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# ON ELLIPTIC EQUATIONS WITH PIECEWISE CONSTANT COEFFICIENTS. II

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**SUMMARY** - In this work we prove an existence and uniqueness theorem for solutions in  $W^{2,p}(R^n)$  of second order linear elliptic equations, whose coefficients are constant-valued in the half-spaces  $R_+^n$  and  $R_-^n$

## 1. Introduction and statement of the problem.

In this paper we are interested in solving a second order linear partial differential equation of elliptic type, whose coefficients are constant-valued in the half-spaces  $R_+^n = \{x \in R^n : x_1 > 0\}$  and  $R_-^n = \{x \in R^n : x_1 < 0\}$ . We carry on our research, begun in [6], where square summable solutions with square summable second derivatives are dealt with: in this work we look for solutions in  $W^{2,p}(R^n)$  ( $1 < p < +\infty$ ). We recall that  $W^{2,p}(R^n)$  denotes the Sobolev space <sup>(1)</sup> of all functions of  $L^p(R^n)$ , that have derivatives in the sense of distributions of the first two orders belonging to  $L^p(R^n)$ ;  $W^{2,p}(R^n)$  is a Banach space with respect to the norm:

$$\|u\|_{W^{2,p}(R^n)} = \left\{ \int_{R^n} \left[ |u|^p + \sum_{r,j=1}^n \left| \frac{\partial^2 u}{\partial x_r \partial x_j} \right|^p \right] dx \right\}^{1/p}$$

Our equation is

$$(1) \quad \begin{cases} L^+ u \equiv \sum_{r,j=1}^n a_{rj}^+ \frac{\partial^2 u}{\partial x_r \partial x_j} + \sum_{j=1}^n a_j^+ \frac{\partial u}{\partial x_j} - h^+ u = f & \text{in } R_+^n \\ L^- u \equiv \sum_{r,j=1}^n a_{rj}^- \frac{\partial^2 u}{\partial x_r \partial x_j} + \sum_{j=1}^n a_j^- \frac{\partial u}{\partial x_j} - h^- u = f & \text{in } R_-^n \end{cases}$$

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<sup>(1)</sup> For the properties of Sobolev spaces see, for instance, [2] or [9].

where  $a_{rj}^+$ ,  $a_{rj}^-$ ,  $a_j^+$ ,  $a_j^-$ ,  $h^+$ ,  $h^-$  ( $r, j = 1, 2, \dots, n$ ) are real constants with the following properties:

i) the  $n \times n$  matrices  $A^+ = (a_{rj}^+)$  and  $A^- = (a_{rj}^-)$  are symmetric and positive definite;

ii)  $h^+ > 0$ ,  $h^- > 0$ ;

and  $f$  is an assigned function in  $L^p(\mathbb{R}^n)$  ( $1 < p < +\infty$ ).

In the following we shall be interested in the case  $n \geq 3$ .

**THEOREM.** *Equation (1) has a unique solution  $u \in W^{2,p}(\mathbb{R}^n)$  for every  $f \in L^p(\mathbb{R}^n)$  ( $1 < p < +\infty$ ). There exists a constant  $C$  independent of  $u$  such that the following estimate holds:*

$$(2) \quad \|u\|_{W^{2,p}(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}$$

From the theorem it follows that, if  $p$  is large enough ( $p > n/2$ ), the solution is continuous across the interface  $x_1 = 0$ : if  $p > n$ , also the first derivatives are continuous across the interface.

The method used to prove the existence of the solution consists in solving the Neumann problems

$$(3) \quad \begin{cases} L^+ u^+ = f \\ u^+ \in W^{2,p}(\mathbb{R}_+^n) \\ \frac{\partial u^+}{\partial x_1}(0^+, \cdot) = g \end{cases} \quad (4) \quad \begin{cases} L^- u^- = f \\ u^- \in W^{2,p}(\mathbb{R}_-^n) \\ \frac{\partial u^-}{\partial x_1}(0^-, \cdot) = g \end{cases}$$

where  $g$  is some function in  $W^{1-\frac{1}{p},p}(\mathbb{R}^n)$ ,  $\frac{\partial u^+}{\partial x_1}(0^+, \cdot)$  and  $\frac{\partial u^-}{\partial x_1}(0^-, \cdot)$  denote respectively the traces of  $\frac{\partial u^+}{\partial x_1}$  and  $\frac{\partial u^-}{\partial x_1}$  on  $x_1 = 0$ . Remember that  $W^{s,p}(\mathbb{R}^n)$ , for  $s > 0$  non integer, is the space of all functions which together with all derivatives of order  $< s$  (in the sense of distributions) are in  $L^p(\mathbb{R}^n)$  and satisfy the inequality

$$(5) \quad \|u\|_{W^{s,p}(\mathbb{R}^n)} = \left\{ \sum_{j=0}^{[s]} \sum_{|\alpha|=j} \left[ \int_{\mathbb{R}^n} |D^\alpha u|^p dx + \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+p(s-[s])}} dy \right]^{1/p} \right\} < +\infty$$

where  $[s]$  is the largest integer  $< s$ .  $W^{s,p}(R^n)$  is a Banach space with respect to the norm defined by the left side of (5).

We observe that  $W^{1-\frac{1}{p},p}(R^n)$  is exactly the space of traces of first derivatives of functions in  $W^{2,p}(R^n)$ .

We shall prove that problems (3) and (4) have a unique solution. Then we shall choose  $g$  so that the function  $u$  so defined :

$$u(x) = \begin{cases} u^+(x) & x \in R_+^n \\ u^-(x) & x \in R_-^n \end{cases}$$

is the wanted solution of (1). To do thus, we shall have to solve an integral equation in  $g$ : the solution of such an equation is obtained by interpolation-techniques.

## 2. Fundamental solution of an operator with constant coefficients.

In this section we shall, for the sake of convenience, denote the variables by  $(x_0, x_1, \dots, x_n)$  ( $n \geq 2$ ).

Consider the second order linear differential operator with real constant coefficients

$$(6) \quad L = \sum_{r,j=0}^n a_{rj} \frac{\partial^2}{\partial x_r \partial x_j} + \sum_{j=0}^n a_j \frac{\partial}{\partial x_j} - h^2$$

where  $A = (a_{rj})$  is a symmetric, positive definite  $(n+1) \times (n+1)$  matrix and  $h > 0$ .

We shall show some properties of the function

$$(7) \quad E(x) = c^{n-1} (\det A)^{-1/2} l_n [cr(x)] \exp \left[ -\frac{1}{2} (A^{-1} a, x) \right] \quad x \neq 0$$

that turns out to be a fundamental solution of  $L$ . In formula (7)

$$(8) \quad l_n(r) = - (2\pi)^{-(n+1)/2} r^{(1-n)/2} K_{(n-1)/2}(r)$$

$K_{(n-1)/2}$  being the modified Hankel function, <sup>(2)</sup>,  $a$  being the vector

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<sup>(2)</sup> For the properties of Bessel functions see [11].

$(a_1 \dots a_n)$ ,

$$(9) \quad c = \left[ h^2 + \frac{1}{4} (A^{-1} a, a) \right]^{1/2} \quad (3)$$

$$(10) \quad r(x) = (A^{-1} x, x)^{1/2}$$

For the following it will be useful to recall the integral representation formula

$$(11) \quad l_n(r) = -\frac{1}{2} (2\pi)^{-(n+1)/2} \int_0^{+\infty} s^{(n-3)/2} \exp\left(-\frac{1}{2} r^2 s - \frac{1}{2s}\right) ds$$

that is an immediate consequence of known formulas for Bessel functions.

Then, we should like to call to mind that, if  $L = \Delta - h^2$ , the fundamental solution (7) becomes the familiar function  $h^{n-1} l_n(h|x|)$ .

Moreover, we observe that

$$(12) \quad E(x) = -\frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi^{(n+1)/2} (\det A)^{1/2} (n-1)} [r(x)]^{1-n} [1 + o(|x|)] \text{ as } x \rightarrow 0,$$

where the function  $-\frac{\Gamma\left(\frac{n+1}{2}\right)}{2\pi^{(n+1)/2} (\det A)^{1/2} (n-1)} [r(x)]^{1-n}$  is a fundamental solution of the operator  $\sum_{r,j=0}^n a_{rj} \frac{\partial^2}{\partial x_r \partial x_j}$ . (12) is easily proved by using the formula

$$(13) \quad K_m(r) = 2^{m-1} \Gamma(m) r^{-m} e^{-r} (1 + o(r)) \text{ as } r \rightarrow 0 \quad (m > 1/2)$$

(see, for instance, [10], appendix) and the inequalities

$$(14) \quad |(A^{-1} a, x)| \leq (A^{-1} x, x)^{1/2} (A^{-1} a, a)^{1/2}$$

(that is valid for all symmetric, positive definite  $(n+1) \times (n+1)$  matrices and for all vectors  $a, x \in R^{n+1}$ )

$$(15) \quad |r(x)| \leq \nu^{-1/2} |x|$$

$\nu$  being the largest eigenvalue of  $A$ .

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(3)  $(\cdot, \cdot)$  denotes the scalar product in  $R^n$ .

Finally, we observe the following estimates for  $E$  and the gradient  $DE$  of  $E$ , that, in particular, imply  $E \in W^{1,1}(R^{n+1})$ :

$$(16) \quad |E(x)| \leq \frac{\nu^{1/2}}{4\pi(\det A)^{1/2}} \cdot \frac{\exp(-\alpha|x|)}{|x|} \quad n=2$$

$$(16) \quad |E(x)| \leq \frac{\nu}{4\pi^2(\det A)^{1/2}} \cdot \frac{\exp(-\alpha|x|)}{|x|^2} \left[ 1 + \left( \frac{c\pi}{2\nu^{1/2}} |x| \right)^{1/2} \right] \quad n=3$$

$$(16) \quad |E(x)| \leq \frac{\Gamma\left(\frac{n+1}{2}\right) \nu^{\frac{n-1}{2}}}{2\pi^{\frac{n+1}{2}} (\det A)^{1/2} (n-1)} \cdot \frac{\exp(-\alpha|x|)}{|x|^{n-1}} \left[ 1 + \frac{c}{C_n \nu^{1/2}} |x| \right]^{\frac{n-2}{2}} \quad n \geq 4$$

$$(17) \quad |DE(x)| \leq \frac{\nu^{3/2}}{2\pi\mu(\det A)^{1/2}} \cdot \frac{\exp(-\alpha|x|)}{|x|^2} \left[ 1 + \frac{2c\nu^{1/2} + |a|}{4\nu} |x| \right] \quad n=2$$

$$(17) \quad |DE(x)| \leq \frac{\nu^2}{2\pi^2\mu(\det A)^{1/2}} \left( 1 + \frac{|a|}{c\nu^{1/2}} \right) \frac{\exp(-\alpha|x|)}{|x|^3} \left[ 1 + \frac{c\pi^{1/3}}{2\nu^{1/2}} |x| \right]^{3/2} \quad n=3$$

$$(17) \quad |DE(x)| \leq \frac{\Gamma\left(\frac{n+1}{2}\right) \nu^{\frac{n+1}{2}}}{2\pi^{\frac{n+1}{2}} \mu (\det A)^{1/2}} \cdot \frac{\exp(-\alpha|x|)}{|x|^n} \left( 1 + \frac{|a|}{2c\nu^{1/2}} \right) \left( 1 + \frac{c}{C_{n+2}\nu^{1/2}} |x| \right)^{n/2} \quad n \geq 4$$

where  $\mu$  and  $\nu$  are the smallest and largest eigenvalues of  $A$  and  $\alpha$ ,  $|a|$ ,  $C_n$  are given respectively by

$$(18) \quad \alpha = \frac{c - \frac{1}{2}(A^{-1}a, a)^{1/2}}{\nu^{1/2}}$$

$$|a| = \left( \sum_{j=0}^n a_j^2 \right)^{1/2}$$

$$(19) \quad C_n = \frac{1}{2} \left[ \pi^{-1/2} \Gamma\left(\frac{n-1}{2}\right) \right]^{\frac{2}{n-2}}$$

Estimates (16) follow immediately from (14), (15) and known inequalities for Bessel functions (see, for instance, [10], appendix). The same arguments

and the formulas

$$DE(x) = \frac{c^n}{(\det A)^{1/2}} \cdot \frac{A^{-1}x}{r(x)} l'_n(cr(x)) \exp\left[-\frac{1}{2}(A^{-1}a, x)\right] - \frac{1}{2} A^{-1}a E(x)$$

(20) 
$$\frac{d}{dr}(r^{-m} K_m(r)) = -r^{-m} K_{m+1}(r)$$

give (17).

Now we can prove two lemmas :

LEMMA 1. *E is a fundamental solution of L, that is*

$$\mathcal{F}E(\xi) = -\frac{1}{(A\xi, \xi) - i(a, \xi) + h^2} \quad \xi \in R^n$$

where  $\mathcal{F}E(\xi) = \int_{R^n} \exp[-i(x, \xi)] E(x) dx$  is the Fourier transform of E.

LEMMA 2. *If  $f \in L^p(R^{n+1})$  ( $1 < p < +\infty$ ), the following properties hold:*

- i)  $E * f \in W^{2,p}(R^{n+1})$ , where \* denotes convolution ;
- ii)  $L(E * f) = f$  ;
- iii)  $\|E * f\|_{W^{2,p}(R^{n+1})} \leq C \|f\|_{L^p(R^{n+1})}$ , C being a constant independent of f.

PROOF OF LEMMA 1 We observe that, for (11), the following formula is easily seen to hold :

$$c^{n-1} \int_{R^{n+1}} \exp[-i(x, \xi)] l_n(c|x|) dx = \frac{-1}{|\xi|^2 + c^2}$$

With the change of variables  $x = M_\tau^{-1}z$ , where M is a  $(n+1) \times (n+1)$  matrix such that  $M_\tau M = A^{(4)}$ , we get easily

$$c^{n-1} \int_{R^{n+1}} \exp[-i(x, \xi)] l_n[cr(x)] dx = -\frac{(\det A)^{1/2}}{(A\xi, \xi) + c^2}$$

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(4)  $M_\tau$  is the transposed matrix of M.

To conclude, we remember that, if  $f$  is a well-behaved function, the Fourier transform of  $f(x) \exp (a, x)$  is  $\mathcal{F} f(\xi - ia)$ : thus the lemma is proved.

PROOF OF LEMMA 2. To prove the lemma we can suppose  $f \in C_0^\infty (R^{n+1})$ . From Young's inequality it follows that

$$(21) \quad \|E * f\|_{L^p (R^{n+1})} \leq \|f\|_{L^p (R^{n+1})} \int_{R^{n+1}} |E(x)| dx,$$

the integral of  $E$  being finite on account of estimates (16). Incidentally, we notice that Lemma 1 and the obvious fact that  $E$  is negative imply the equation

$$\int_{R^{n+1}} |E(x)| dx = 1/h^2$$

The formula

$$\frac{\partial^2 (E * f)}{\partial x_r \partial x_j} = - \mathcal{F}^{-1} [\xi_r \xi_j \mathcal{F} E \mathcal{F} f] = (\mathcal{F}^{-1} \varphi_{rj}(\xi) \mathcal{F}) f,$$

where

$$\varphi_{rj}(\xi) = \frac{\xi_r \xi_j}{(A \xi, \xi) - i(a, \xi) + h^2} \quad (r, j = 0, \dots, n),$$

and a theorem of Hörmander-Mihlin show that the second derivatives of  $E * f$  are in  $L^p (R^{n+1})$  and that, taking into account (21), iii) is fulfilled. For, it is easy to see that the functions  $\varphi_{rj}$  are, following Hörmander's terminology, multipliers of type  $(p, p)$  for every  $1 < p < +\infty$ . For more details, see the appendix of this paper.

Then, from lemma 1 it follows that

$$\mathcal{F}[L(E * f)](\xi) = - [(A \xi, \xi) - i(a, \xi) + h^2] \mathcal{F} E(\xi) \mathcal{F} f(\xi) = \mathcal{F} f(\xi),$$

that is ii). The lemma is proved.

For the following we need to have the expression of the Fourier transform of  $E(t, \cdot)$ , where we have put, for the sake of convenience,  $t = x_0$ ,  $x = (x_1, \dots, x_n)$ : moreover,  $\xi$  will denote the dual variable of  $x$  and, occasionally,  $\mathcal{F}_x$  will denote the Fourier transform with respect to  $x$ .

The following lemma holds:

LEMMA 3. We have

$$(22) \quad - \int_{R^n} \exp[-i(x, \xi)] E(t, x) dx = \psi(t, \xi) = \begin{cases} \frac{\exp [tz_1(\xi)]}{\sqrt{H}(\xi)} & t \geq 0 \\ \frac{\exp [tz_2(\xi)]}{\sqrt{H}(\xi)} & t < 0 \end{cases}$$



where  $z_1(\xi)$  and  $z_2(\xi)$  are the roots of the equation

$$(23) \quad a_{00} z^2 + \alpha(\xi) z - \beta(\xi) = 0$$

respectively with negative and positive real part and

$$(24) \quad H(\xi) = \alpha^2(\xi) + 4a_{00} \beta(\xi)$$

is the discriminant <sup>(5)</sup>,  $\alpha(\xi)$  and  $\beta(\xi)$  being defined as follows :

$$(25) \quad \alpha(\xi) = a_0 + 2i \sum_{j=1}^n a_{0j} \xi_j$$

$$(26) \quad \beta(\xi) = h^2 + \sum_{r,j=1}^n a_{rj} \xi_r \xi_j - i \sum_{j=1}^n a_j \xi_j$$

REMARK 1. Observe that the discriminant  $H(\xi)$  has the following properties :

- i)  $\operatorname{Re} H(\xi) \geq a_0^2 + 4a_{00} h^2$
- ii)  $\operatorname{Re} \sqrt{H(\xi)} \geq \sqrt{\operatorname{Re} H(\xi)}$
- iii)  $|H(\xi)| \geq C(1 + |\xi|^2)$ ,  $\left[ C \text{ being a strictly positive constant, } |\xi| = \left( \sum_{j=1}^n \xi_j^2 \right)^{1/2} \right]$ .

Moreover, the roots  $z_1(\xi)$  and  $z_2(\xi)$  have the properties :

$$(27) \quad \operatorname{Re} z_1(\xi) \leq \frac{-a_0 - \sqrt{a_0^2 + 4a_{00} h^2}}{2a_{00}} < 0$$

$$(28) \quad \operatorname{Re} z_2(\xi) \geq \frac{-a_0 + \sqrt{a_0^2 + 4a_{00} h^2}}{2a_{00}} > 0$$

$$(29) \quad C'(1 + |\xi|^2)^{1/2} \leq |\operatorname{Re} z_j(\xi)| \leq |z_j(\xi)| \leq C''(1 + |\xi|^2)^{1/2} \quad (j = 1, 2)$$

$C'$  and  $C''$  being positive constants.

For the proof, see [6], lemmas 1 and 3.

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<sup>(5)</sup> In this paper the square root of a complex number is the one with non-negative real part.

REMARK 2. The function  $-\psi$  is a fundamental solution of the operator

$$a_{00} \frac{\partial^2}{\partial t^2} + \alpha(\xi) \frac{\partial}{\partial t} - \beta(\xi)$$

that admits (23) as its characteristic equation.

PROOF OF LEMMA 3. From the properties of integrability of  $E$  (see inequalities (16)) it follows that

$$(30) \quad \int_{R^n} \exp[-i(x, \xi)] E(t, x) dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp(it\tau) \mathcal{F} E(\tau, \xi) d\tau$$

For lemma 1 and the notations introduced before the statement of lemma 3, we can write

$$(31) \quad \mathcal{F} E(\tau, \xi) = - \frac{1}{a_{00} \tau^2 - i\alpha(\xi)\tau + \beta(\xi)}$$

where  $\alpha(\xi)$  and  $\beta(\xi)$  are defined by (25), (26).

Applying Cauchy's theorem on residues, from (30) and (31) we get

$$\int_{R^n} \exp[-i(x, \xi)] E(t, x) dx = \begin{cases} - \frac{1}{a_{00}} \cdot \frac{\exp[tz_1(\xi)]}{z_2(\xi) - z_1(\xi)} & t \geq 0 \\ - \frac{1}{a_{00}} \cdot \frac{\exp[tz_2(\xi)]}{z_2(\xi) - z_1(\xi)} & t < 0 \end{cases}$$

i. e. the assertion.

For the following, it will be useful to estimate all the derivatives of  $E$ : such estimates imply, in particular that  $E$  is analytic in  $R^n - \{0\}$ .

LEMMA 4. Let  $\gamma$  be any multi-index: then

$$(32) \quad |D^\gamma E(x)| \leq (|\gamma|!) C^{|\gamma|+1} |x|^{1-n-|\gamma|} \exp(-\delta|x|) \quad x \neq 0$$

where  $D^\gamma = \frac{\partial^{\gamma_0}}{\partial x_0^{\gamma_0}} \frac{\partial^{\gamma_1}}{\partial x_1^{\gamma_1}} \dots \frac{\partial^{\gamma_n}}{\partial x_n^{\gamma_n}}$ ,  $|\gamma| = \sum_{j=0}^n \gamma_j$ ,  $c$  is defined by (9),

$$(33) \quad \delta = \frac{h^2}{2\nu^{1/2} [(A^{-1}a, a) + 2h^2]^{1/2}}$$

and  $C$  is a constant independent of  $\gamma$ .

PROOF. Leibniz's rule for the derivative of a product, inequalities (14), (15), formulas (9) and (33) imply that it is enough to prove the following inequality

$$(34) \quad |D^\nu l_n[cr(x)]| \leq (|\gamma|!) C^{|\nu|+1} [r(x)]^{1-n-|\nu|} \exp[-\varepsilon cr(x)]$$

where

$$(35) \quad \varepsilon = \frac{[(A^{-1}a, a) + 2h^2]^{1/2}}{[(A^{-1}a, a) + 4h^2]^{1/2}}$$

For the derivation of (32) from (34) it is useful to bear in mind the inequality

$$(36) \quad \varepsilon c - \frac{1}{2} (A^{-1}a, a)^{1/2} = \frac{h^2}{[(A^{-1}a, a) + 2h^2]^{1/2} + (A^{-1}a, a)^{1/2}} > \nu^{1/2} \delta$$

Since there exists an orthogonal  $(n+1) \times (n+1)$  matrix that maps the quadratic form  $r^2(x)$  into the canonical form  $\varrho^2(x) = \sum_{j=0}^n \lambda_j x_j^2$ , it suffices to prove (34) with  $l_n[cr(x)]$  substituted by  $l_n[c\varrho(x)]$ . For the integral representation formula (11) we get

$$(37) \quad l_n[c\varrho(x)] = -\frac{1}{2(2\pi)^{\frac{n+1}{2}}} \int_0^{+\infty} s^{\frac{n-3}{2}} \exp\left[-\frac{1}{2} s c^2 \sum_{j=0}^n \lambda_j x_j^2 - \frac{1}{2s}\right] ds$$

First we prove the following estimate:

$$(38) \quad \left| \frac{d^k}{dx^k} \exp(-ax^2) \right| \leq k! \left(\frac{k}{2}\right)^{-k/2} e^{k/2} a^{k/2} (1-\eta^2)^{-k/2} \exp(-\eta^2 ax^2)$$

for every  $a > 0$ ,  $0 \leq \eta < 1$ ,  $k = 0, 1, 2, \dots$ .

In fact, since  $\exp(-ax^2)$  is an entire function, we can write the inequality

$$\left| \frac{d^k}{dx^k} \exp(-ax^2) \right| \leq \frac{k! r^{-k}}{2\pi} \int_0^{2\pi} \exp[-a \operatorname{Re}(x + re^{i\varphi})^2] d\varphi \quad r \in (0, +\infty)$$

where

$$\operatorname{Re}(x + re^{i\varphi})^2 = x^2 + 2rx \cos \varphi + r^2 \cos^2 \varphi - r^2 \sin^2 \varphi$$

From the inequality

$$|2rx \cos \varphi| \leq (1 - \eta^2) x^2 + \frac{r^2 \cos^2 \varphi}{1 - \eta^2}$$

valid for every  $\eta \in [0, 1)$ , we infer that

$$\operatorname{Re} (x + r e^{i\varphi})^2 \geq \eta^2 x^2 - r^2 \left[ \frac{\eta^2}{1 - \eta^2} \cos^2 \varphi + \sin^2 \varphi \right] \geq \eta^2 x^2 - \frac{r^2}{1 - \eta^2}$$

Hence

$$(39) \quad \left| \frac{d^k}{dx^k} \exp(-ax^2) \right| \leq k! r^{-k} \exp \frac{ar^2}{1 - \eta^2} \exp(-\eta^2 ax^2)$$

(38) follows from (39) minimizing with respect to  $r \in (0, +\infty)$ .

Then, (37) and (38) imply that, for  $\eta \in (0, 1)$ , denoting  $|\gamma|$  by  $\sigma$

$$(40) \quad |D^\nu l_n[\varrho(x)]| \leq \frac{C^\sigma \left(\frac{\sigma}{2}\right)^{-\sigma/2} e^{\sigma/2} \sigma!}{2 (2\pi)^{\frac{n+1}{2}}} \int_0^{+\infty} s^{\frac{\sigma+n-3}{2}} \exp \left[ -\frac{s\eta^2 \varrho^2(x)}{2} - \frac{1}{2s} \right] ds = \\ = \left(\frac{\sigma}{2}\right)^{-\sigma/2} e^{\sigma/2} (\sigma!) |l_{n+\sigma}[\eta \varrho(x)]|$$

Use has been made of the inequalities  $\lambda_j \leq \frac{1}{\mu}$  ( $j = 0, \dots, n$ ),  $\mu$  being the smallest eigenvalue of  $A$ , of formula (11) and the estimates :

$$\prod_{j=0}^n \gamma_j! \leq \sigma! \quad \prod_{j=0}^n \left(\frac{\gamma_j}{2}\right)^{-\gamma_j/2} \leq (n+1)^{\sigma/2} \left(\frac{\sigma}{2}\right)^{-\sigma/2}.$$

Then, the estimate

$$|K_m(r)| \leq 2^{m-1} r^{-m} \Gamma(m) (1 - \eta^2)^{-m} \exp(-\eta r) \quad r > 0, \quad 0 < \eta < 1$$

(see, for instance, [10], appendix), Stirling's formula

$$\lim_{m \rightarrow +\infty} \frac{\Gamma(m + \varrho)}{m^\varrho \Gamma(m)} = 1 \quad \varrho > 0$$

and inequality (41) imply the assertion, if  $\eta$  is chosen to be equal to  $\varepsilon^{1/2}$ .

### 3. The Poisson kernel.

In this section, and from now on, we denote, as in lemma 3, the variables by  $(t, x)$ , where  $t \in R, x = (x_1, \dots, x_n) \in R^n$ .

Now, consider the real analytic function

$$(42) \quad P(t, x) = 2 D_{\zeta} E(t, x) + a_0 E(t, x) \quad (t, x) \neq (0, 0)$$

where  $\zeta = (a_{00}, \dots, a_{0n})$  and  $D_{\zeta} = \sum_{j=1}^n a_{0j} \frac{\partial}{\partial x_j} + a_{00} \frac{\partial}{\partial t}$  is the differentiation along the conormal direction  $\zeta$ . From (7) and (11) it follows that

$$(43) \quad P(t, x) = \frac{2c^n t}{r(t, x) (\det A)^{1/2}} l'_n [cr(t, x)] \exp\left(\tilde{a}_0 t + \sum_{j=1}^n \tilde{a}_j x_j\right) = \\ = \frac{c^{n+1} t \exp\left(\tilde{a}_0 t + \sum_{j=1}^n \tilde{a}_j x_j\right)}{(2\pi)^{\frac{n+1}{2}} (\det A)^{1/2}} \int_0^{+\infty} s^{\frac{n-1}{2}} \exp\left[-\frac{sc^2}{2} r^2(t, x) - \frac{1}{2s}\right] ds$$

where

$$(44) \quad \tilde{a} = -\frac{1}{2} A^{-1} a.$$

It is easy to recognize from (43) that  $P$  has the sign of  $t$ .

From formulas (18), (19), (20), (43), inequalities (14), (15) and known estimates for Bessel functions (see, for instance, [10], appendix), we infer

$$(45) \quad |P(t, x)| \leq \frac{\Gamma\left(\frac{n+1}{2}\right) c^{\frac{n+1}{2}}}{2\pi^{\frac{n+1}{2}} (\det A)^{1/2}} \cdot \frac{|t| \exp[-\alpha(t^2 + |x|^2)^{1/2}]}{(t^2 + |x|^2)^{\frac{n+1}{2}}} \left[1 + \frac{c}{C_{n+2} c^{1/2}} (t^2 + |x|^2)^{1/2}\right]^{n/2} \quad n \geq 2$$

where  $\alpha$  is defined by (18).

(45) establishes that  $P \in L^1(R^{n+1})$ ,  $P(t, \cdot) \in L^1(R^n)$  for all  $t \in R$  and  $P(\cdot, x) \in L^1(R)$  for all  $x \in R^n - \{0\}$ . Moreover, observe that (13), (20), (43), imply that

$$P(t, x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}} (\det A)^{1/2}} \frac{t}{[r(t, x)]^{n+1}} \{1 + o[(t^2 + x^2)^{1/2}]\} \text{ as } (t, x) \rightarrow (0, 0)$$

We notice that, when  $L = \Delta$ ,  $P$  coincides with  $\frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}} \cdot \frac{t}{(t^2 + x^2)^{\frac{n+1}{2}}}$ , the usual Poisson kernel.

Finally, we point out that the Fourier transform of  $P(t, \cdot)$  is given by

$$(46) \quad [\mathcal{F}_x P(t, \cdot)](\xi) = \psi(t, \xi) \sqrt{H(\xi)} \operatorname{sgn} t \quad t \neq 0$$

Indeed, from (42) and lemma 3, recalling (17) and (25), we get

$$[\mathcal{F}_x P(t, \cdot)](\xi) = -2a_{00} \frac{\partial \psi}{\partial t}(t, \xi) - \alpha(\xi) \psi(t, \xi) = \psi(t, \xi) \sqrt{H(\xi)} \operatorname{sgn} t$$

Obviously  $P$  is a solution of the equation  $Lu = 0$  in  $R^{n+1} - \{0\}$ .

Then, consider the convolution

$$(47) \quad v(t, x) = \int_{R^n} P(t, x - z) g(z) dz \quad t > 0$$

the properties of which are stated in the following lemmas :

LEMMA 5. If  $g \in L^p(R^n)$  ( $1 < p < +\infty$ ), then

- i)  $\|v(t, \cdot)\|_{L^p(R^n)} \leq \psi(t, 0) \sqrt{H(0)} \|g\|_{L^p(R^n)} =$   
 $= \|g\|_{L^p(R^n)} \exp\left\{\frac{t}{2a_{00}} [-a_0 - (a_0^2 + 4a_{00}h^2)^{1/2}]\right\}$
- ii)  $\|v(t, \cdot) - g\|_{L^p(R^n)} \rightarrow 0 \quad \text{as } t \rightarrow 0 +$

LEMMA 6. If  $g \in W^{s,p}(R^n)$  ( $0 < s < 1$ ,  $1 < p < +\infty$ ), then

- i)  $\left(\int_0^{+\infty} t^{p(1-s)-1} dt \int_{R^n} \left|\frac{\partial v}{\partial t}(t, x)\right|^p dx\right)^{1/p} \leq$   
 $\leq C_1 [|z_1(0)|^s \|g\|_{L^p(R^n)} + |g|_{W^{s,p}(R^n)}]$
- ii)  $\left(\int_0^{+\infty} t^{p(1-s)-1} dt \int_{R^n} \left|\frac{\partial v}{\partial x_j}(t, x)\right|^p dx\right)^{1/p} \leq C_2 |g|_{W^{s,p}(R^n)}$

where

$$|g|_{W^{s,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} dz \int_{\mathbb{R}^n} \frac{|g(x-z) - g(x)|^p}{|z|^{n+ps}} dx \right)^{1/p}$$

and  $C_1$  and  $C_2$  are constants independent of  $g$ .

Moreover,  $v$  is an analytic function in the half-space  $\mathbb{R}_+^{n+1}$  and is a solution of the equation  $Lu = 0$ .

In particular, if  $s = 1 - \frac{1}{p}$ ,  $v \in W^{1,p}(\mathbb{R}_+^{n+1})$ .

REMARK 4. From Lemmas 5 and 6 it follows easily that, if  $g \in W^{s,p}(\mathbb{R}^n)$  ( $0 < s < 1$ ,  $1 < p < +\infty$ ),  $v(0+, \cdot) = g$ , where the left side denotes the trace of  $v$ , defined in the usual way.

We shall premise the proof of lemma 5 with

LEMMA 7. *The function  $P$  has the following properties :*

$$\text{i) } \int_{\mathbb{R}^n} P(t, x) dx = \psi(t, 0) \sqrt{H(0)} \operatorname{sgn} t =$$

$$= \begin{cases} \exp \left\{ \frac{t}{2a_{00}} [-a_0 - (a_0^2 + 4a_{00} h^2)^{1/2}] \right\} & t > 0 \\ -\exp \left\{ \frac{t}{2a_{00}} [-a_0 + (a_0^2 + 4a_{00} h^2)^{1/2}] \right\} & t < 0 \end{cases}$$

$$\text{ii) } \int_{|x| > \lambda} |P(t, x)| dx \leq C \frac{|t|}{\lambda}$$

where  $\lambda$  is any positive number and the constant  $C$  is independent of  $t$  and  $\lambda$ .

PROOF OF LEMMA 7. i) follows from (46), substituting  $\xi = 0$ . Estimate (45) implies the chain of inequalities

$$|P(t, x)| \leq C' \frac{|t|}{(t^2 + |x|^2)^{\frac{n+1}{2}}} \leq C' \frac{|t|}{|x|^{n+1}}$$

where  $C'$  is a constant; hence ii) is easily obtained by integration.

Now, we can prove lemma 5: we observe that i) is an immediate consequence of Young's inequality on convolution and i) in lemma 7.

In order to show ii) we proceed as follows :

$$\|v(t, \cdot) - g\|_{L^p(\mathbb{R}^n)} \leq \|v(t, \cdot) - \psi(t, 0)\sqrt{H(0)}g\|_{L^p(\mathbb{R}^n)} + [1 - \psi(t, 0)\sqrt{H(0)}]\|g\|_{L^p(\mathbb{R}^n)}$$

It is evident, from the definition (22) of  $\psi$ , that it is enough to prove that

$$\|v(t, \cdot) - \psi(t, 0)\sqrt{H(0)}g\|_{L^p(\mathbb{R}^n)} \rightarrow 0 \text{ as } t \rightarrow 0 +$$

Making use again of i) in lemma 5, we get

$$v(t, x) - \psi(t, 0)\sqrt{H(0)}g(x) = \int_{\mathbb{R}^n} P(t, z)[g(x-z) - g(x)] dz.$$

From Minkowski's inequality and ii) in lemma 7 we infer that

$$\begin{aligned} \|v(t, \cdot) - \psi(t, 0)\sqrt{H(0)}g\|_{L^p(\mathbb{R}^n)} &\leq \int_{\mathbb{R}^n} |P(t, z)| \left( \int_{\mathbb{R}^n} |g(x-z) - g(x)|^p dx \right)^{1/p} dz \leq \\ &\leq w(\lambda) \int_{|x| \leq \lambda} |P(t, z)| dz + 2\|g\|_{L^p(\mathbb{R}^n)} \int_{|x| > \lambda} |P(t, z)| dz \leq \\ &\leq \psi(t, 0)\sqrt{H(0)}w(\lambda) + 2C\|g\|_{L^p(\mathbb{R}^n)} \frac{|t|}{\lambda} \end{aligned}$$

where  $\lambda$  is any positive number and  $w(\lambda) = \text{Sup}_{|z| \leq \lambda} \left( \int_{\mathbb{R}^n} |g(x-z) - g(x)|^p dx \right)^{1/p}$  tends to 0 as  $\lambda$  tends to  $0 +$ : hence ii) follows at once.

PROOF OF LEMMA 6. The analyticity of  $v$  is a consequence of the estimates

$$(48) \quad |D^\gamma P(t, x)| \leq [(\gamma + 1)!] C^{|\gamma|+2} (t^2 + |x|^2)^{\frac{-n-|\gamma|}{2}} \exp[-\theta\delta(t^2 + x^2)^{1/2}]$$

where  $\delta$  is defined by (33) and  $0 < \theta < 1$ : they follow from (42) and lemma 4. Clearly  $v$  is a solution of the equation  $Lv(t, x) = 0 \quad t > 0$ .

For the proof of i) and ii) it is necessary to observe that (46) asserts that  $\frac{\partial P}{\partial t}(t, \cdot), \frac{\partial P}{\partial x_j}(t, \cdot)$  belong to  $L^1(\mathbb{R}^n)$  ( $j = 1, \dots, n$ ) for all  $t \neq 0$ . Moreover

$$(49) \quad \int_{\mathbb{R}^n} \frac{\partial P}{\partial x_j}(t, x) dx = 0 \quad \text{for all } t \neq 0 \quad (j = 1, \dots, n).$$



For, from the equation

$$\left[ \mathcal{F}_x \frac{\partial P}{\partial x_j}(t, \cdot) \right] (\xi) = i \xi_j [\mathcal{F}_x P(t, \cdot)] (\xi) = i \xi_j \psi(t, \xi) \sqrt{\overline{H(\xi)}} \operatorname{sgnt}$$

(49) follows, substituting  $\xi = 0$ . Hence

$$(50) \quad \frac{\partial v}{\partial x_j}(t, x) = \int_{\mathbb{R}^n} \frac{\partial P}{\partial z_j}(t, z) g(x - z) dz = \int_{\mathbb{R}^n} \frac{\partial P}{\partial z_j}(t, z) [g(x - z) - g(x)] dz$$

( $j = 1, \dots, n$ )

From the identity

$$v(t, x) = \psi(t, 0) \sqrt{\overline{H(0)}} g(x) + \int_{\mathbb{R}^n} P(t, z) [g(x - z) - g(x)] dz$$

we get

$$(51) \quad \frac{\partial v}{\partial t}(t, x) = \frac{\partial \psi}{\partial t}(t, 0) \sqrt{\overline{H(0)}} g(x) + \int_{\mathbb{R}^n} \frac{\partial P}{\partial t}(t, z) [g(x - z) - g(x)] dz$$

From (48), (49), (50) and Minkowski's inequality we infer that

$$(52) \quad \left\| \frac{\partial v}{\partial x_j}(t, \cdot) \right\|_{L^p(\mathbb{R}^n)} \leq C' \int_{\mathbb{R}^n} \frac{L(z)}{(t^2 + |z|^2)^{\frac{n+1}{2}}} dz$$

$$(53) \quad \left\| \frac{\partial v}{\partial t}(t, \cdot) \right\|_{L^p(\mathbb{R}^n)} \leq \left| \frac{\partial \psi}{\partial t}(t, 0) \right| \sqrt{\overline{H(0)}} \|g\|_{L^p(\mathbb{R}^n)} + C' \int_{\mathbb{R}^n} \frac{L(z)}{(t^2 + |z|^2)^{\frac{n+1}{2}}} dz$$

where  $L(z) = \left( \int_{\mathbb{R}^n} |g(x - z) - g(x)|^p dx \right)^{1/p}$  and  $C'$  is a constant.

Now we show that

$$(54) \quad I = \left[ \int_0^{+\infty} t^{p(1-s)-1} \left( \int_{\mathbb{R}^n} \frac{L(z)}{(t^2 + z^2)^{\frac{n-1}{2}}} dz \right)^p dt \right]^{1/p} \leq C'' \|g\|_{W^{s,p}(\mathbb{R}^n)}$$

where  $C''$  is a constant depending only on  $n, p, s$ . For, 'Minkowski and

Hölder's inequalities yield the following chain of inequalities

$$\begin{aligned}
 I &= \left[ \int_0^{+\infty} \left( \int_{\mathbb{R}^n} t^{-s-\frac{1}{p}} \frac{L(tz)}{(1+z^2)^{\frac{n+1}{2}}} dz \right)^p dt \right]^{1/p} \leq \\
 &\leq \int_{\mathbb{R}^n} \left( \int_0^{+\infty} t^{-sp-1} L^p(tz) dt \right)^{1/p} \frac{dz}{(1+|z|^2)^{\frac{n+1}{2}}} \leq \\
 &\leq \int_{\mathbb{R}^n} \left( \int_0^{+\infty} r^{-sp-1} L^p\left(r \frac{z}{|z|}\right) dr \right)^{1/p} \frac{|z|^s}{(1+|z|^2)^{\frac{n+1}{2}}} dz = \\
 &= \int_0^{+\infty} \frac{\varrho^{s+n-1}}{(1+\varrho^2)^{\frac{n+1}{2}}} d\varrho \int_{|\zeta|=1} \left( \int_0^{+\infty} r^{-sp-1} L^p(r\zeta) dr \right)^{1/p} \mu(d\zeta) \leq \\
 &\leq \omega_n^{1-\frac{1}{p}} \left( \int_0^{+\infty} \frac{\varrho^{s+n-1}}{(1+\varrho^2)^{\frac{n+1}{2}}} d\varrho \right) \cdot \left( \int_{|\zeta|=1} \mu(d\zeta) \int_0^{+\infty} r^{-sp-1} L^p(r\zeta) dr \right)^{1/p} = \\
 &= C'' \left( \int_{\mathbb{R}^n} \frac{L^p(z)}{|z|^{n+sp}} dz \right)^{1/p} = C'' |g|_{W^{s,p}(\mathbb{R}^n)}
 \end{aligned}$$

where  $\mu(d\zeta)$  denotes the Lebesgue measure on  $|\zeta|=1$  and

$$C'' = \omega_n^{1-\frac{1}{p}} \int_{\mathbb{R}^n} \frac{\varrho^{s+n-1}}{(1+\varrho^2)^{\frac{n+1}{2}}} d\varrho.$$

Now, ii) is an immediate consequence of (52) and (54) with  $C_2 = C''C'$ , while i) follows from Minkowski's inequality with measure  $t^{p(1-s)-1} dt$ , applied to (51) and from the equation

$$\begin{aligned}
 \left( \int_0^{+\infty} t^{p(1-s)-1} \left| \frac{\partial \psi}{\partial t}(t, 0) \sqrt{H(0)} \right|^p dt \right)^{1/p} &= \left( \int_0^{+\infty} t^{p(1-s)-1} \exp(tpz_1(0)) dt \right)^{1/p} |z_1(0)| = \\
 &= p^{s-1} \Gamma(p-ps) |z_1(0)|^s.
 \end{aligned}$$

The constant  $C_1$  is given by  $\max [p^{s-1} \Gamma(p-ps), C_2]$ .

4. The Neumann problem.

Consider the function

$$(55) \quad N(t, x) = - \int_t^{+\infty} P(s, x) ds.$$

Since  $P(t, x) > 0$ , if  $t > 0$ , it follows that

$$(56) \quad N(t, x) < 0 \quad t > 0$$

Moreover, the following estimate holds :

$$(57) \quad |N(t, x)| \leq C_\theta (t^2 + |x|^2)^{-\frac{n-1}{2}} \exp [-\theta\alpha (t^2 + |x|^2)^{1/2}] \quad t \geq 0$$

where  $\alpha$  is defined by (18),  $0 < \theta < 1$  and  $C_\theta$  is a constant. For, (45) implies that, for every  $0 < \theta < 1$ , there exists a constant  $C'_\theta$  such that

$$|P(t, x)| \leq C'_\theta \frac{t}{(t^2 + |x|^2)^{\frac{n+1}{2}}} \exp [-\theta\alpha (t^2 + |x|^2)^{1/2}] \quad t > 0$$

Hence

$$\begin{aligned} |N(t, x)| &\leq C'_\theta \int_t^{+\infty} \frac{s}{(s^2 + |x|^2)^{\frac{n+1}{2}}} \exp [-\theta\alpha (s^2 + |x|^2)^{1/2}] ds \leq \\ &\leq C'_\theta \exp [-\theta\alpha (t^2 + |x|^2)^{1/2}] \int_t^{+\infty} \frac{s}{(s^2 + |x|^2)^{\frac{n+1}{2}}} ds = \\ &= \frac{C'_\theta}{n-1} (t^2 + |x|^2)^{\frac{1-n}{2}} \exp [-\theta\alpha (t^2 + |x|^2)^{1/2}] \end{aligned}$$

From (57) one recognizes easily  $N \in L^1(\mathbb{R}_+^{n+1})$  and  $N(t, \cdot) \in L^1(\mathbb{R}^n)$  for all  $t > 0$ . Moreover, the Fourier transform of  $N(t, \cdot)$  is given by

$$(58) \quad [\mathcal{F}_x N(t, \cdot)](\xi) = \frac{\exp(tz_1(\xi))}{z_1(\xi)} \quad t \geq 0$$

This follows from (46) and the fact that  $P \in L^1(\mathbb{R}^{n+1})$ .

We remark also that, for (48),  $N$  is a real analytic function, that satisfies the equation  $LN = 0$  in  $\mathbb{R}_+^{n+1}$ .

In particular, if  $L = \sum_{r,j=1}^n a_{rj} \frac{\partial^2}{\partial x_r \partial x_j} + \frac{\partial^2}{\partial t^2} + \sum_{j=1}^n a_j \frac{\partial}{\partial x_j} - h^2$ , (i. e., if the coefficients  $a_{0j}$  ( $j = 1 \dots n$ ) and  $a_0$  vanish), then  $P$  coincides with  $2 \frac{\partial E}{\partial t}$  and, hence,  $N$  with  $2E$ .

Consider the convolution

$$(59) \quad u(t, x) = \int_{R^n} N(t, x - z) g(z) dz \quad t > 0$$

The following lemma holds:

**LEMMA 8.** *If  $g \in W^{1-\frac{1}{p}, p}(R^n)$ , then*

- i)  $u \in W^{2, p}(R_+^{n+1})$ ;
- ii)  $\|u\|_{W^{2, p}(R_+^{n+1})} \leq C \|g\|_{W^{1-\frac{1}{p}, p}(R^n)}$ ,

where  $C$  is a constant independent of  $g$ :

- iii)  $u(0+, \cdot) = N(0, \cdot) * g$ ;
- iv)  $\frac{\partial u}{\partial t}(0+, \cdot) = g$

Moreover,  $u$  is an analytic function in the half-space  $R_+^{n+1}$  and is a solution of the equation  $Lu = 0$ .

**PROOF.** The analyticity of  $u$  follows, as before, from estimates for the kernel: they are

$$(60) \quad |D^\gamma N(t, x)| \leq [(|\gamma| + 1)!] C_\theta^{|\gamma|+2} (t^2 + |x|^2)^{\frac{1-n-|\gamma|}{2}} \exp[-\theta\delta(t^2 + |x|^2)^{1/2}] \text{ if } |\gamma| \geq 1$$

where  $\delta$  is defined by (33) and  $0 < \theta < 1$ . For the derivation of (60) we use the formula

$$D^\gamma N(t, x) = \frac{\partial^{|\gamma|} N}{\partial t^{\gamma_0} \partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}(t, x) = \int_0^{+\infty} \frac{\partial^{|\gamma|} P}{\partial s^{\gamma_0} \partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}}(s, x) ds$$

and estimates (48) for the derivatives of  $P$ .

Clearly  $Lu(t, x) = 0$  if  $t > 0$ .

To show i) it suffices, for the usual reasons of density, to suppose  $g \in C_0^\infty(\mathbb{R}^n)$ . Young's inequality and (58) imply

$$\|u(t, \cdot)\|_{L^p(\mathbb{R}^n)} \leq \|N(t, \cdot)\|_{L^1(\mathbb{R}^n)} \|g\|_{L^p(\mathbb{R}^n)} = \frac{\exp(tz_1(0))}{|z_1(0)|} \|g\|_{L^p(\mathbb{R}^n)};$$

hence

$$(61) \quad \|u\|_{L^p(\mathbb{R}_+^{n+1})} \leq p^{-1/p} |z_1(0)|^{-1-\frac{1}{p}} \|g\|_{L^p(\mathbb{R}^n)}$$

In order to prove that the second derivatives of  $u$  with respect to  $x$  are in  $L^p(\mathbb{R}_+^{n+1})$ , we use the equation

$$(62) \quad \frac{\partial^2 u}{\partial x_r \partial x_j} = (\mathcal{F}_x^{-1} \varphi_j \mathcal{F}_x) \frac{\partial v}{\partial x_r} \quad (r, j = 1, 2, \dots, n),$$

where  $v$  is defined by (47) and

$$\varphi_j(\xi) = \frac{i\xi_j}{z_1(\xi)} \quad (j = 1, 2, \dots, n).$$

(62) is a consequence of (46), (58) and i) in lemma 5: for

$$\frac{\partial^2 u}{\partial x_r \partial x_j} = -\mathcal{F}_x^{-1}[\xi_r \xi_j \mathcal{F}_x N \mathcal{F}_x g] = \mathcal{F}_x^{-1}[i \xi_r \varphi_j \mathcal{F}_x P \mathcal{F}_x g] = (\mathcal{F}_x^{-1} \varphi_j \mathcal{F}_x) \frac{\partial v}{\partial x_r}$$

Since the functions  $\varphi_j$  are multipliers of type  $(p, p)$  for every  $1 < p < +\infty$  (see proposition 2 in appendix) and the functions  $\frac{\partial v}{\partial x_r}$ , for ii) in lemma 6 with  $s = 1 - \frac{1}{p}$ , satisfy the inequality

$$\left\| \frac{\partial v}{\partial x_r} \right\|_{L^p(\mathbb{R}_+^{n+1})} \leq C_2 \|g\|_{W^{1-\frac{1}{p}, p}(\mathbb{R}^n)} \quad (r = 1, 2, \dots, n),$$

then, there exists a constant  $C'$  such that

$$(63) \quad \left\| \frac{\partial^2 u}{\partial x_r \partial x_j} \right\|_{L^p(\mathbb{R}_+^{n+1})} \leq C' \|g\|_{W^{1-\frac{1}{p}, p}(\mathbb{R}^n)}.$$

Since

$$(64) \quad \frac{\partial u}{\partial t}(t, x) = \int_{\mathbb{R}^n} \frac{\partial N}{\partial t}(t, x - z) g(z) dz = \int_{\mathbb{R}^n} P(t, x - z) g(z) dz = v(t, x),$$

lemma 6, with  $s = 1 - \frac{1}{p}$ , implies

$$(65) \quad \left\| \frac{\partial^2 u}{\partial t^2} \right\|_{L^p(\mathbb{R}_+^{n+1})} \leq C_1 \left[ |z_1(0)|^{1-\frac{1}{p}} \|g\|_{L^p(\mathbb{R}^n)} + \|g\|_{W^{1-\frac{1}{p}, p}(\mathbb{R}^n)} \right]$$

$$(66) \quad \left\| \frac{\partial^2 u}{\partial x_j \partial t} \right\|_{L^p(\mathbb{R}_+^{n+1})} \leq C_2 \|g\|_{W^{1-\frac{1}{p}, p}(\mathbb{R}^n)}.$$

Therefore, all the derivatives of  $u$  are in  $L^p(\mathbb{R}_+^{n+1})$ : moreover, the inequalities (61), (63), (65), (66) show that  $u \in W^{2,p}(\mathbb{R}_+^{n+1})$  and there exists a constant  $C$  independent of  $g$  such that

$$\|u\|_{W^{2,p}(\mathbb{R}_+^{n+1})} \leq C \|g\|_{W^{1-\frac{1}{p}, p}(\mathbb{R}^n)}.$$

Thus, also ii) is proved.

iv) is an immediate consequence of (64) and remark 4 after lemma 6, while iii) follows from the property

$$\|N(t, \cdot) - N(0, \cdot)\|_{L^1(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } t \rightarrow 0 +$$

that implies

$$\|N(t, \cdot) * g - N(0, \cdot) * g\|_{L^p(\mathbb{R}^n)} \rightarrow 0 \quad \text{as } t \rightarrow 0 +.$$

Hence the trace of  $N(t, \cdot) * g$  must coincide with  $N(0, \cdot) * g$ .

At last we state the following

LEMMA 9. If  $(f, g) \in L^p(\mathbb{R}_+^{n+1}) \times W^{1-\frac{1}{p}, p}(\mathbb{R}^n)$  ( $1 < p < +\infty$ ), the Neumann problem

$$(67) \quad \begin{cases} Lu = f \\ u \in W^{2,p}(\mathbb{R}_+^{n+1}) \\ \frac{\partial u}{\partial t}(0+, \cdot) = g \end{cases}$$

$L$  being defined by (6), admits a unique solution  $u$  given by

$$(68) \quad u(t, x) = \int_{\mathbb{R}^n} N(t, x - z) \left[ g(z) - \frac{\partial w}{\partial t}(0, z) \right] dz + w(t, x)$$

where

$$w(t, x) = \int_{R_+^{n+1}} E(t - s, x - z) f(s, z) ds dz$$

and  $\frac{\partial w}{\partial t}(0, \cdot)$  stands for the trace of  $\frac{\partial w}{\partial t}$ .

PROOF. Lemma 2 and remark 3 show that  $w \in W^{2,p}(R_+^{n+1})$  and satisfies the equation  $Lw = f$ . Hence, for lemma 8,  $u \in W^{2,p}(R_+^{n+1})$  and satisfies the equation  $Lu = f$  and the condition  $\frac{\partial u}{\partial t}(0 +, \cdot) = g$ .

Now we prove that  $u$  is the unique solution of (67). Let  $u$  be an arbitrary function in  $C_0^\infty(\overline{R_+^{n+1}})$ : put

$$\begin{aligned} Lu &= f \\ \frac{\partial u}{\partial t}(0 +, \cdot) &= g \\ [\mathcal{F}_x u(t, \cdot)](\xi) &= v(t, \xi) \\ [\mathcal{F}_x f(t, \cdot)](\xi) &= \widehat{f}(t, \xi) \\ \mathcal{F}_x g(\xi) &= \widehat{g}(\xi). \end{aligned}$$

Then  $v$  is a solution of the problem

$$(70) \quad \begin{cases} a_{00} \frac{\partial^2 v}{\partial t^2} + \alpha(\xi) \frac{\partial v}{\partial t} - \beta(\xi) v = \widehat{f}(t, \xi) \\ \frac{\partial v}{\partial t}(0, \xi) = \widehat{g}(\xi) \end{cases}$$

where  $\alpha(\xi)$  and  $\beta(\xi)$  are defined respectively by (25) and (26).

Since  $v \in L^2(R_+^{n+1})$ , from (70) we infer that  $v$  can be represented as follows:

$$v(t, \xi) = c(\xi) \exp(tz_1(\xi)) - \int_0^{+\infty} \psi(t - s, \xi) \widehat{f}(s, \xi) ds,$$

where  $z_1(\xi)$  is the root with real negative part of equation (23),  $\psi(t, \xi)$  is defined by (22) and  $c(\xi)$  is a suitable  $\xi$  — function, that is determined by

imposing the condition  $\frac{\partial v}{\partial t}(0, \xi) = \widehat{g}(\xi)$ . We get

$$c(\xi) = \frac{\widehat{g}(\xi)}{z_1(\xi)} + \frac{1}{z_1(\xi)} \int_0^{+\infty} \frac{\partial \psi}{\partial t}(-s, \xi) \widehat{f}(s, \xi) ds :$$

hence

$$(71) \quad v(t, \xi) = \frac{\exp(tz_1(\xi))}{z_1(\xi)} \widehat{g}(\xi) + \frac{\exp(tz_1(\xi))}{z_1(\xi)} \int_0^{+\infty} \frac{\partial \psi}{\partial t}(-s, \xi) \widehat{f}(s, \xi) ds + \\ - \int_0^{+\infty} \psi(t-s, \xi) \widehat{f}(s, \xi) ds.$$

From the formula

$$\left[ \mathcal{F}_x \frac{\partial E}{\partial t}(t, \cdot) \right](\xi) = - \frac{\partial \psi}{\partial t}(t, \xi),$$

that is a consequence of lemmas 3 and 4, we infer

$$(72) \quad \int_0^{+\infty} \frac{\partial \psi}{\partial t}(-s, \xi) \widehat{f}(s, \xi) ds = \\ = - \int_0^{+\infty} ds \int_{\mathbb{R}^n} \exp[-i(x, \xi)] dx \int_{\mathbb{R}^n} \frac{\partial E}{\partial t}(-s, x-z) f(s, z) dz = \\ = - \int_{\mathbb{R}^n} \exp[-i(x, \xi)] dx \int_{\mathbb{R}_+^{n+1}} \frac{\partial E}{\partial t}(-s, x-z) f(s, z) ds dz = \\ = - \int_{\mathbb{R}^n} \exp[-i(x, \xi)] \frac{\partial w}{\partial t}(0, x) dx$$

$w$  being defined by (69).

Moreover, for (22)

$$(73) \quad \int_0^{+\infty} \psi(t-s, \xi) \widehat{f}(s, \xi) ds =$$



$$\begin{aligned} &= - \int_0^{+\infty} ds \int_{R^n} \exp[-i(x, \xi)] dx \int_{R^n} E(t-s, x-z) f(s, z) dz = \\ &= - \int_{R^n} \exp[-i(x, \xi)] w(t, x) dx. \end{aligned}$$

(58), (71), (72), (73) imply that  $u$  is of the form (68): since  $C_0^\infty(\overline{R_+^{n+1}})$  is dense in  $W^{2,p}(R_+^{n+1})$  the uniqueness is proved.

**5. Proof of the theorem.**

The proof of the theorem follows easily from lemmas 10 and 11 stated below.

Notations: we denote by  $E^+, P^+, N^+$ , respectively the fundamental solution and the Poisson kernels related with the operator  $L^+$  in (1). The functions  $E^-$  and  $P^-$  are analogously defined with regard to  $L^-$  in (1); while  $N^-$  is defined as follows:

$$(74) \quad N^-(t, x) = - \int_{-\infty}^t P^-(s, x) ds.$$

Moreover, the functions  $\alpha^\pm, \beta^\pm, H^\pm$  are connected with  $L^\pm$ , according to formulas (24), (25), (26);  $z_1^+$  and  $z_2^-$  denote the roots of the equations

$$a_{00}^\pm z^2 + \alpha^\pm(\xi)z - \beta^\pm(\xi) = 0$$

respectively with negative and positive real parts.

LEMMA 10. *Let  $u \in W^{2,p}(R^{n+1})$  ( $1 < p < +\infty$ ) be a solution of (1). The assertions stated below are true:*

i) *the following representation formula holds:*

$$(75) \quad u(t, x) = \begin{cases} \int_{R^n} N^+(t, x-z) \left[ g(z) - \frac{\partial w^+}{\partial t}(0, z) \right] dz + w^+(t, x) & t > 0 \\ \int_{R^n} N^-(t, x-z) \left[ g(z) - \frac{\partial w^-}{\partial t}(0, z) \right] dz + w^-(t, x) & t < 0 \end{cases}$$

where  $g$  is the trace of the normal derivative of  $u$  on the hyperplane  $t = 0$

and

$$(76) \quad w^\pm(t, x) = \int_{R_\pm^{n+1}} E^\pm(t - s, x - z) f(s, z) ds dz;$$

ii)  $g = \frac{\partial u}{\partial t}(0, \cdot)$  is a solution belonging to  $W^{1-\frac{1}{p}, p}(R^n)$  of the integral equation

$$(77) \quad \int_{R^n} N(x - z) g(z) dz = (Uf)(x)$$

where

$$(78) \quad N(x) = N^-(0, x) - N^+(0, x)$$

and

$$(79) \quad (Uf)(x) = \int_{R^n} N^-(0, z) \frac{\partial w^-}{\partial t}(0, z) dz - \int_{R^n} N^+(0, z) \frac{\partial w^+}{\partial t}(0, z) dz - w^-(0, z) + w^+(0, z).$$

Vice versa, if there exists  $g \in W^{1-\frac{1}{p}, p}(R^n)$  ( $1 < p < +\infty$ ) that satisfies equation (77), then the function defined by (75) belongs to  $W^{2, p}(R^{n+1})$  and is a solution of (1).

The operator  $U$ , defined by (79), is bounded from  $L^p(R^{n+1})$  into  $W^{2-\frac{1}{p}, p}(R^n)$  ( $1 < p < +\infty$ ).

LEMMA 11. The integral equation

$$(80) \quad \int_{R^n} N(x - z) g(z) dz = f(x),$$

where  $N$  is defined by (78) and  $f$  is any given function in  $W^{2-\frac{1}{p}, p}(R^n)$ , admits a unique solution  $g \in W^{1-\frac{1}{p}, p}(R^n)$  ( $1 < p < +\infty$ ). Moreover,  $g$  verifies the inequality

$$(81) \quad \|g\|_{W^{1-\frac{1}{p}, p}(R^n)} \leq C \|f\|_{W^{2-\frac{1}{p}, p}(R^n)},$$

$C$  being a constant independent of  $f$ .

PROOF OF LEMMA 10. Let  $u$  be a solution of (1): we denote by  $u^+$  and  $u^-$  respectively its restrictions to  $R_+^{n+1}$  and  $R_-^{n+1}$ . Then  $u^+$  and  $u^-$  are solutions of the Neumann problems

$$(82) \quad \begin{cases} L^+ u^+ = f \\ u^+ \in W^{2,p}(R_+^{n+1}) \\ \frac{\partial u^+}{\partial t}(0+, \cdot) = g \end{cases} \quad \begin{cases} L^- u^- = f \\ u^- \in W^{2,p}(R_-^{n+1}) \\ \frac{\partial u^-}{\partial t}(0-, \cdot) = g \end{cases}$$

Lemma 9, applied to  $u^+$  and  $u^-$ , implies that  $u$  can be represented as in (75). Moreover, from the equation  $u^+(0+, \cdot) = u^-(0-, \cdot)$ , (79) and iii) in lemma 8, it follows easily that  $g$  is a solution of (77).

Vice versa, if  $g \in W^{1-\frac{1}{p},p}(R^n)$  is a solution of (77) and we denote as before by  $u^+$  and  $u^-$  the restrictions to  $R_+^{n+1}$  and  $R_-^{n+1}$  of the function  $u$  defined by (75), lemma 9 implies that  $u^+$  and  $u^-$  are solutions of problems (82). Consequently,  $u$  satisfies equation (1): it remains to show that  $u \in W^{2,p}(R^{n+1})$ . This property follows from (82) and equations  $\frac{\partial u^+}{\partial t}(0+, \cdot) = \frac{\partial u^-}{\partial t}(0-, \cdot)$ ,  $u^+(0+, \cdot) = u^-(0-, \cdot)$ : the former is an immediate consequence of iv) in lemma 8, while the latter is nothing else but a rearrangement of (77).

PROOF OF LEMMA 11. The existence of a solution of equation (80) belonging to  $W^{1-\frac{1}{p},p}(R^n)$  follows from the property :

i) the operator  $G$ , inverse of the convolution with kernel  $N$ , is a bounded operator from  $W^{2-\frac{1}{p},p}(R^n)$  into  $W^{1-\frac{1}{p},p}(R^n)$ .  $G$  is defined for  $f \in C_0^\infty(R^n)$  by the equation

$$(83) \quad (Gf)(x) = \frac{1}{(2\pi)^n} \int_{R^n} \exp[i(x, \xi)] \frac{\widehat{f}(\xi)}{\widehat{N}(\xi)} d\xi,$$

where  $\widehat{f}$  and  $\widehat{N}$  denote the Fourier transforms of  $f$  and  $N$ .

The uniqueness of the solution is an obvious consequence of property i) and the following one:

ii) the convolution with kernel  $N$  is a bounded operator from  $W^{1-\frac{1}{p},p}(R^n)$  into  $W^{2-\frac{1}{p},p}(R^n)$ .

Both i) and ii) can be shown by interpolation, using a theorem of Hörmander-Mihlin and well-known properties of interpolation spaces  $W^{s,p}(\mathbb{R}^n)$  ( $1 < s < 2$ ). For the sake of brevity we prove i) only.

The quoted interpolation property (see, for instance, [4], p. 399, [5], chap. VII, § 2, n. 4 or [7] theor. 2.1, 2.8) follows from the assertions :

iii) the operator  $G$ , defined by (83), can be extended with a bounded operator from  $W^{1,p}(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$ ;

iv) the operator  $G$  can be extended with a bounded operator from  $W^{2,p}(\mathbb{R}^n)$  into  $W^{1,p}(\mathbb{R}^n)$ .

Clearly, iv) follows from iii), since  $G$  commutes with differentiations. Then we focus our attention on iii). Observe that from the equations

$$\widehat{N} = \widehat{N}^-(0, \cdot) - \widehat{N}^+(0, \cdot)$$

$$\widehat{N}^-(0, \xi) = \frac{1}{z_2^-(\xi)}$$

$$\widehat{N}^+(0, \xi) = \frac{1}{z_1^+(\xi)}$$

it follows that

$$\widehat{N}(\xi) = \frac{1}{z_2^-(\xi)} - \frac{1}{z_1^+(\xi)}$$

where  $z_1^+$  and  $z_2^-$  are defined at the beginning of this section.

We have

$$(Gf)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp[i(x, \xi)] M(\xi) [(1 - \Delta)^{1/2} f]^\wedge(\xi) d\xi \quad f \in C_0^\infty(\mathbb{R}^n)$$

where

$$M(\xi) = \frac{z_1^+(\xi) z_2^-(\xi)}{[z_1^+(\xi) - z_2^-(\xi)](1 + |\xi|^2)^{1/2}}$$

and

$$(1 - \Delta)^{1/2} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \exp[i(x, \xi)] (1 + |\xi|^2)^{1/2} \widehat{f}(\xi) d\xi \quad f \in C_0^\infty(\mathbb{R}^n).$$

In appendix it is shown that  $M$  verifies the following inequalities (that are consequences of the properties of  $z_1^+$  and  $z_2^-$ ):

$$(84) \quad \text{Sup}_{\xi \in \mathbb{R}^n} |\xi|^{|\gamma|} |D^\gamma M(\xi)| \leq C_\gamma,$$

$\gamma$  being any multi-index and  $C_\gamma$  some constant depending on  $\gamma$ .

For the theorem of Hörmander-Mihlin inequalities (84) imply that  $M$  is a multiplier of type  $(p, p)$  for every  $1 < p < +\infty$ . (For terminology see appendix below).

Moreover, the operator  $(1 - \Delta)^{1/2}$  is bounded from  $W^{1,p}(\mathbb{R}^n)$  into  $L^p(\mathbb{R}^n)$ . This property, that can be easily seen by a further application of the theorem of Hörmander-Mihlin, is a particular case of a theorem of Calderón on spaces of Bessel potentials  $L_s^p(\mathbb{R}^n)$ . Then, property iii) is proved.

Finally, the estimate (81) follows from i).

## APPENDIX

For the convenience of the reader we recall, following [3], the definition of multipliers and some criteria that enable to ascertain whether a given function is a multiplier.

**DEFINITION.**  $M_p^q(\mathbb{R}^n)$  ( $p, q \geq 1$ ) is the set of Fourier transforms  $\widehat{T}$  of all temperate distributions such that

$$\sup_{u \in C_0^\infty(\mathbb{R}^n)} \frac{\|T * u\|_{L^q(\mathbb{R}^n)}}{\|u\|_{L^p(\mathbb{R}^n)}} < +\infty.$$

The elements in  $M_p^q(\mathbb{R}^n)$  are called multipliers of type  $(p, q)$ .

**THEOREM.** ([3], p. 120) *Let  $f \in L^\infty(\mathbb{R}^n)$  and assume that*

$$\frac{1}{r^n} \int_{\frac{r}{2} \leq |\xi| \leq r} |r^{|\gamma|} D^\gamma f(\xi)|^2 d\xi \leq B \quad 0 < r < +\infty, \quad |\gamma| \leq k,$$

where  $B$  is a constant and  $k$  is the least integer  $> \frac{n}{2}$ .

Then  $f$  belongs to  $M_p^p(\mathbb{R}^n)$  for every  $1 < p < +\infty$ .

In particular we shall use the following corollary (Mihlin's Theorem, see [8]).

**COROLLARY 1.** *If  $f \in C^n(\mathbb{R}^n - \{0\})$  and*

$$\sup_{\xi \in \mathbb{R}^n} |\xi|^{|\gamma|} |D^\gamma f(\xi)| \leq B \quad \text{for } |\gamma| \leq n,$$

where  $B$  is a constant, then  $f \in M_p^p(\mathbb{R}^n)$  for every  $1 < p < +\infty$ .

We prove

PROPOSITION 1. Let  $P$  and  $Q$  be two polynomials in  $(\xi, \eta)$  ( $\xi \in R^n, \eta \in R^m, n \geq 1, m \geq 1$ ) of the same degree  $q$ : let  $H(\xi) = (H_1(\xi), \dots, H_m(\xi))$  be a vector in  $R^m$  with components that are polynomials of degree 2 in  $\xi$  and satisfy the following inequalities:

$$i) \quad |H_r(\xi)| \geq C'(1 + |\xi|^2) \quad (r = 1, \dots, m)$$

$$ii) \quad |Q(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})| \geq C''(1 + |\xi|^2)^{q/2},$$

$C'$  and  $C''$  being positive constants.

Then the function

$$R(\xi) = \frac{P(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})}{Q(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})}$$

is a multiplier of type  $(p, p)$  for every  $1 < p < +\infty$ .

PROOF. We observe that for i) and ii)  $R$  is in  $C^\infty(R^n)$ . Moreover

$$\begin{aligned} \frac{\partial R}{\partial \xi_j} &= \left[ \frac{\partial P}{\partial \xi_j} + \frac{1}{2} \sum_{r=1}^m \frac{\partial P}{\partial \eta_r} \frac{1}{H_r^{1/2}} \frac{\partial H_r}{\partial \xi_j} \right] Q^{-1} - \left[ \frac{\partial Q}{\partial \xi_j} + \frac{1}{2} \sum_{r=1}^m \frac{\partial Q}{\partial \eta_r} \frac{1}{H_r^{1/2}} \frac{\partial H_r}{\partial \xi_j} \right] P Q^{-2} = \\ &= \left( \prod_{r=1}^m H_r \right)^{-1} Q^{-2} R_j \end{aligned}$$

where

$$\begin{aligned} R_j &= \left[ \left( \prod_{r=1}^m H_r \right) \frac{\partial P}{\partial \xi_j} + \frac{1}{2} \sum_{r=1}^m \frac{\partial P}{\partial \eta_r} \frac{\partial H_r}{\partial \xi_j} \left( \prod_{s \neq r} H_s \right) H_r^{1/2} \right] Q + \\ &\quad - \left[ \left( \prod_{r=1}^m H_r \right) \frac{\partial Q}{\partial \xi_j} + \frac{1}{2} \sum_{r=1}^m \frac{\partial Q}{\partial \eta_r} \frac{\partial H_r}{\partial \xi_j} \left( \prod_{s \neq r} H_s \right) H_r^{1/2} \right] P \end{aligned}$$

is a polynomial in  $(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})$  of degree less or equal to  $2m + 2q - 1$ .

CLAIM: for all multi-index  $\gamma$  the following equation holds:

$$(A1) \quad D^\gamma R = \left( \prod_{r=1}^m H_r \right)^{-|\gamma|} Q^{-1-|\gamma|} R_\gamma,$$

$R_\gamma$ , being a polynomial in  $(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})$  of degree less or equal to  $|\gamma|(2m + q - 1) + q$ .

The proof proceeds by induction: we suppose that (A1) is true for all  $\gamma$  with  $|\gamma| = r$  and we show that it is valid for all  $\gamma$  with  $|\gamma| = r + 1$ . For,

$$\begin{aligned} \frac{\partial}{\partial \xi_j} D^\gamma R &= -|\gamma| \left( \prod_{r=1}^m H_r \right)^{-1-|\gamma|} Q^{-1-|\gamma|} R_\gamma \sum_{r=1}^m \left( \prod_{s \neq r} H_s \right) \frac{\partial H_r}{\partial \xi_j} + \\ &- (1 + |\gamma|) \left( \prod_{r=1}^m H_r \right)^{-|\gamma|} Q^{-2-|\gamma|} R_\gamma \left[ \frac{\partial Q}{\partial \xi_j} + \frac{1}{2} \sum_{r=1}^m \frac{\partial Q}{\partial \eta_r} \frac{1}{H_r^{1/2}} \frac{\partial H_r}{\partial \xi_j} \right] + \\ &+ \left( \prod_{r=1}^m H_r \right)^{-|\gamma|} Q^{-1-|\gamma|} \left[ \frac{\partial R_\gamma}{\partial \xi_j} + \frac{1}{2} \sum_{r=1}^m \frac{\partial R_\gamma}{\partial \eta_r} \frac{1}{H_r^{1/2}} \frac{\partial H_r}{\partial \xi_j} \right] = \left( \prod_{r=1}^m H_r \right)^{-1-|\gamma|} Q^{-2-|\gamma|} R_{\gamma,j} \end{aligned}$$

where

$$\begin{aligned} R_{\gamma,j} &= -|\gamma| Q R_\gamma \sum_{r=1}^m \left( \prod_{s \neq r} H_s \right) \frac{\partial H_r}{\partial \xi_j} - \\ &- (1 + |\gamma|) R_\gamma \left[ \left( \prod_{r=1}^m H_r \right) \frac{\partial Q}{\partial \xi_j} + \frac{1}{2} \sum_{r=1}^m \frac{\partial Q}{\partial \eta_r} \left( \prod_{s \neq r} H_s \right) H_r^{1/2} \frac{\partial H_r}{\partial \xi_j} \right] + \\ &+ \left( \prod_{r=1}^m H_r \right) Q \frac{\partial R_\gamma}{\partial \xi_j} + \frac{1}{2} Q \sum_{r=1}^m \frac{\partial R_\gamma}{\partial \eta_r} \left( \prod_{s \neq r} H_s \right) H_r^{1/2} \frac{\partial H_r}{\partial \xi_j}. \end{aligned}$$

Clearly  $R_{\gamma,j}$  is a polynomial in  $(\xi, H_1(\xi)^{1/2}, \dots, H_m(\xi)^{1/2})$  of degree less or equal to  $(1 + |\gamma|)(2m + q - 1) + q$  for the hypothesis of induction: the proof of (A1) is fulfilled.

From (A1) and i), ii) and the hypothesis on  $P$  and  $Q$  we infer that there exists a constant  $C_\gamma$  such that

$$(A2) \quad |D^\gamma R(\xi)| \leq C_\gamma (1 + |\xi|^2)^{-|\gamma|/2} \quad |\gamma| \geq 0.$$

From (A2) and an application of corollary 1, the assertion follows.

From proposition 1, iii) in remark 1 after lemma 3, (29), the definitions of  $z_1^+$  and  $z_2^-$ , the fact that  $A$  is a symmetric positive definite matrix and  $\alpha^\pm(\xi)$ , defined as in (25), are linear functions in  $\xi$ , there follows easily:

PROPOSITION 2. The following functions are in  $M_p^p(\mathbb{R}^n)$  for every  $1 < p < +\infty$ :

$$\varphi_{rj}(\xi) = \frac{\xi_r \xi_j}{(A(\xi, \xi) - i(a, \xi) + \hbar^2)} \quad (r, j = 1, \dots, n)$$

$$\varphi_j^+(\xi) = \frac{i \xi_j}{z_j^+(\xi)} \quad (j = 1, \dots, n)$$

$$\varphi_j^-(\xi) = \frac{i \xi_j}{z_j^-(\xi)} \quad (j = 1, \dots, n)$$

$$M(\xi) = \frac{z_1^+(\xi) z_2^-(\xi)}{[z_1^+(\xi) - z_2^-(\xi)] (1 + |\xi|^2)^{1/2}}.$$



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