to the above equation to get

$$\sum_{i=1}^{\langle a \rangle} \mu^{(m'-i)(p-1)+p(p-2)-p'} \nu\theta_t ((-1)^i P_p^i a - D_p^{i(p-1)} a) = 0.$$

Notice that $(m'-i)(p-1) + p(p-2) - p' = lp' - 1 - i(p-1)$. By (3.2) and (3.10), $(-1)^i P_p^i a = D_p^{i(p-1)} a$ and hence the $P_p^i a$'s are uniquely determined. Also $\lambda_2 = -1$.

(2) Let $p = 2$. By (2), (3) and (4) of Corollary 4.9, we get

$$Sq^i \nu_0 = \begin{cases} \nu_0, & \text{for } i = 0 \\ \nu_0 \cup \nu_0, & \text{for } i = 1 \\ 0, & \text{for } i \neq 0, 1. \end{cases}$$

By (5) of (4.9) we also have $\nu Sq^i = \sum_{j=0}^{i} \nu^j Sq^{i-j} \nu$ just as in (4.14.3). Let

$a \in H^m(X; Z_2)$ and $\bar{a} = a \otimes 1$, then

$$\nu Sq^m(\gamma_0' \bar{a}) = \sum_{i=0}^{m} \nu^j Sq^{m-j} \nu\gamma_0' \bar{a}$$

$$= \sum_{i=0}^{m} \nu^{m-i} Sq^i \theta a \qquad \text{by (3.1) and } i = m - j$$

$$= \sum_{i=0}^{m} \nu^{m-i} \theta Sq^i a \qquad \text{by (4.10)}$$

$$= \nu \sum_{i=0}^{m-1} \nu^{m-1-i} \theta Sq^i a + \theta Sq^m a$$

$$= \nu \sum_{i=0}^{m-1} \nu^{m-1-i} \theta Sq^i a .$$

Now, we have $\theta Sq^m a = \theta(a \cup a) = 0$ and $\nu Sq^m(\gamma_0' \bar{a}) = \nu(\gamma_0' \bar{a} \cup \gamma_0' \bar{a}) = \delta_0(\gamma_0' \bar{a} \cup \gamma_0' \bar{a}) = \delta_0 \gamma_0(\bar{a} \cup \gamma_0' \bar{a}) = 0$. Hence $\nu \sum_{i=0}^{m-1} \nu^{m-1-i} \theta Sq^i a = 0$, i. e. $\sum_{i=0}^{m-1} \nu^{m-1-i} \theta Sq^i a \in \operatorname{im} \gamma_0 \subset \operatorname{im} \gamma_0'$. By (3.10.2), $\sum_{i=0}^{m-1} \nu^{m-1-i} \theta Sq^i a = \gamma_0'(a \otimes a)$. The uniqueness of the representation of $\gamma_0'(a \otimes a)$ implies that the Sq^i's are uniquely determined. Q. E. D.

COROLLARY 4.19. Let $p \geq 3$. Then

$$\lambda_{2m} = (-1)^m .$$

PROOF. The relation $\lambda_{m+n} = (-1)^{mnp'} \lambda_m \lambda_n$ of (4.7) implies

$$(*) \qquad \lambda_m = (-1)^{\frac{1}{2} m(m-1) p'} \lambda_1^m ,$$

and

$$\lambda_{2m} = \lambda_2^m .$$

In the proof of (4.18), we noticed $\lambda_2 = -1$. Hence

$$\lambda_{2m} = (-1)^m . \qquad \text{Q. E. D.}$$

REMARK. (*) and $\lambda_2 = -1$ yield $\lambda_2 = -1 = (-1)^{p'} \lambda_1^2$, hence $\lambda_1^2 = (-1)^{p'+1} = (p'!)^2$ since $(p-1)! = -1 \bmod p$ (Wilson's theorem). Thus

$\lambda_1 = \pm p'!$. Furthermore, (*) implies $\lambda_m + \lambda_n = 0$ if $m \equiv 1 \mod 4$, $n \equiv 3 \mod 4$.

A careful analysis of the proof of (4.8) reveals the following:

(*a*) We did not use the requirement that Sq^i and P_p^i are additive homomorphisms.

(*b*) For $p \geqq 3$, the Cartan formula was used for $a \cup b$ with $\dim a \leqq 2$, as (5) in (4.16) was only used to prove the commutativity of P_p^i with the coboundary operator θ_t and in the case were $a = \mu_0$ or $a = \nu_0$.

(*c*) When $p = 2$, a similar discussion holds as in (*b*) above.

(*d*) We can start with a homomorphism $D_p^i : H^m \longrightarrow H^{m+2i}$ with (1), (2), (3), (4), (5). Then, by proving first the divisibility (4.15), we get P_p^i. Notice that in this case D_p^i must be an additive homomorphism.

Hence, we have the following theorem:

THEOREM 4.20. *The following axiom systems uniquely characterize the Steenrod powers P_p^i for Z_p and the Steenrod Squares Sq^i for Z_2.*

(1) A 1 : $P_p^i : H^m \longrightarrow H^{m+2i(p-1)}$ *is a natural transformation.*

A 2 : $P_p^0 = 1$.

A 3 : *If* $\dim a = 2m$, *then* $P_p^m a = (\cup a)^p$.

A 4 : *If* $\dim a < 2j$ *or* $j < 0$, $P_p^j a = 0$.

A 5 : *Cartan formula : for any two cohomology classes a and b such that* $\dim a \leqq 2$,

$$P_p^i(a \cup b) = \sum_{j=0}^{i} P_p^{i-j} a \cup P_p^j b.$$

When $p = 2$,

A 1′ : $Sq^i : H^m \longrightarrow H^{m+i}$ *is a natural transformation of homology functors.*

A 2′ : $Sq^0 = 1$.

A 3′ : *If* $\dim a = m$, *then* $Sq^m a = a \cup a$.

A 4′ : *If* $\dim a < j$ *or* $j < 0$, $Sq^i a = 0$.

A 5′ : *Cartan formula: for any two cohomology classes a and b over* Z_2 *such that* $\dim a = 1$,

$$Sq^i(a \cup b) = \sum_{j=0}^{i} Sq^{i-j} a \cup Sq^j b.$$

(2) K 1 : *$D_p^i : H^m \longrightarrow H^{m+2i}$ is an additive homomorphism which is natural.*

K 2 : $D_p^0 = 1$.

K 3 : *If $i = mp'$, then $D_p^i a = (\cup a)^p$.*

K 4 : *If $i > mp'$ or $i < 0$, $D_p^i a = 0$.*

K 5 : *The Cartan formula holds, i. e.,*

$$D_p^i (a \cup b) = \sum_{j=0}^{i} D_p^{i-j} a \cup D_p^j b.$$

K 6 : (Notation) $P_p^i = D_p^{i(p-1)}$.

The following theorem is immediate from what we have shown:

THEOREM 4.21 (*Main Theorem: Thom-Bott formulas*). *Let $a \in H^m(X)$, $m > 0$.*

(1) *Let $p \geqq 3$, $\alpha = mp' - 1$, $\langle \alpha \rangle = \left[\frac{\alpha}{p-1} \right]$* and $2 \leqq k \leqq p-1$. *Then there is a unique representation for $\gamma_t' a^p$ and $\gamma_k' a^p$:*

$$\lambda_m \gamma_t' a^p = \sum_{j=0}^{\langle \alpha \rangle} (-1)^j \mu_{p-1}^{\alpha - j(p-1)} (\theta_s \beta + \delta_s \theta_t) P_p^j a$$

$$= \sum_{j=0}^{\langle \alpha \rangle} (-1)^{j+1} (\cup \mu_0)^{\alpha+1-j(p-1)} \cup \gamma_t' P_p^j \overline{a}$$

$$+ \sum_{j=0}^{\langle \alpha \rangle} (-1)^j (\cup \mu_0)^{\alpha - j(p-1)} \cup \nu_0 \cup \gamma_t' \beta P_p^j \overline{a}$$

and

$$\lambda_m \gamma_k' a^p = \sum_{j=0}^{\langle \alpha \rangle} (-1)^j \mu_{p-k}^{\alpha - j(p-1)} \delta_{p-k} \theta_k P_p^j a$$

$$= \sum_{j=0}^{\langle \alpha \rangle} (-1)^{j+1} (\cup \mu_0)^{\alpha+1-j(p-1)} \cup \gamma_k' P_p^j \overline{a},$$

where $P_p^j : H^m \longrightarrow H^{m+2j(p-1)}$ are the Steenrod powers for Z_p, $p \geqq 3$. Moreover, when $k = p-1$, we have

$$\lambda_m \gamma_s' a^p = \sum_{j=0}^{\langle \alpha \rangle} (-1)^j \mu^{\alpha - j(p-1)} \nu \theta_t P_p^j a.$$

(2) *Let* $p=2$. *Then*

$$\gamma_0'(a\otimes a)=\sum_{j=0}^{m-1}\nu^{m-1-j}\,\theta\,Sq^j a$$

$$=\sum_{j=0}^{m-1}(\cup\,\nu_0)^{m-j}\cup\gamma_0'\,Sq^j(a\otimes 1)$$

is the unique representation for $\gamma_0'(a\otimes a)$.

§ 5. The Adem Relations.

The Adem relations are known to follow from the axioms of cohomology operations described in (4.9) and (4.16). In this §, however, we shall prove them in the framework of Smith theory.

LEMMA 5.1. Let R be a sum of compositions of P_p^i and Bockstein homomorphism β for $0\to Z_p\to Z_{p^2}\to Z_p\to 0$ (or of Sq^i). Let $\{n_j\}$ be a sequence of positive integers strictly increasing with j. If $R\equiv 0$ on H^{n_j} for all j, then $R\equiv 0$.

PROOF [7 ; p. 114]. Let X be a space. Suppose that $R\equiv 0$ on H^r. Then for $u\in H^{r-1}X$ and $v\in H^1 S^1$, we have $0=R(u\otimes v)=(Ru)\otimes v$. Hence $Ru=0$ and $R\equiv 0$ on $H^{r-1}X$. Q. E. D.

LEMMA 5.2. Let p be a prime, and let $a=\sum_{k=0}^{m} a_i p^i$ and $b=\sum_{i=0}^{m} b_i p^i$ with $0\leqq a_i, b_i\leqq p-1$. Then

$$\binom{b}{a}\equiv\prod_{i=0}^{m}\binom{b_i}{a_i}\bmod p.$$

PROOF (cf. [7]; pp. 5-6). Since $(1+x)^p=1+x^p$, $(1+x)^{p^i}=1+x^{p^i}$. Hence $(1+x)^b=\prod_{i=0}^{m}(1+x^{p^i})^{b_i}=\prod_{i=0}^{m}\sum_{s=0}^{b_i}\binom{b_i}{s}x^{sp^i}\bmod p$. The coefficient of $x^a=x^{\Sigma a_i p^i}$ is $\binom{b}{a}$. But from the above equation, it is also $\prod_{i=0}^{m}\binom{b_i}{a_i}$. Q.E.D.

First, let us concentrate ourselves on the case where $p=2$ and show the Adem relations for the Steenrod squares Sq^i.

LEMMA 5.3. Let k and r be integers, $r \geqq 0$. Then

$$Sq^k \nu^r = \sum_{j \geqq 0} \binom{r}{j} \nu^{r+j} Sq^{k-j}$$

where $\nu = \delta_0$ is the Smith operation for Z_2.

PROOF. We use induction on r. If $r = 1$

$$Sq^k \nu x = Sq^k (\nu_0 \cup x) = \sum_{i=0}^{k} Sq^i \nu_0 \cup Sq^{k-i} x =$$

$$= \nu_0 \cup Sq^k x + \nu_0 \cup \nu_0 \cup Sq^{k-1} x = (\nu Sq^k + \nu^2 Sq^{k-1}) x,$$

which proves the formula for $r = 1$. When $r = 0$ it is trivial. Suppose that the formula is true for r, $0 \leqq r \leqq m$. Then,

$$Sq^k \nu^{m+1} = (Sq^k \nu^m) \nu$$

$$= \sum_{j \geqq 0} \binom{m}{j} \nu^{m+j} Sq^{k-j} \nu$$

$$= \sum_{j \geqq 0} \binom{m}{j} \nu^{m+j} (\nu Sq^{k-j} + \nu^2 Sq^{k-j-1})$$

$$= \sum_{j \geqq 0} \binom{m}{j} \nu^{m+1+j} Sq^{k-j} + \sum_{j \geqq 1} \binom{m}{j-1} \nu^{m+1+j} Sq_{k-j}$$

$$= \sum_{j \geqq 0} \binom{m+1}{j} \nu^{m+1+j} Sq^{k-j}.$$

Q. E. D.

LEMMA 5.4. Let a and b be integers, $a \geqq 1$, and let

$$A = \sum_{i=1}^{a} \binom{2i + 2b - a - 1}{i} \binom{i + b - 1}{a - i}.$$ Then, $A \equiv 0 \bmod 2$.

PROOF. We use induction on a. When $a = 1$, $A = \binom{2b}{1}\binom{b}{0} = 2b = 0 \bmod 2$. Assume that $A \equiv 0$ for all b and for $1, 2, \dots, a - 1$. We will prove the formula for a. In the proof, we shall make use of the following relation (congruence mod 2):

$$(*)\qquad \binom{m}{n} = \begin{cases} 0 & \text{if } m \text{ is even and } n \text{ is odd} \\ \binom{\left[\frac{m}{2}\right]}{\left[\frac{n}{2}\right]} & \text{otherwise.} \end{cases}$$

(1) Let $a = 2a_1$ and $b = 2b_1$. Then

$$A = \sum_{i=1}^{2a_1} \binom{2i + 4b_1 - 2a_1 - 1}{i} \binom{i + 2b_1 - 1}{2a_1 - i}$$

$$= \sum_{\substack{i=1 \\ i \text{ even}}}^{2a_1} \binom{2i + 4b_1 - 2a_1 - 1}{i} \binom{i + 2b_1 - 1}{2a_1 - i} + \sum_{\substack{i=1 \\ i \text{ odd}}}^{2a_1} \binom{2i + 4b_1 - 2a_1 - 1}{i} \binom{i + 2b_1 - 1}{2a_1 - i}$$

$$= \sum_{\substack{i=1 \\ i \text{ even}}}^{2a_1} \binom{2i + 4b_1 - 2a_1 - 1}{i} \binom{i + 2b_1 - 1}{2a_1 - i} \qquad \text{by } (*)$$

$$= \sum_{i=1}^{a_1} \binom{4i + 4b_1 - 2a_1 - 1}{2i} \binom{2i + 2b_1 - 1}{2a_1 - 2i}$$

$$= \sum_{i=1}^{a_1} \binom{2i + 2b_1 - a_1 - 1}{i} \binom{i + b_1 - 1}{a_1 - i} \qquad \text{by } (*)$$

$= 0$, by induction hypothesis.

(2) Let $a = 2a_1 + 1$ and $b = 2b_1$. Then,

$$A = \sum_{i=1}^{2a_1+1} \binom{2i + 4b_1 - 2a_1 - 2}{i} \binom{i + 2b_1 - 1}{2a_1 + 1 - i}$$

$$= \sum_{\substack{i=1 \\ i \text{ even}}}^{2a_1+1} \binom{2i + 4b_1 - 2a_1 - 2}{i} \binom{i + 2b_1 - 1}{2a_1 + 1 - i} + \sum_{\substack{i=1 \\ i \text{ odd}}}^{2a_1+1} \binom{2i + 4b_1 - 2a_1 - 2}{i} \binom{i + 2b_1 - 1}{2a_1 + 1 - i}$$

$$= \sum_{\substack{i=1 \\ i \text{ even}}}^{2a_1} \binom{2i + 4b_1 - 2a_1 - 2}{i} \binom{i + 2b_1 - 1}{2a_1 + 1 - i} \qquad \text{by } (*)$$

$$= \sum_{i=1}^{a_1} \binom{4i + 4b_1 - 2a_1 - 2}{2i} \binom{2i + 2b_1 - 1}{2a_1 + 1 - 2i}$$

$$= \sum_{j=1}^{a_1} \binom{2i + 2b_1 - a_1 - 1}{i} \binom{i + b_1 - 1}{a_1 - i} \qquad \text{by } (*)$$

$= 0$ by induction hypothesis.

(3) Let $a = 2a_1 + 1$ and $b = 2b_1 + 1$. Then $A = \sum_{i=1}^{2a_1+1} \binom{2i + 4b_1 - 2a_1}{i} \cdot \binom{i + 2b_1}{2a_1 + 1 - i}$. If i is even $\binom{i + 2b_1}{2a_1 + 1 - i} \equiv 0 \bmod 2$, and if i is odd $\binom{2i + 4b_1 - 2a_1}{i} \equiv 0 \bmod 2$. Hence $A \equiv 0$.

(4) Let $a = 2a_1$ and $b = 2b_1 + 1$. Then

$$A = \sum_{i=1}^{2a_1} \binom{2i + 4b_1 - 2a_1 + 1}{i} \binom{i + 2b_1}{2a_1 - i}$$

$$= \sum_{\substack{i=1 \\ i \text{ even}}}^{2a_1} \binom{2i + 4b_1 - 2a_1 + 1}{i} \binom{i + 2b_1}{2a_1 - i} + \sum_{\substack{i=1 \\ i \text{ odd}}}^{2a_1} \binom{2i + 4b_1 - 2a_1 + 1}{i} \binom{i + 2b_1}{2a_1 - i}$$

$$= \sum_{i=1}^{a_1} \binom{4i + 4b_1 - 2a_1 + 1}{2i} \binom{2i + 2b_1}{2a_1 - 2i} + \sum_{i=1}^{a_1} \binom{4i + 4b_1 - 2a_1 - 1}{2i - 1} \binom{2i + 2b_1 - 1}{2a_1 - 2i + 1}$$

$$= \sum_{i=1}^{a_1} \binom{2i + 2b_1 - a_1}{i} \binom{i + b_1}{a_1 - i} + \sum_{i=1}^{a_1} \binom{2i + 2b_1 - a_1 - 1}{i - 1} \binom{i + b_1 - 1}{a_1 - i} \text{ by } (*)$$

$$= \sum_{i=1}^{a_1} \left\{ \binom{2i + 2b_1 - a_1 - 1}{i} + \binom{2i + 2b_1 - a_1 - 1}{i - 1} \right\} \cdot$$

$$\cdot \left\{ \binom{i + b_1 - 1}{a_1 - i} + \binom{1 + b_1 - 1}{a_1 - i - 1} \right\} + \sum_{i=1}^{a_1} \binom{2i + 2b_1 - a_1 - 1}{i - 1} \binom{i + b_1 - 1}{a_1 - i}$$

$$= \sum_{i=1}^{a_1} \binom{2i + 2b_1 - a_1 - 1}{i} \binom{i + b_1 - 1}{a_1 - i}$$

$$+ \sum_{i=1}^{a_1} \binom{2i + 2b_1 - a_1 - 1}{i} \binom{i + b_1 - 1}{a_1 - i - 1} +$$

$$+\sum_{i=1}^{a_1}\binom{2i+2b_1-a_1-1}{i-1}\binom{i+b_1-1}{a_1-i-1}$$

$$=\sum_{i=1}^{a_1}\binom{2i+2b_1-a_1-1}{i}\binom{i+b_1-1}{a_1-i}$$

$$+\sum_{i=1}^{a_1}\left\{\binom{2i+2b_1-a_1-1}{i}+\binom{2i+2b_1-a_1-1}{i-1}\right\}\binom{i+b_1-1}{a_1-i-1}$$

$$=\sum_{i=1}^{a_1}\binom{2i+2b_1-a_1-1}{i}\binom{i+b_1-1}{a_1-i}$$

$$+\sum_{i=1}^{a_1}\binom{2i+2b_1-a_1}{i}\binom{i+b_1-1}{a_1-1-i}$$

$$=\sum_{i=1}^{a_1-1}\binom{2i+2b_1-a_1}{i}\binom{i+b_1-1}{a_1-1-i} \qquad \text{by induction hypothesis}$$

$$=\sum_{i=1}^{a_1-1}\binom{2i+2b_1-(a_1-1)-1}{i}\binom{i+b_1-1}{(a_1-1)-i}=0, \text{ by induction hypothesis.}$$

Q. E. D.

LEMMA 5.5. Let a, b be integers, $a \geqq 1$, and let j be a nonnegative integer. Then

$$A=\sum_{i=1}^{a-2j}\binom{2i+2b-a-1}{i}\binom{i+b-1-j}{a-i-2j}\equiv 0 \mod 2.$$

PROOF. By induction on a. Let $a=1$, then the formula can be easily checked:

$$A=\sum_{i=1}^{1-2j}\binom{2i+2b-2}{i}\binom{i+b-1-j}{1-i-2j}=0 \mod 2, \text{ for all } j\geqq 0$$

and for any b. Suppose that the formula is true for $1, 2, \dots, a-1$, $j\geqq 0$, and for all b. We shall prove it for a. If $j=0$, then $A\equiv 0$ follows from (5.4). Hence we assume $j\geqq 1$. Then

$$A=\sum_{i=1}^{a-2j}\binom{2i+2b-a-1}{i}\binom{i+b-1-j}{a-i-2j}$$

$$=\sum_{i=1}^{a'-2j'}\binom{2i+2b'-a'-1}{i}\binom{i+b'-1-j'}{a'-i-2j'}$$

where $a' = a - 2$, $b' = b - 1$ and $j' = j - 1 \geqq 0$
(note $a' - 2j' = a - 2j$),
hence $A = 0$, by induction hypothesis. Q. E. D.

THEOREM 5.6. *The Steenrod squares Sq^i for Z_2 satisfy the Adem relations,* i. e., *for* $0 < a < 2b$,

$$\text{(AR)} \qquad Sq^a\, Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j}\, Sq^j .$$

PROOF. Let X be a space and let $x \in H^m(X; Z_2)$, $m > 0$, m odd. By (4.21.2) we have the representation

$$\lambda_0'(x \otimes x) = \sum_{i=0}^{m-1} \nu^{m-1-i}\theta Sq^i x.$$

Hence,

$$Sq^1 \gamma_0'(x \otimes x) = Sq^1 \left(\sum_{i=0}^{m-1} \nu^{m-1-i}\, \theta Sq^i x \right)$$

$$= \sum_{i=0}^{m-1} \nu^{m-1-i}\theta Sq^1\, Sq^i x + \sum_{i=0}^{m-1} (m+1+i)\, \nu^{m-i}\, \theta Sq^i x \qquad \text{by (5.3)}$$

$$= \sum_{i=0}^{m-1} \nu^{m-1-i}\theta Sq^1\, Sq^i x + \sum_{i=-1}^{m-2} (m+i+2)\, \nu^{m-1-i}\, \theta Sq^{i+1} x$$

$$= \sum_{i=0}^{m-1} \nu^{m-1-i}\theta\, (Sq^1\, Sq^i + (1+i)\, Sq^{i+1})\, x \qquad \text{because } m \text{ is odd.}$$

Since

$$Sq^1\, \gamma_0(x \otimes x) = \gamma_0(Sq^1 x \otimes x + x \otimes Sq^1 x) = \gamma_0\, t^*(Sq^1 x \otimes x) = 0,$$

we get

$$\sum_{i=0}^{m-1} \nu^{m-1-i}\theta\, (Sq^1\, Sq^i + (1+i)\, Sq^{i+1})\, x = 0.$$

By the Thom direct decomposition, we have

$$Sq^1\, Sq^i = (1+i)\, Sq^{i+1}$$

on $H^m X$, where m is odd > 0. By (5.1) $Sq^1\, Sq^i = (1+i)\, Sq^{i+1}$. This proves (AR) for $a = 1$ and $b = i > 0$.

Suppose that (AR) holds for $k = 1, 2, \ldots, a-1$ and for suitable b's satisfying the inequality $2b > k > 0$. We shall show (AR) for $k = a$ and

b such that $2b > a$. Since $Sq^{2b-1}\gamma_0'(x \otimes x) = 0$ as before, we have

$$0 = Sq^{2b-1} \sum_{i=0}^{m-1} \nu^{m-1-i} \theta Sq^i x$$

$$= \sum_{i=0}^{m-1} \sum_{j \geqq 0} \binom{m-1-i}{j} \nu^{m-1-i+j} \theta Sq^{2b-1-j} Sq^i x \qquad \text{by (5.3).}$$

Let n be chosen such that $2b < 2^n$ and let $m = 2^n + b$. Then the above equation becomes

$$\sum_{i=0}^{m-1} \sum_{j=0}^{2b-1} \binom{2^n + b - 1 - i}{j} \nu^{m-1-i+j} \theta Sq^{2b-1-j} Sq^i x = 0.$$

If $0 \leqq i \leqq b-1$, then $2^n + b - 1 - i = 2^n + \sum_{i=0}^{n-1} a_i 2^i$ with $a_i = 0$ or 1, and $j = \sum_{i=0}^{n-1} b_i 2^i$ with $b_i = 0$ or 1 because $j \leqq 2b-1 < 2^n$. Hence, $\binom{2^n+b-1-i}{j} = \binom{b-1-i}{j}$ mod 2 by (5.2), when $0 \leqq i \leqq b-1$.

If $i = b$, then $\binom{2^n + b - 1 - i}{j} = \binom{2^n - 1}{j} = \binom{1 + 2 + \dots + 2^{n-1}}{j} = \sum_{i=0}^{n-1} \binom{1}{b^i} = 1$ mod 2, again by (5.2).

Let $i \geqq b+1$. Then, $\binom{2^n+b-1-i}{j} = \dfrac{(2^n + b - 1 - i)\dots(2^n + b - i - j)}{j!} = \dfrac{(i+j-b)(i+j-b-1)\dots(i+1-b)}{j!}$ (if n is sufficiently large) $= \binom{i+j-b}{j}$ mod 2.

Hence, we have

$$0 = \sum_{i=0}^{b-1} \sum_{j=0}^{b-1-i} \binom{b-1-i}{j} \nu^{m-1-i+j} \theta Sq^{2b-1-j} Sq^i x$$

$$+ \sum_{j=0}^{2b-1} \nu^{m-1-i+j} \theta Sq^{2b-1-j} Sq^b x$$

$$+ \sum_{i=b+1}^{m-1} \sum_{j=0}^{2b-1} \binom{i+j-b}{j} \nu^{m-1-i+j} \theta Sq^{2b-1-j} Sq^i x.$$

In the above equation, the dimension of the cohomology class is $2m + 2b - 1$, and the exponents of ν are bounded by $m + b - 2$ so that the above expression is in the Thom direct sum decomposition range of ${}^tN^{2m+2b-1}$. Hence, corresponding to $\nu^{m+b-a-2}$, we have

$$\sum_{i=0}^{b-1}\binom{b-1-i}{b-a-1+i} Sq^{a+b-i}\, Sq^i x + Sq^a\, Sq^b\, x +$$

$$+ \sum_{i=b+1}^{m-1}\binom{2i-a-1}{b-a-1+i} Sq^{a+b-i}\, Sq^i\, x = 0.$$

This equality shows that it is enough to prove that

$$\Phi = \sum_{i=b+1}^{m-1}\binom{2i-a-1}{b-a-1+i} Sq^{a+b-i}\, Sq^i x = 0.$$

Now,

$$\Phi = \sum_{i\geqq 1}\binom{2i+2b-a-1}{2b-a-1+i} Sq^{a-i}\, Sq^{i+b}\, x$$

$$= \sum_{i\geqq 1}\binom{2i+2b-a-1}{i} Sq^{a-i}\, Sq^{i+b}\, x$$

$$= \sum_{i\geqq 1}\sum_{j\geqq 0}\binom{2i+2b-a-1}{i}\binom{i+b-1-j}{a-i-2j} Sq^{a+b-j}\, Sq^j\, x$$

by induction hypothesis

$$= \sum_{j}\left(\sum_{i=1}^{a-2j}\binom{2i+2b-a-1}{i}\binom{i+b-1-j}{a-i-2j}\right) Sq^{a+b-j}\, Sq^j\, x$$

$= 0$ by (5.5). Q. E. D.

Now let us prove the Adem relations for $p \geqq 3$. First we generalize (5.4).

LEMMA 5.7. Let a and b be integers and p be a prime. Set

$$A = \sum_{i=1}^{a}\binom{pi+pb-a-1}{i(p-1)}\binom{(p-1)(i+b)-1}{a-i}.$$

Then $A = 0 \bmod p$.

REMARK. If $p = 2$, (5.7) reduces to (5.4).

PROOF. (*a*) Suppose $b = b'p$. Let $a = a'p + a''$ and $i = jp + i_0$ where $0 \leqq a''$, $i_0 < p$. If $a'' < i_0 < p$, then

$$\binom{(p-1)(i+b)-1}{a-i} = \binom{\text{multiple of } p + (p - i_0 - 1)}{\text{multiple of } p + (p + a'' - i_0)}$$

$$= (\text{integer}) \binom{p - i_0 - 1}{p + a'' - i_0} = 0 \mod p,$$

$$\text{because } p - i_0 - 1 < p + a'' - i_0 .$$

If $1 \leqq i_0 \leqq a'' < p$, then

$$\binom{pi + pb - a - 1}{i(p-1)} = \binom{\text{multiple of } p + (p - a'' - 1)}{\text{multiple of } p + (p - i_0)} =$$

$$= (\text{integer}) \binom{p - a'' - 1}{p - i_0} = 0$$

mod p, because $p - a'' - 1 < p - i_0$. Hence,

$$A = \sum_{\substack{i \geqq 1 \\ i_0 = 0}} \binom{pi + pb - a - 1}{i(p-1)} \binom{(p-1)(i+b)-1}{a-i}$$

$$= \sum_{j=1}^{a'} \binom{pj + pb' - a' - 1}{j(p-1)} \binom{(p-1)(j+b')-1}{a'-j} \binom{p-1}{a''}$$

$$= (-1)^{a''} \sum_{j=1}^{a'} \binom{pj + pb' - a' - 1}{j(p-1)} \binom{(p-1)(j+b')-1}{a'-j} . \quad (*)$$

It is easy to check directly that $A = 0 \mod p$ for $a \leqq 1$. Hence by induction on a, $A = 0$ because $a' < a$ in (*).

(*b*) Suppose b is arbitrary. Then

$$0 = \sum_{j=1}^{ap} \binom{pi + p(pb) - (ap) - 1}{i(p-1)} \binom{(p-1)(bp+i)-1}{(ap)-i} \quad \text{by } (a)$$

$$= \sum_{j=1}^{a} \binom{pj + pb - a - 1}{j(p-1)} \binom{(p-1)(b+j)-1}{a-j},$$

which shows (5.7). Q. E. D.

COROLLARY 5.8. Let $\widetilde{A}=\sum_{i=1}^{a}(-1)^{i}\binom{pi+pb-a-1}{i(p-1)}\binom{(p-1)(i+b)-1}{a-i}$; then $\widetilde{A}=0 \bmod p$.

PROOF. It is the same as that of (5.7), noting that $(-1)^{i}=(-1)^{ip}=(-1)^{j+i_0}=(-1)^{j}$ for $i_0=0$. Q. E. D.

COROLLARY 5.9. For any $j \geqq 0$,

$$(1)\ A=\sum_{i\geqq 1}\binom{ip+pb-a-1}{i(p-1)}\binom{(p-1)(b+i-j)-1}{a-i-pj}=0 \bmod p,$$

$$(2)\ \widetilde{A}=\sum_{i\geqq 1}(-1)^{i}\binom{ip+pb-a-1}{i(p-1)}\binom{(p-1)(b+i-j)-1}{a-i-pj}=0 \bmod p.$$

PROOF. (1) If $j=0$, $A=0$ by (5.7). Suppose $j\geqq 1$.

$$A=\sum_{i\geqq 1}\binom{ip+pb'-a'-1}{i(p-1)}\binom{(p-1)(b'+i-j')-1}{a'-i-pj'}$$

where $a'=a-p$, $b'=b-1$ and $j'=j-1\geqq 0$.

After finitely many steps as above, j decreases to 0 and the situation reduces to (5.7).

(2) Similar proof. Q. E. D.

THEOREM 5.10. If $a<pb$, then

$$(A1)\ P^{a}P^{b}=\sum_{j=0}^{[a/p]}(-1)^{a+j}\binom{(p-1)(b-j)-1}{a-pj}P^{a+b-j}P^{j}.$$

PROOF. Let $x\in H^{m}X$, $m>0$, $\alpha=mp'-1$, $\langle\alpha\rangle=[\alpha/p-1]$. The Thom-Bott formula (Theorem 4.21) for γ'_s says

$$\lambda_m\gamma'_s x^{p}=\sum_{i=0}^{\langle\alpha\rangle}(-1)^{i}\mu^{\alpha-i(p-1)}\nu\theta_t P^{i}x.$$

Applying $P^{pb-1}\mu$ to both sides of the above equation, we see that

$$0=\sum_{i=0}^{\langle\alpha\rangle}\sum_{j\geqq 0}(-1)^{i}\binom{mp'-i(p-1)}{j}\mu^{mp'-(i-j)(p-1)}\nu\theta_t P^{pb-1-j}P^{i}x$$

(this is justified by Lemma 1 in [5 ,vi]). By specifying $m=2(1+p+p^{2}+\cdots+$

$+ p^{n-1} + b)$ for some large n,

$$0 = \sum_{i=0}^{\langle a \rangle} \sum_{j} (-1)^i \binom{p^n - 1 + (b-i)(p-1)}{j} \mu^{mp'-(i-j)(p-1)} \nu\theta_t P^{pb-1-j} P^i x.$$

We check that

$$\binom{p^n - 1 + (b-i)(p-1)}{j} = \begin{cases} \binom{(b-i)(p-1)-1}{j} & \text{if } 0 \leqq i \leqq b-1 \\ (-1)^j & \text{if } i = b \\ (-1)^j \binom{(i-b)(p-1)+j}{j} & \text{if } i > b. \end{cases}$$

Therefore,

$$\sum_{i=0}^{b-1} \sum_{j=0} (-1)^i \binom{(b-i)(p-1)-1}{j} \mu^{mp'-(i-j)(p-1)} \nu\theta_t P^{pb-1-j} P^i x$$

$$+ \sum_{j} (-1)^{b+j} \mu^{mp'-(b-j)(p-1)} \nu\theta_t P^{pb-1-j} P^b x$$

$$+ \sum_{i \geqq b+1} \sum_{j} (-1)^{b+j} \binom{(i-b)(p-1)+j}{j} \mu^{mp'-(i-j)(p-1)} \nu\theta_t P^{pb-1-j} P^i x$$

$$= 0.$$

The dimension of the above cohomology class is $mp + 2 + 2(pb-1)(p-1) = p(m + 2pb - 2 - 2b) + 4$, and everything is in the direct sum decomposition range. Letting $j = i - k$, we have

$$\sum_{i=0}^{b-1} \sum_{k} (-1)^i \binom{(b-i)(p-1)-1}{i-k} \mu^{mp'-k(p-1)} \nu\theta_t P^{pb-1-i+k} P^i x$$

$$+ \sum_{k} (-1)^k \mu^{mp'-k(p-1} \nu\theta_t P^{pb-1-b+k} P^b x$$

$$+ \sum_{i \geqq b+1} \sum_{k} (-1)^{b+i-k} \binom{(i-b)(p-1)+i-k}{i-1} \mu^{mp'-k(p-1)} \nu\theta_t P^{pb-1-i+k} P^i x$$

$$= 0.$$

Let $k = b + 2 - pb$. Then

$$\sum_{i=0}^{b-1} (-1)^i \binom{(b-i)(p-1)-1}{i-b-2+pb} P^{1+b-i} P^i x + (-1)^{b+2-pb} P^1 P^b x$$

$$+ \sum_{i \geqq b+1} (-1)^{i-2-pb} \binom{(i-b)(p-1)+i+pb-2-b}{i+pb-2-b} P^{1+b-i} P^i x = 0.$$

Now

$$\sum_{i=0}^{b-1}(-1)^i\binom{(b-i)(p-1)-1}{i-b-2+pb}P^{1+b-i}P^i x$$

$$=\sum_{i=0}^{b-1}(-1)^i\binom{(b-i)(p-1)-1}{1-ip}P^{1+b-i}P^i x$$

$$=\binom{b(p-1)-1}{1}P^{1+b}x$$

$$=(bp-b-1)P^{1+b}x$$

$$=-(b+1)P^{b+1}x,$$

and

$$\sum_{i\geqq b+1}(-1)^{i-2-pb}\binom{(i-b)(p-1)+i+pb-2-b}{i+pb-b-2}P^{1+b-i}P^i x=$$

$$=\pm\binom{pb-p}{pb-2}P^{b+1}x=0$$

because $pb-p<pb-2$ when $p\geqq 3$. Hence $P^1 P^b=(b+1)P^{1+b}$, which is (A1) for $a=1$. Now we assume that (A1) is true for $1,2,\dots,a-1$ and prove it for a. Let $k=a+b+1-pb$. Then

$$\sum_{i=0}^{b-1}(-1)^i\binom{(b-i)(p-1)-1}{i-b-a+pb-1}P^{a+b-i}P^i x+(-1)^{a+b+1-pb}P^a P^b x$$

$$+\sum_{i\geqq b+1}(-1)^{a+b+1+i}\binom{(i-b)(p-1)+i-a-b-1+pb}{i-a-b-1+pb}P^{a+b-i}P^i x$$

$$(*)\begin{cases}=\sum_{i=0}^{[a/p]}(-1)^i\binom{(b-i)(p-1)-1}{a-ip}P^{a+b-i}P^i x\\ +(-1)^{a+1}P^a P^b x+\sum_{i\geqq b+1}(-1)^{a+1+b+i}\binom{i(p-1)+i-a-1}{i-a-b-1+pb}P^{a+b-i}P^i x=0.\end{cases}$$

Thus, it suffices to show the third term in the above equation vanishes.

$$\sum_{i\geqq b+1}(-1)^{a+1+b+i}\binom{i(p-1)+i-a-1}{i-a-b-1+pb}P^{a+b-i}P^i x$$

$$=\sum_{i\geqq 1}(-1)^{a+1+i}\binom{i(p-1)+i-a-1+pb}{i-a-1+pb}P^{a-i}P^{b+i}x$$

$$=\sum_{i=1}^{a}(-1)^{a+1+i}\binom{ip+pb-a-1}{i(p-1)}P^{a-i}P^{b+i}x$$

$$=(-1)^{a+1}\sum_{i\geqq 1}\sum_{j}(-1)^{i}(-1)^{a-i+j}\binom{ip+pb-a-1}{i(p-1)}\cdot$$

$$\cdot\binom{(p-1)(b+i-j)-1}{a-i-pj}P^{a+b-j}P^{j}x \qquad \text{by induction hypothesis}$$

$$=\sum_{j}(-1)^{j+1}\left(\sum_{i\geqq 1}\binom{ip+pb-a-1}{i(p-1)}\cdot\right.$$

$$\left.\cdot\binom{(p-1)(b+i-j)-1}{a-i-pj}\right)P^{a+b-j}P^{j}x=0 \qquad \text{by (5.9).} \qquad \text{Q. E. D.}$$

REMARK. If we assume (A1), then (5.7) is its easy consequence. In the expression (*) in the proof of (5.10), we see that

$$\sum_{i\geqq b+1}(-1)^{a+1+b+i}\binom{i(p-1)+i-a-1}{i-a-b-1+pb}P^{a+b-i}P^{i}x=0 \pmod p$$

by (A1); i. e.,

$$(**)\qquad \sum_{j\geqq 0}(-1)^{j}\left(\sum_{i\geqq 1}\binom{ip+pb-a-1}{i(p-1)}\binom{(p-1)(b+1-j)-1}{a-i-pj}\right)P^{a+b-j}P^{j}x=0.$$

When $a\leqq 0$, clearly $A=\sum_{i\geqq 1}\binom{ip+pb-a-1}{i(p-1)}\binom{(p-1)(b+i-j)-1}{a-i-pj}\Big|_{(j\geqq 0)}=0.$
Assume $A=0$ for $<a$. Then, for $j\geqq 1$

$$\sum_{i\geqq 1}\binom{ip+pb-a-1}{i(p-1)}\binom{(p-1)(b+i-j)-1}{a-i-pj}$$

$$=\sum_{i\geqq 1}\binom{ip+pb'-a'-1}{i(p-1)}\binom{(p-1)(b'+i-j')-1}{a'-i-pj'}$$

(in the same notation as in the proof of (5.7))

$= 0$ by assumption.

Hence, only the term corresponding to $j=0$ is left in (**); i. e.,

$$\left(\sum_{i\geqq 1}\binom{ip+pb-a-1}{i(p-1)}\binom{(p-1)(b+i)-1}{a-i}\right)P^{a+b}x=0 \text{ for all } x$$

which implies (5.7).

Let $x \in H^m X$, $m > 0$; let $\alpha = mp' - 1$, $\langle \alpha \rangle = [\alpha/p - 1]$. By Thom-Bott,

$$\lambda_m \gamma_t' x^p = \sum_{i=0}^{\langle\alpha\rangle} (-1)^i \mu_{p-1}^{\alpha - i(p-1)} (\theta_s \beta + \delta_s \theta_t) P^i x.$$

Let $y = \delta_t(\lambda_m \gamma_t x^p)$, then

$$y = \sum_{i=0}^{\langle\alpha\rangle} (-1)^i \mu^{\alpha - i(p-1)} (\nu\theta_t \beta + \mu\theta_t) P^i x = 0\,.$$

LEMMA 5.11. Let $l \in Z^+$; then

$$P^l y = \delta_t \sum_{i=0}^{\langle\alpha\rangle + l} \mu_{p-1}^{\alpha + (l-i)(p-1)} \left\{ \sum_{r=0}^{i} (-1)^r \binom{\alpha - r(p-1)}{l + r - i} \theta_s P^{i-r} \beta P^r x + \right.$$

$$\left. + \sum_{r=0}^{i} (-1)^r \binom{\alpha + 1 - r(p-1)}{l + r - i} \delta_s \theta_t P^{i-r} P^r x \right\} = 0\,.$$

PROOF. See [5,vii], Lemma 1, pp. 3-6.
Specify $m = 2(p^n + b)$ and $l = pb$ where n is large. Then,

LEMMA 5.12.

$$P^l y = \delta_t \sum_{i=0}^{\langle\alpha\rangle + l} \mu_{p-1}^{\alpha + (l-i)(p-1)} \left\{ \sum_{r=0}^{b-1} (-1)^r \binom{(b-r)(p-1)-1}{i - b - rp - 1} \theta_s P^{i-r} \beta P^r x + \right.$$

$$+ (-1)^{i+b} \theta_s P^{i-b} \beta P^b x + \sum_{r=b+1}^{i} (-1)^{b+1} \binom{rp + b - i}{bp + r - i} \theta_s P^{i-r} \beta P^r x +$$

$$\left. + \sum_{r=0}^{i} (-1)^r \binom{p^{n+1} - p^n + (b-r)(p-1)}{bp + r - i} \delta_s \theta_t P^{i-r} P^r x \right\} = 0\,.$$

PROOF. Directly from (5.11) and from the relation

$$\binom{\alpha - r(p-1)}{l + r - i} = \begin{cases} \binom{(b-r)(p-1)-1}{i - b - rp - 1} & \text{for } r < b \\ (-1)^i & \text{for } r = b \\ (-1)^{b+r+i} \binom{rp + b - i}{bp + r - i} & \text{for } r > b. \end{cases}$$

Q. E. D.

LEMMA 5.13. Let

$$\psi = \sum_{i=0}^{\langle\alpha\rangle + l} \sum_{r=0}^{i} (-1)^r \binom{p^{n+1} - p^n + (b-r)(p-1)}{bp + r - i} \mu^{\alpha + 1 + (l-i)(p-1)} \nu\theta_t P^{i-r} P^r x.$$

Then

(1) $\psi = 0$

(2) $$\beta\psi = \delta_t \mu_{p-1} \sum_{i=0}^{\langle a\rangle+l} \sum_{r=0}^{i} (-1)^r \cdot$$
$$\cdot \binom{p^{n+1} - p^n + (b-r)(p-1)}{bp + r - i} \mu_{p-1}^{a+(l-i)(p-1)} (\delta_s \theta_t + \theta_s \beta) P^{i-r} P^r x$$
$$= 0.$$

(3) $$\beta\psi = \delta_t \mu_{p-1} \sum_{i=0}^{\langle a\rangle+l} \mu_{p-1}^{a+(l-i)(p-1)} \circ$$
$$\circ \left\{ \sum_{r=0}^{b-1} (-1)^r \binom{(b-r)(p-1)}{i-b-rp} \theta_s \beta P^{i-r} P^r x + \right.$$
$$+ \sum_{r=b+1}^{i} (-1)^{b+i} \binom{rp+b-i-1}{bp+r-i} \theta_s \beta P^{i-r} P^r x +$$
$$\left. + \sum_{r=0}^{i} (-1)^r \binom{p^{n+1} - p^n + (b-r)(p-1)}{bp+r-i} \delta_s \theta_t P^{i-r} P^r x \right\} = 0.$$

PROOF. For (1): See [5, vi], Lemma 4, pp. 4-5.
For (2): See [5, vii], pp. 7,8.
For (3): Enough to observe

$$\binom{p^{n+1} - p^n + (b-r)(p-1)}{bp+r-i} = \begin{cases} \binom{(b-r)(p-1)}{i-b-rp} & \text{for } r < b \\ 0 & \text{for } r = b \\ (-1)^{b+r+i} \binom{rp+b-i-1}{bp+r-i} & \text{for } r > b. \end{cases}$$

Q. E. D.

$$\dim \beta\psi = 2p^{n+1} + 2bp^2 + 3 \text{ and } \dim P^l y = 2p^{n+1} + 2bp^2 + 1.$$

Hence the following sum makes sense.

$$\Phi = \mu_{p-1} P^l y - \beta\psi = \delta_t \mu_{p-1} \left[\sum_{i=0}^{\langle a\rangle+l} \mu_{p-1}^{a+(l-i)(p-1)} \theta_s \{Y\}_i x \right] = 0$$

where

$$\{Y\}_i = \sum_{r=0}^{b-1} (-1)^r \binom{(b-r)(p-1)-1}{i-b-rp-1} P^{i-r} \beta P^r$$
$$+ (-1)^{i+b} P^{i-b} \beta P^b + \sum_{r=b+1}^{i} (-1)^{b+i} \binom{rp+b-i}{bp+r-i} P^{i-r} \beta P^r +$$

$$- \sum_{r=0}^{b-1} (-1)^r \binom{(b-r)(p-1)}{i-b-rp} \beta P^{i-r} P^r$$
$$- \sum_{r=b+1}^{i} (-1)^{b+i} \binom{rp+b-i-l}{bp+r-i} \beta P^{i-r} P^r .$$

It is easy to check $\{Y\}_i(x) = 0$ for $i \leqq b$. Since $\dim \Phi = 2p^{n+1} + 2bp^2 + 3$ and $p \geqq 3$, Proposition 3.10 and Lemma 4.17 imply

$$\sum_{i=0}^{\langle a\rangle + l} \mu_{p-1}^{a+(l-i)(p-1)} \theta_s \{Y\}_i x = \lambda_{2p^n+2bp} \gamma_t' (\{Y\}_b x)^p = 0.$$

By directness of the decomposition,

$$\{Y\}_i(x) = 0 \text{ for all } i \text{ and } X \in H^m X.$$

Hence $\{Y\}_i = 0$ for all i.

THEOREM 5.14. P^i and β satisfy the Adem relations (A2):
(A2) If $a \leqq pb$, then

$$P^a \beta P^b = \sum_{r=0}^{[a/p]} (-1)^{a+r} \binom{(p-1)(b-r)}{a-pr} \beta P^{a+b-r} P^r$$
$$+ \sum_{r=0}^{[(a-1)/p]} (-1)^{a+r-1} \binom{(p-1)(b-r)-1}{a-pr-1} P^{a+b-r} \beta P^r .$$

PROOF (by induction on a). Let $i = b+1$, and calculate $\{Y\}_{b+1}$ to get (easy calculation)

$$0 = \{Y\}_{b+1} = b\beta P^{b+1} - P^1 \beta P^b + P^{b+1} \beta,$$

so that $P^1 \beta P^b = b\beta P^{b+1} + P^{b+1}\beta$. This equation is (A2) for $a = 1$.

Assume (A2) holds for $1, 2, \dots, a-1$. Then for a we have

$$0 = \{Y\}_{a+b}$$
$$= \sum_{r=0}^{[(a-1)/p]} (-1)^r \binom{(b-r)(p-1)-1}{a-rp-1} P^{a+b-r} \beta P^r$$
$$+ (-1)^a P^a \beta P^b$$
$$+ \sum_{r=0}^{[a/p]} (-1)^{r+1} \binom{(b-r)(p-1)}{a-rp} \beta P^{a+b-r} P^r +$$

$$+(-1)^a \sum_{r=b+1}^{a+b} \binom{rp-a}{bp+r-a-b} P^{a+b-r} \beta P^r$$

$$+(-1)^{a+1} \sum_{r=b+1}^{a+b} \binom{rp-a-1}{bp+r-a-b} \beta P^{a+b-r} P^r$$

$$=(-1)^a \Big[P^a \beta P^b x - \sum_{r=0}^{[a/p]} (-1)^{a+r} \binom{(b-r)(p-1)}{a-rp} \beta P^{a+b-r} P^r$$

$$- \sum_{r=0}^{[(a-1)/p]} (-1)^{a+r-1} \binom{(b-r)(p-1)-1}{a-rp-1} P^{a+b-r} \beta P^r \Big]$$

$$+(-1)^a \sum_{r=b+1}^{a+b} \left[\binom{rp-a}{bp+r-a-b} P^{a+b-r} \beta P^r - \binom{rp-a-1}{bp+r-a-b} \beta P^{a+b-r} P^r \right].$$

Therefore, it suffices to show

$$\xi = \sum_{r=b+1}^{a+b} \left[\binom{rp-a}{bp+r-a-b} P^{a+b-r} \beta P^r - \binom{rp-a-1}{bp+r-a-b} \beta P^{a+b-r} P^r \right] = 0.$$

Now

$$\xi = \sum_{r=1}^{a} \left[\binom{(b+r)p-a}{bp-a+r} P^{a-r} \beta P^{b+r} - \binom{(r+b)p-a-1}{bp+r-a} \beta P^{a-r} P^{b+r} \right]$$

$$= \sum_{r=1}^{a} \binom{(b+r)p-a}{r(p-1)} P^{a-r} \beta P^{b+r} - \sum_{r=1}^{a} \binom{(b+r)p-a-1}{r(p-1)-1} \beta P^{a-r} P^{b+r}$$

$$= \beta \sum_{r=1}^{a} \sum_{j} (-1)^{a-r+j} \binom{(b+r)p-a}{r(p-1)} \binom{(b+r-j)(p-1)}{a-r-1-pj} P^{a+b-j} P^j$$

$$+ \sum_{r=1}^{a} \sum_{j} (-1)^{a-r+j-1} \binom{(b+r)p-a}{r(p-1)} \binom{(b+r-j)(p-1)-1}{a-r-1-pj} P^{a+b-j} \beta P^j$$

$$- \beta \sum_{r=1}^{a} \binom{(b+r)p-a-1}{r(p-1)-1} P^{a-r} P^{b+r} \quad \text{by induction hypothesis.}$$

The middle term in the above sum is 0, as it is equal to

$$\sum_j (-1)^{a+j} \left\{ \sum_{r \geq 1} (-1)^r \binom{bp+rp-a'-1}{r(p-1)} \binom{(p-1)(b+r-j)-1}{a'-r-pj} \right\} P^{a+b-j} \beta P^j$$

which is 0 by (5.9). $a' = a - 1$. Hence $\xi = \beta \sum_j (-1)^{a+j} \Omega_j P^{a+b-j} P^j$, where

$$\Omega_j = \sum_{r\geqq 1} (-1)^r \binom{bp + rp - a}{r(p-1)} \binom{(b+r-j)(p-1)}{a-r-pj}$$

$$- \sum_{r\geqq 1} (-1)^r \binom{(b+r)p - a - 1}{r(p-1)-1} \binom{(b+r-j)(p-1)-1}{a-r-pj} \quad \text{by (5.10)}$$

$$= \sum_{r\geqq 1} \Bigg[(-1)^r \binom{bp + rp - a' - 1}{r(p-1)} \binom{(b+r-j)(p-1)-1}{a-r-pj} \qquad (a' = a-1)$$

$$+ (-1)^r \binom{bp + rp - a' - 1}{r(p-1)} \binom{(b+r-j)(p-1)-1}{a'-r-pj}$$

$$- (-1)^r \binom{bp + rp - a' - 1}{r(p-1)} \binom{(b+r-j)(p-1)-1}{a-r-pj}$$

$$+ (-1)^r \binom{bp + rp - a - 1}{r(p-1)} \binom{(b+r-j)(p-1)-1}{a-r-pj} \Bigg] = 0 \bmod p \text{ by (5.9).}$$

Q. E. D.

REFERENCES

[1] R. BOTT, *On symmetric products and the Steenrod squares,* Ann. of Math. 57 (1953) 579-590.

[2] S. T. HU, *Immersions of compact metric spaces into Euclidean spaces,* Notes UCLA, Nov. 1961.

[3] J. MILNOR, *On axiomatic homology theory,* Pacific J. Math. 12 (1962) 337-341.

[4] M. NAKAOKA, *Cohomology theory of a complex with a transformation of prime period and its applications,* J. Inst. Polytech. Osaka City Univ. 7 No. 1 and 2 (1956) Ser. A, 51-102.

[5, i] Seminar Notes, *Smith theory and Steenrod squares,* Feb. 1966, Minneapolis.

[5, ii-vii] Seminar Notes, *Smith theory and Steenrod powers for* Z_p; I, II, III, IV, V, VI; 1967, Minneapolis.

[6] P. A. SMITH, *Fixed points of periodic transformations,* AMS Colloquium Publications 27 (1942), Algebraic Topology, Appendix B, 350-472.

[7] N. E. STEENROD and D. B. A. EPSTEIN, *Cohomology operations,* Annals of Mathematics Studies Number 50, Princeton, N. J. (1962).

[8] R. THOM, *Une théorie intrinsèque des puissances de Steenrod,* Colloque de topologie de Strasbourg (1951).

[9] WU WEN-TSÜN, *A theory of imbedding, immersion and isotopy of polytopes in a Euclidean space,* Science Press, Peking (1965).

[10] WU WEN-TSÜN, *Sur les puissances de Steenrod,* Colloque de Topologie de Strasbourg (1951).

[11] WU WEN-TSÜN, *On the relations between Smith operations and Steenrod powers,* Fund. Math. 44 (1957) 262-269.

CONTENTS