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TRACES OF POTENTIALS ARISING FROM TRANSLATION INVARIANT OPERATORS

by D. R. Adams

For a function u in the space $W^{\alpha,p}(E_n)$ (the usual Sobolev space of L_p functions on Euclidean n-space E_n with distribution derivatives of orders $\leq \alpha$ in L_p), it is possible to characterize the « restriction » or trace of u (call it u^*) to certain lower dimensional manifolds M provided their dimension d, satisfies $d > n - \alpha p$.

In this paper the characterization is given in terms of the Lebesgue or L_p class of u^*, where the norm of u^* is taken with respect to an appropriate measure μ concentrated on M. In particular, it is known that if $n - d < < \alpha p < n$ and $p > 1$, then $u^* \in L_r(M)$, $1 \leq r \leq dp/(n - \alpha p)$, when M is « smooth » and μ is the surface area measure on M. The usual procedure for proving this is to first obtain the result for a subset of a d-dimensional hyperplane in E_n and then extend via a change of variables to manifolds which are diffeomorphic images of d-dimensional coordinate patches. In such a method, the essence is to work coordinate wise, from E_n down to the hyperplane. This paper presents a new method for achieving this, which in addition allows an extension of the trace result to sets M of fractional Hausdorff dimension d, $0 < d \leq n$.

Since every $u \in W^{\alpha\, p}$ can be represented as a Bessel potential of an L_p function (see [1]), we will consider functions u in the form of potentials $T(f)$, where $f \in L_p$ and $T \in S_\alpha$, S_α being the class of translation invarient operators of smoothness α, $\alpha > 0$ (see section 1 for the definitions). Theorem 1 then states that for each class S_α there is a corresponding class of « appropriate » measures $\mu\,(\mathcal{L}^+_{1;\,d})$ for which $T(f) \in L_{p^*}(\mu)$, $p^* = dp/(n - \alpha p)$,

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$n - d < \alpha p < n$. Here the set M is the support of μ. When T is the Riesz potential operator, Theorem 1 can be improved (Theorem 2). In this case, the condition $\mu \in \mathcal{L}^+_{1;d}$ is both necessary and sufficient for the map T: $L_p \longrightarrow L_{p^*}(\mu)$ to be continuous.

The class of measures $\mathcal{L}^+_{1;d}$ is closely related to the Hausdorff d-dimensional measure H_d by a well known theorem of O. Frostman (see [3]). In particular if $H_d(M) > 0$, then there is a measure μ concentrated on M such that $\mu \in \mathcal{L}^+_{1;d}$ and $\mu \not\equiv 0$.

Theorem 1 can also be viewed another way: if μ_0 is given then the condition — μ_0 restricted to M belongs to $\mathcal{L}^+_{1;d}$ — will be a sufficient condition on M to insure that $T(f)$ has a $L_{p^*}(\mu_0)$ trace on M. For example, if $\mu_0 = H_d$, d a positive integer, then any smooth compact manifold in E_n satisfies this condition.

I am grateful to G. Stampacchia for pointing out the result appearing in the appendix of [9]. It is the forerunner of lemmas 1 and 2.

Section 1 contains the preliminaries and section 2, the statements and proofs of the main results. In section 3, I have attempted to point out relationships between Theorem 1 and various results appearing in the literature.

1. Preliminaries.

1.1. Let Λ_γ denote the usual Banach space of all bounded Hölder continuous functions of exponent $\gamma > 0$ defined on E_n (see for example [8] for a precise definition). A linear transformation T which maps Λ_γ into $\Lambda_{\gamma+\alpha}$, $\alpha > 0$, boundedly and which commutes with translations will be termed a linear translation invarient operator of smoothness α and the class of all such T will be denoted by S_α. Here we will be content to list the various properties of $T \in S_\alpha$ needed for this paper.

If $T \in S_\alpha$, then it is known (see [8]) that for $0 < \alpha < 1$, T applied to any smooth function f is given by $T(f)(x) = \int k(x-y) f(y)\, dy$ where k, the kernel of T, satisfies

$$\int |k(x)|\, dx < \infty, \tag{1}$$

$$\int |k(x-y) - k(x)|\, dx \leq Q\, |y|^\alpha, \tag{2}$$

Q a constant independent of y. Here the symbol $\int \ldots dx$ denotes integration over E_n with respect to n-dimensional Lebesgue measure m_n.

Our main interest lies with the Riesz potential operator whose kernel is $h^\alpha(x) = |x|^{\alpha-n}$ and the Bessel potential operator J^α. In general, J^α is defined for all real α by: J^α is the mapping given by the convolution with a tempered distribution and whose Fourier Transform is $(2\pi)^{-n/2} \cdot (1 + |x|^2)^{-\alpha/2}$. When $\alpha > 0$, the kernel of J^α is denoted by g_α and satisfies, in addition to (1) and (2)

$$0 < g_\alpha(x) \leq Q\, h_\alpha(x), \quad \text{for all} \quad x \in E_n . \tag{3}$$

Here Q is a constant independent of x. For additional properties of g_α see [1].

A basic feature of the map J^β is the fact that it is a bicontinuous isomorphism of Λ_γ to $\Lambda_{\gamma+\beta}$ as long as $\gamma + \beta > 0$. From this, it easily follows that for any $T \in S_\alpha$, $T = J^\beta\, T J^{-\beta}$, i.e. T commutes with J^β.

1.2. By $\mathcal{M}$ we will understand the collection of all completions of Borel measures on E_n and by $\mathcal{L}_1$ those $\mu \in \mathcal{M}$ for which $\|\mu\|_1 =$ total variation of $\mu < \infty$. We will use the Morrey space notation $\mathcal{L}_{1;d}$ to denote those $\mu \in \mathcal{M}$ for which

$$V_x(\mu, r) = |\mu|(\{y : |x - y| < r\}) \leq A r^d , \tag{4}$$

for all $x \in E_n$ and all $r \geq 0$. Here $0 < d \leq n$ and $|\mu|$ denotes the sum of the positive and negative parts of μ. A is a constant independent of x and r.

The notation $\|\cdot\|_p$ will represent the usual Lebesgue p-norm, $1 \leq p \leq \infty$, with respect to m_n. For any other measure $\mu \in \mathcal{M}^+$, the symbol $\|\cdot\|_{p,\mu}$ will be used. L_p and $L_p(\mu)$ will denote the corresponding Lebesgue function spaces. $\mathcal{M}_0$ denotes the measures with compact support.

The superscript « + » is used to indicate the subclass of non-negative elements. The letter Q will denote various constants, possibly not the same constant in any one proof, whereas A, A_1, etc. will denote specific constants.

2. The main results.

2.1. The results of principal interest are Theorems 1 and 2 below

THEOREM 1: For $T \in S_\alpha$ and $\mu \in \mathcal{L}^+_{1;d}$, there exists a constant Q such that for all f in L_p,

$$\| T(f) \|_{p^*,\mu} \leq Q\, \|f\|_p$$

provided $n - d < \alpha p < n$, $0 < d \leq n$, $p > 1$. Here $p^* = dp/(n - \alpha p)$ and Q is independent of f.

REMARK 1: Special cases of Theorem 1 are known, e.g. when $\mu = m_n$, it is the Theorem of Stein-Zygmund (see Section 2.4); when $\mu = m_d$, d an integer, and T the Riesz potential operator, it becomes the imbedding result of II′ in [5].

The program for proving Theorem 1 will be: (*a*) to establish necessary and sufficient conditions on a non-negative Borel measure μ in order that the above inequality holds for the Riesz potential operator, (*b*) to show then that Theorem 1 holds for the Bessel potential operator (using (3)), and finally (*c*), to establish Theorem 1 for general T (using (*b*) and the Theorem of Stein-Zygmund). Thus the main burden of the proof of Theorem 1 is in establishing (*a*). This can be stated as follows:

THEOREM 2: The necessary and sufficient condition for

$$\| h_\alpha * f \|_{p^*, \mu} \leq Q \| f \|_p$$

to hold for all $f \in L_p$, Q a constant independent of f, with $\mu \in \mathfrak{M}^+$ ($p^* = dp/n - \alpha p$, $0 < d \leq n$, $n - d < \alpha p < n$, $p > 1$) is that $\mu \in \mathcal{L}^+_{1;d}$.

2.2. The proof of the sufficiency for Theorem 2 involves an estimate on the number $\| h_\alpha * \mu^k \|_{p'}$, where $p' = p/(p-1)$ and μ^k denotes μ restricted to K, K a compact set in E_n of positive μ measure. To obtain the desired estimate, two main cases are considered, namely $1 < p \leq 2$ and $p > 2$. In the first case, $h_{\alpha p} * \mu^K$ is estimated in the L_∞ norm and then $h_\alpha * \mu^K$ in the $L_{p'}$ norm (lemmas 1 and 2).

For the second case we note that

$$\| h_\alpha * \mu^K \|_{p'}^{p'} = \int h_\alpha * (h_\alpha * \mu^K)^{1/(p-1)} \, d\mu^K = \int u^K \, d\mu^K \tag{5}$$

where $u^K(x) = h_\alpha * f^K(x)$, $f^K(y) = [h_\alpha * \mu^K(y)]^{1/(p-1)}$. Hence it suffices to estimate u^K in the L_∞ norm. To do this, observe that

$$u^K(x) = \int h_\alpha(x-y) f^K(y) \, dy = \int_0^\infty h_\alpha(r) \, d V_x(f^K, r). \tag{6}$$

Here $V_x(f^K, r)$ is given by (4), for a measure with density f^K. In lemmas 3-5, estimates for the functions $f^K(y)$ and $V_x(f^K, r)$ are obtained. Finally, lemma 6 is the desired estimate on u^K.

LEMMA 1 : $h_{\alpha p} * \mu^K(x) \leq A_1 \, \mu(K)^{(\alpha p - n + d)/d}$, for all $x \in E_n$, $A_1 = 1 + A(n - \alpha p)/(\alpha p - n + d)$.

PROOF : $h_{\alpha p} * \mu^K(x) = \int_0^\infty h_{\alpha p}(r) \, d\, V_x(\mu^K, r)$ and altho the function $V_x(\mu^K, r)$ is not in general continuous in r (for each fixed x), it is non decreasing and left continuous. This formula follows from the definitions of the integrals involved.

Integrating by parts we get

$$\int_0^\infty h_{\alpha p}(r) \, d\, V_x(\mu^K, r) = -\int_0^\infty V_x(\mu^K, r) \, d h_{\alpha p}(r)$$

$$= (n - \alpha p) \int_0^\infty V_x(\mu^K, r) \, r^{\alpha p - n - 1} \, dr,$$

since as $r \to 0$, $h_{\alpha p}(r) \cdot V_x(\mu^K, r) \leq A r^{\alpha p - n + d}$, $\alpha p - n + d > 0$ whereas $h_{\alpha p}(r) \cdot V_x(\mu^K, r) \leq \mu(K) \, r^{\alpha p - n}$, as $r \to \infty$, $\alpha p - n < 0$. Thus

$$h_{\alpha p} * \mu^K(x) \leq (n - \alpha p) \left(\int_0^\sigma + \int_\sigma^\infty \right) V_x(\mu^K, r) \cdot r^{\alpha p - n - 1} \, dr = (n - \alpha p)(I_1 + I_2).$$

$$I_1 \leq A \int_0^\sigma r^{d + \alpha p - n - 1} \, dr = \frac{A}{\alpha p - n + d} \cdot \sigma^{\alpha p - n + d},$$

$$I_2 \leq \mu(K) \int_\sigma^\infty r^{\alpha p - n - 1} \, dr = \frac{1}{n - \alpha p} \cdot \mu(K) \cdot \sigma^{\alpha p - n}.$$

The result now follows by choosing $\sigma = \mu(K)^{1/d}$.

LEMMA 2 : For $1 < p \leq 2$,

$$\| h_\alpha * \mu^K \|_{p'} \leq A_2 \cdot \mu(K)^{(\alpha p - n + dp)/dp},$$

where $A_2 = C\left(\frac{\alpha p}{2}, \frac{\alpha p}{2}\right)^{1/p'} \cdot A_1^{1/p}$; $C(\alpha, \beta)$ is the Riesz convolution constant, i. e. $h_\alpha * h_\beta = C(\alpha, \beta) \cdot h_{\alpha + \beta}$ for $\alpha + \beta < n$.

PROOF: For $1 < p < 2$ (the case $p = 2$ can be handled by a trivial modification) choose $\theta : 0 < \theta < 1$ and $1 = \theta p/2 + (1-\theta) p$, then

$$h_\alpha(x) = h_{\alpha p/2}(x)^\theta \, h_{\alpha p}(x)^{1-\theta}.$$

Thus from Hölder's inequality, we have

$$h_\alpha * \mu^K(y) \leq [h_{\alpha p/2} * \mu^K(y)]^\theta \cdot [h_{\alpha p} * \mu^K(y)]^{1-\theta},$$

and

$$\| h_\alpha * \mu^K \|_{p'} \leq \| h_{\alpha p} * \mu^K \|_\infty^{1-2/p'} \cdot \| h_{\alpha p/2} * \mu^K \|_2^{2/p'} \tag{7}$$

by the choice of θ, i. e. $\theta p' = 2$. But

$$\| h_{\alpha p/2} * \mu^K \|_2^2 = C\left(\frac{\alpha p}{2}, \frac{\alpha p}{2}\right) \int h_{\alpha p} * \mu^K \, d\mu^K \tag{8}$$

$$\leq C\left(\frac{\alpha p}{2}, \frac{\alpha p}{2}\right) \| h_{\alpha p} * \mu^K \|_\infty \cdot \mu(K).$$

(7) and (8), together with lemma 1, now give the desired result.

LEMMA 3: For $p > 2$,

$$V_x(f^K, r) \leq A_3 \, \mu(K)^{1/(p-1)} \cdot r^{n-(n-\alpha)/(p-1)}$$

for all $x \in E_n$ and all $r \geq 0$. $A_3 = \omega_n (1 + 3^\alpha/\alpha)^{1/(p-1)}$, $\omega_n =$ area of the unit sphere in E_n.

PROOF: Since $p > 2$, Hölder's inequality gives

$$V_x(f^K, r) \leq (\omega_n r^n)^{1-1/(p-1)} \cdot \{ V_x(h_\alpha * \mu^K, r) \}^{1/(p-1)} \tag{9}$$

$$= (\omega_n r^n)^{1-1/(p-1)} \cdot \{ I_3 + I_4 \}^{1/(p-1)},$$

where

$$I_3 = \int_{K \cap [|x-z| > 2r]} d\mu(z) \int_{|x-y| < r} h_\alpha(y - z) \, dy$$

which never exceeds

$$\int_{K \cap [|x-z| > 2r]} d\mu(z) \int_{|x-y| < r} h_\alpha(r) \, dy$$

since $|y - z| \geq r$. Thus $I_3 \leq r^{\alpha-n} \cdot \omega_n r^n \cdot \mu(K)$.

And

$$I_4 = \int_{K \cap [\,|x-z| \le 2r]} d\mu(z) \int_{|x-y|<r} h_\alpha(y-z)\,dy$$

$$\le \int_{K \cap [\,|x-z| \le 2r]} d\mu(z) \int_{|y-z| \le 3r} h_\alpha(y-z)\,dy,$$

Since now $|y-z| \le 3r$. Thus $I_4 \le \omega_n\, \mu(K) \cdot (3r)^\alpha / \alpha$.

LEMMA 4: For $p > 2$ and $0 < \alpha < n - d$,

$$V_x(f^K, r) \le A_4\, r^{n-(n-\alpha-d)/(p-1)}$$

for all $x \in E_n$ and all $r \ge 0$. $A_4 = \omega_n A^{1/(p-1)} \cdot \left(\dfrac{n-\alpha}{n-\alpha-d} + \dfrac{2^d\, 3^\alpha}{\alpha}\right)^{1/(p-1)}$.

PROOF: Equivalently this lemma asserts that f^K belongs to the Morrey class $\mathcal{L}_{1;b}$, with $b = n - (n-\alpha-d)/(p-1)$ when $\alpha < n - d$ (compare this to lemma 5).

For fixed x, we consider y such that $|x-y| < r$, then

$$h_\alpha * \mu^K(y) \le \left(\int_{|x-z| \ge 2r} + \int_{|x-z|<2r}\right) h_\alpha(y-z)\,d\mu(z) = I_5 + I_6\,.$$

Let

$$\varphi_y(\varrho) = \int_{[\,|x-z| \ge 2r] \cap [\,|y-z| < \varrho]} d\mu(z)$$

and note:

(i) $\qquad \varphi_y(\varrho) = 0, \quad \text{when} \quad 0 \le \varrho \le r;$

(ii) $\qquad \varphi_y(\varrho) \le V_y(\mu, \varrho) \le A\varrho^d$, for all $y \in E_n$ and $\varrho \ge 0$;

(iii) For each fixed y, $\varphi_y(\varrho)$ is non-decreasing in ϱ and left continuous.

$$I_5 = \int_r^\infty h_\alpha(\varrho)\,d\varphi_y(\varrho) = (n-\alpha)\int_r^\infty \varphi_y(\varrho)\,\varrho^{\alpha-n-1}\,d\varrho$$

using (i) and then integrating by parts.

Also note that $h_\alpha(\varrho)\,\varphi_y(\varrho) \longrightarrow 0$, as $\varrho \longrightarrow \infty$ by (ii). Thus

$$I_5 \leq (n-\alpha)\,A \int_r^\infty \varrho^{\alpha-n+d-1}\,d\varrho = \frac{(n-\alpha)}{(n-\alpha-d)}\,A r^{\alpha-n+d}.$$

$$\int_{|x-y|<r} I_6\,dy \leq \int_{|x-z|<2r} d\mu(z) \int_{|y-z|<3r} h_\alpha(y-z)\,dy = \omega_n\,(3r)^\alpha/\alpha \cdot V_x(\mu, 2r).$$

With these estimates and (9) of lemma 3, the result follows.

LEMMA 5: For $p > 2$ and $0 \leq n-d < \alpha$,

$$f^K(y) \leq A_5 \cdot \mu(K)^{(d+\alpha-n)/d(p-1)}$$

for all $y \in E_n$. $A_5 = [A(1+(n-\alpha)/(\alpha-n+d))+1]^{1/(p-1)}$.

PROOF: In contrast to lemma 4, f^K is no longer in a Morrey class, but in a Hölder class with exponent $(d+\alpha-n)/(p-1)$.

$$[f^K(y)]^{p-1} = \int h_\alpha(y-z)\,d\mu^K(z)$$

$$= \left(\int_{|y-z|>\sigma} + \int_{|y-z|\leq\sigma}\right) h_\alpha(y-z)\,d\mu^K(z) = I_7 + I_8.$$

Again with $\sigma = \mu(K)^{1/d}$,

$$I_7 \leq \sigma^{\alpha-n}\,\mu(K),$$

$$I_8 \leq \int_0^\sigma h_\alpha(\varrho)\,d V_y(\mu, \varrho)$$

$$\leq h_\alpha(\sigma)\,V_y(\mu,\sigma) + (n-\alpha)\int_0^\sigma V_y(\mu,\varrho)\,\varrho^{\alpha-n-1}\,d\varrho$$

$$\leq \left(A + \frac{(n-\alpha)}{(\alpha-n+d)}\,A\right)\sigma^{\alpha-n+d},$$

the result now follows easily.

LEMMA 6: For $p > 2$, there is a constant A_6 independent of the set K such that

$$u^K(x) \leq A_6\, \mu(K)^{(\alpha p - n + d)/d(p-1)}.$$

Hence by (5), $\| h_\alpha * \mu^K \|_{p'} \leq A_6^{1/p'}\, \mu(K)^{(\alpha p - n + dp)/dp}$.

PROOF: *case* (1) $0 < \alpha < n - d$: Integrating by parts in (6),

$$u^K(x) = (n - \alpha) \int_0^\infty V_x(f^K, r)\, r^{\alpha - n - 1}\, dr$$

since by lemma 4, $V_x(f^K, r) \cdot h_\alpha(r)$ is $O(r^{(\alpha p - n + d)/(p-1)})$ as $r \to 0$, and is $O(r^{(\alpha p - n)/(p-1)})$ as $r \to \infty$, by lemma 3. Thus

$$u^K(x) = (n - \alpha) \left(\int_0^\sigma + \int_\sigma^\infty \right) V_x(f^K, r)\, r^{\alpha - n - 1}\, dr = (n - \alpha)(I_9 + I_{10}).$$

Applying lemma 4 to I_9 and lemma 3 to I_{10}, we have

$$I_9 \leq \frac{A_4(p-1)}{(\alpha p - n + d)} \cdot \sigma^{(\alpha p - n + d)/(p-1)},$$

and

$$I_{10} \leq \frac{A_3(p-1)}{(n - \alpha p)} \cdot \mu(K)^{1/(p-1)} \cdot \sigma^{(\alpha p - n)/(p-1)}.$$

The result follows taking $\sigma = \mu(K)^{1/d}$.

case (2) $0 \leq n - d < \alpha$:

$$u^K(x) = \left(\int_{|x-y| \leq \sigma} + \int_{|x-y| > \sigma} \right) h_\alpha(x - y) f^K(y)\, dy = I_{11} + I_{12}.$$

Applying lemma 5 to I_{11} and lemma 3 to I_{12}, we have

$$I_{11} \leq \frac{A_5\, \omega_n}{\alpha}\, \mu(K)^{(d + \alpha - n)/d(p-1)} \cdot \sigma^\alpha,$$

$$I_{12} \leq \frac{(n - \alpha)(p - 1)}{(n - \alpha p)}\, A_3\, \mu(K)^{1/(p-1)} \cdot \sigma^{(\alpha p - n)/(p-1)},$$

with the same choice of σ.

case (3) $0 < \alpha = n - d$: This case is resolved by interpolating between cases (1) and (2) as follows: choose pairs (α_i, p) with $n - d < \alpha_i p < n$, $i = 0, 1$ but $0 < \alpha_0 < n - d < \alpha_1 < n$. Let $\alpha = \theta \alpha_0 + (1 - \theta) \alpha_1$, $0 < \theta < 1$.

As before $h_\alpha(x) = h_{\alpha_0}(x)^\theta \cdot h_{\alpha_1}(x)^{1-\theta}$ and upon applying Hölder's inequality, we have

$$f^K(y) \leq [f_0^K(y)]^\theta \cdot [f_1^K(y)]^{1-\theta}$$

where $f_i^K(y) = [h_{\alpha_i} * \mu^K(y)]^{1/(p-1)}$, $i = 0, 1$. Thus

$$u^K(x) \leq [h_{\alpha_0} * f_0^K(x)]^\theta \cdot [h_{\alpha_1} * f_1^K(x)]^{1-\theta} = [u_0^K(x)]^\theta \cdot [u_1^K(x)]^{1-\theta}.$$

Case (1) gives $u_0^K(x) \leq A_6' \mu(K)^{(\alpha_0 p - n + d)/d(p-1)}$ and case (2) gives $u_1^K(x) \leq$ $\leq A_0'' \mu(K)^{(\alpha_1 p - n + d)/d(p-1)}$. Hence it is now clear that a finite constant A_6 may be chosen with the required properties.

2.3. PROOF OF THEOREM 2: For the sufficiency, lemmas 2 and 6 are used to show that for fixed α, the Riesz potential operator is of weak type $(L_p, L_{p^*}(\mu))$, when $n - d < \alpha p < n$.

Let $E_t = \{x : |h_\alpha * f(x)| > t\}$, $t > 0$ and $f \in L_p$.

$$t\mu(E_t) \leq \int h_\alpha * |f|(x)\, d\mu^{E_t}(x) = \int h_\alpha * \mu^{E_t}(x)\, |f(x)|\, dx \leq \| h_\alpha * \mu^{E_t} \|_{p'} \|f\|_p.$$

Now since lemmas 2 and 6 hold for all compact sets K, and the constants A_2 and A_6 are independent to K, these estimates must also hold for K replaced by E_t, a G_δ-set, since μ is a Borel measure. Thus

$$t\, \mu(E_t) \leq Q' \|f\|_p\, \mu(E_t)^{(\alpha p - n + dp)/dp}$$

or

$$\mu(E_t) \leq \left(\frac{Q' \|f\|_p}{t}\right)^{p^*}, \qquad p^* = dp/(n - \alpha p).$$

We now apply the well known interpolation theorem of Marcinkiewicz to deduce the strong type estimate required.

To prove the necessity, we choose a particular L_p function, namely the characteristic function of the ball $B_r(x_0) = \{x : |x - x_0| < r\}$, $r > 0$ and $x_0 \in E_n$ arbitrary. Denote this function by $\chi^r(x)$. $\|\chi_r\|_p = (\omega_n r^n)^{1/p}$. On the other hand

$$\| h_\alpha * \chi_r \|_{p^*, \mu} \geq \left\{ \int_{|x - x_0| < r} \left(\int_{|y - x_0| < r} h_\alpha(x - y)\, dy \right)^{p^*} d\mu(x) \right\}^{1/p^*}$$

$$\geq h_\alpha(2r)\, \omega_n r^n [V_{x_0}(\mu, r)]^{1/p^*}.$$

Thus if $h_\alpha : L_p \to L_{p^*}(\mu)$ is continuous, we immediately get

$$V_{x_0}(\mu, r) \leq A' r^d .$$

The proof of Theorem 2 is now complete.

REMARK 2: It is interesting to note that the region in the (p, α) plane $1 < p < \infty$, $0 < \alpha < n$, for which the above result holds is the region between the two hyperbolas $\alpha p = n - d$ and $\alpha p = n$. It is possible to « shift » this region to obtain a result of additional interest (see Remark 6).

Making the changes: $\mu \to \mu_2$, $m_n \to \mu_1$ and $h_\alpha \to h_{\alpha+d_1}$, where $0 \leq d_1 < < d_2 \leq n$, $\mu_1 \in \mathcal{L}^+_{1; n-d_1}$ and $\mu_2 \in \mathcal{L}^+_{1; d_2-d_1}$, we easily get

$$\| h_{\alpha+d_1} * f \mu_1 \|_{p^*, \mu_2} \leq Q \| f \|_{p, \mu_1} \tag{10}$$

where $p^* = (d_2 - d_1) p/(n - \alpha p - d_1)$, $n - d_2 < \alpha p < n - d_1$, $p > 1$. Here $f \mu_1$ denotes a measure with density f.

2.4. To see that Theorem 1 holds for h_α replaced by g_α, it is only necessary to combine (3) with Theorem 2.

Necessary and sufficient conditions on μ for g_α are possible only if the variation of μ is allowed to grow more rapidly at infinity.

PROOF OF THEOREM 1: This extension of Theorem 2 can now be established by applying the theorem of Stein-Zygmund [8]. This result may be stated as follows:

THEOREM: If $T \in S_\alpha$, then there is a constant Q such that

$$\| T(f) \|_q \leq Q \| f \|_p, \qquad q = np/(n - \alpha p)$$

for all $f \in L_p$; Q is independent of f. $1 < p < \infty$, $0 < \alpha p < n$.

For $T \in S_\alpha$, choose β satisfying $\dfrac{(n-d)(n-\alpha p)}{dp} < \beta < \alpha$. Note that this is always possible since $\alpha p > n - d$. From 1.1, $T \cdot J^{-\beta} \in S_{\alpha-\beta}$, thus the theorem of Stein-Zygmund yields

$$\| T \cdot J^{-\beta}(f) \|_q \leq Q_1 \| f \|_p$$

where $q = np/[n - (\alpha - \beta) p]$. Using Theorem 2 (which we now know is true for the Bessel potential operators),

$$\| T(f) \|_{q^*, \mu} = \| J^\beta (T \cdot J^{-\beta}(f)) \|_{q^*, \mu} \leq Q_2 \| T \cdot J^{-\beta}(f) \|_q$$

where $q^* = dq/(n - \beta q) = dp/(n - \alpha p) = p^*$. Hence

$$\| T(f) \|_{p^*, \mu} \leq Q_2 \cdot Q_1 \| f \|_p .$$

Note that the conditions $1 < q < q^* < \infty$ are satisfied when $1 < p < \infty$, $n - d < \alpha p < n$, and by the choice of β.

REMARK 3: It might be noted that if the more general interpolation theorem of R. Hunt [4] had been used in place of the theorem of Marcinkiewicz, it would be possible to deduce that any $T \in S_\alpha$ maps the Lorentz space $L(p, q)$ continuously into $L(p^*, s)(\mu)$, with $q \leq s$, the usual restrictions on α, p and d. In particular when $q = s = \infty$, T maps weak-L_p continuously into weak-$L_{p^*}(\mu)$.

3. Related results.

3.1. We begin by giving potential versions of two classical trace theorems.

THEOREM 3: Let $T \in S_\alpha$ and $\mu \in \mathcal{L}^+_{1;d}$, then there is a constant Q such that for all $f \in L_p$

$$\| \Delta_t T(f) \|_{r, \mu} \leq Q \, | t |^{\alpha - n/p + d/r} \, \| f \|_p$$

where $\max \{dp/[n - (\alpha - 1)p], p\} < r < p^*$, $n - d < \alpha p < n$. Here Δ_t denotes the first difference; Q is to be independent of f and t.

PROOF: The restrictions on r insure that the exponent of $|t|$ is always positive and less than 1. It is easy to see that $\Delta_t T(f)(x) = (\Delta_t k) * f(x)$, where k is the kernel of T.

We write $T = (J^{-\theta} \cdot T) \cdot J^\theta = k_{\alpha-\theta} * g_\theta$ where $k_{\alpha-\theta}$ is the kernel of $J^{-\theta}$. T, θ chosen to satisfy initially $(n - d)/p < \theta < \alpha$. Then

$$\Delta_t T(f)(x) = \int \Delta_t k_{\alpha-\theta}(z) \cdot g_\theta * f_z(x) \, dz$$

since $k = k_{\alpha-\theta} * g_\theta$. Here $f_z(y)$ denotes $f(y - z)$. By the inequality of Minkowski and Theorem 1, we have

$$\| \Delta_t T(f) \|_{r, \mu} \leq \int | \Delta_t k_{\alpha-\theta}(z) | \cdot \| g_\theta * f_z \|_{r, \mu} dz$$

$$\leq \int | \Delta_t k_{\alpha-\theta}(z) | \cdot Q \, \| f_z \|_p \, dz$$

where $r = dp/(n - \theta p)$. Since $\| f_z \|_p = \| f \|_p$, we have, using (2)

$$\| \Delta_t T(f) \|_{r,\mu} \leq Q' \| f \|_p \, | t |^{\alpha - \theta}, \qquad 0 < \alpha - \theta < 1.$$

But $\theta = n/p - d/r$, hence the theorem follows.

THEOREM 4: For $T \in S_\alpha$, $f \to T(f)$ is a compact mapping of L_p into $L_r(\mu)$ for any $\mu \in \mathcal{M}_1^+ \cap \mathcal{L}_{1;d}^+$, $1 \leq r < p^*$, $n - d < \alpha p < n$.

PROOF: Let $\{f_k\}$ be a bounded sequence in L_p, then there exists a weakly convergent subsequence $f'_k \to f, f \in L_p$. By (1) and (2) $k_{\alpha-\theta} * f'_k \to$ $\to k_{\alpha-\theta} * f$ strongly in L_p locally, by the familiar Riesz compacteness criterion. But since $\mu \in \mathcal{M}_0$, $T(f'_k) \to T(f)$ in μ measure and thus using Theorem 1 the result follows by a standard argument.

3.2. We now consider a « dual » to Theorem 1 and then apply it to obtain an extension of a theorem of Campanato [2].

THEOREM 5: Let $T \in S_\alpha$ and $\mu \in \mathcal{L}_{1;d}^+$, then there is a constant Q such that for all $g \in L_q(\mu)$,

$$\| T(g\mu) \|_{\bar{q}} \leq \| g \|_{q,\mu} \, Q$$

where $\bar{q} = nq/[d + q(n - \alpha - d)]$ and $1 < q < \bar{q} < \infty$. Here Q is independent of g.

PROOF: Let $g \in L_{p^{*\prime}}(\mu)$ and $f \in L_p$, $p^* = dp/(n - \alpha p)$, then

$$\int T(g\mu) f \, dx = \int T(f)\, g \, d\mu \leq \| T(f) \|_{p^*,\mu} \| g \|_{p^{*\prime},\mu} \leq Q \| f \|_p \cdot \| g \|_{p^{*\prime},\mu},$$

the last inequality following from Theorem 1. The result now follows by taking $q = p^{*\prime}$ and $\bar{q} = p'$.

Let $\mathcal{L}_{t;n-\lambda}$, $1 \leq t < \infty$, $0 < \lambda < n$, denote the class of measures $\mu \in$ $\in \mathcal{L}_{1;n-\lambda}$ which are absolutely continuous with respect to m_n with density f satisfying $|f|^t \cdot m_n \in \mathcal{L}_{1;n-\lambda}^+$.

THEOREM 6: If $\mu \in \mathcal{L}_{1;n-\lambda}$, then $h_\alpha * \mu \in \mathcal{L}_{t;n-t(\lambda-\alpha)}$ where $1 \leq t < \lambda/(\lambda - \alpha)$, $0 < \alpha < \lambda$.

PROOF: In Theorem 5, take $g(x) = \chi_{2r}(x)$, the characteristic function of the ball $B_{2r}(x_0)$, and $t = \bar{q}$, $\bar{q}/q = (n - t(\lambda - \alpha))/(n - \lambda)$, $d = n - \lambda$, then

$$\int_{|x-x_0|<r} |h_\alpha * \mu_{2r}|^t \, dx \leq Q' \, r^{n-t(\lambda-\alpha)}$$

where μ_{2r} is μ restricted to $B_{2r}(x_0)$. The condition $1 < q < \bar{q} < \infty$ is equivalent to $n/(n - \alpha) < t < \lambda/(\lambda - \alpha)$.

It now remains to estimate the t-power of the variation over $B_r(x_0)$ of $h_\alpha * (\mu - \mu_{2r})$. However, this quality is just the variation of $|I_5|^t$ over $B_r(x_0)$ (I_5 as in lemma 4) with x_0 playing the role of x and $\alpha < \lambda$.

Next, if $1 \leq t < n/(n - \alpha)$, then

$$\left\{\int_{|x-x_0|<r} |h_\alpha * \mu_{2r}|^t \, dx\right\}^{1/t} \leq \int_{|y-x_0|<2r} \left\{\int_{|x-x_0|<r} h_\alpha(x - y)^t \, dx\right\}^{1/t} d\,|\mu|\,(y)$$

$$\leq Q r^{[n-t(\lambda-\alpha)]/t} .$$

The remaining integral is handled as before.

Finally, for any t in the interval $[1, \lambda/(\lambda - \alpha)$, a simple interpolation argument gives the result.

REMARK 4: The result of Campanato [2] can be stated as follows: If $f \in \mathcal{L}_{p;n-\lambda}$, then $h_\alpha * f \in \mathcal{L}_{t;n-\sigma}$ where $\alpha p < \lambda$, $1 \leq t < \lambda p/(\lambda - \alpha p)$, and $\lambda > \sigma > t(\lambda - \alpha p)/p$, $1 < p < \infty$.

His proof fails when $p = 1$, which Theorem 6 now treats. Note, when $p = 1$, it is possible to take $\sigma = t(\lambda - \alpha)$.

REMARK 5: It might be interesting to find conditions on μ for which $t = \lambda/(\lambda - \alpha)$ is allowed in Theorem 6, for in general, it is known that $\mu \in \mathcal{L}_{1;n-\lambda}$ is *not* sufficient. Indeed, if $\lambda = \alpha p$, then $\lambda/(\lambda - \alpha) = p'$ and such a μ does not even insure that $h_\alpha * \mu \in L_{p'}$ locally (for this see [6]). From the proof of Theorem 6, it appears that any condition on μ which insures $\| h_\alpha * \mu_{2r} \|_t^t \leq Q r^{n-\lambda}$, $t = \lambda/(\lambda - \alpha)$ will be sufficient. Two such conditions are:

(i) $h_\alpha : L_{\lambda/\alpha} \longrightarrow L_{\lambda/\alpha}(\mu)$ continuously, $0 < \alpha < \lambda$,

and

(ii) $h_\alpha * (h_\alpha * \mu)^{\alpha/(\lambda-\alpha)}$ is bounded on E_n.

The proofs require no new ideas.

REMARK 6: Finally, we observe that (10) is an extension of a theorem of Stein and Weiss [7] — see in particular their Theorem B^* in the case $p < q$, which corresponds to our case $p < p^*$. The example referred to in the above remark easily shows that no extension of this generality is possible when $p = p^*$.

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