ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

L. FUCHS
G. J. HAUPTFLEISCH

Note on pure-projectivity of modules

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 27, nº 2 (1973), p. 339-344
<http://www.numdam.org/item?id=ASNSP_1973_3_27_2_339_0>

NOTE ON PURE-PROJECTIVITY OF MODULES

by L. Fuchs and G. J. Hauptfleisch

1. Our aim in this note is to generalize the following well-known theorem, due to Pontryagin (see e. g. [4]), from abelian groups to modules:

Pontryagin's Theorem: *A countable torsion-free abelian group is free if every subgroup of finite rank is free.*

A slightly different, but equivalent form is more suitable for our purpose (ω will denote the first infinite ordinal):

If A is a torsion-free abelian group such that $A = \bigcup_{n<\omega} A_n$ where $0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ is an ascending chain of finitely generated and pure subgroups A_n of A, then A is free.

One can try to generalize this result to (unital left) modules A over a ring R with 1, by replacing torsion-freeness and freeness by flatness and projectivity, respectively. One naturally expects that an extension of Pontryagin's theorem is possible only if the class of rings is strongly restricted. It was to our surprise to learn that it suffices to replace finite generation by finite presentation in order to obtain a result which holds for modules over arbitrary rings:

Suppose A is a flat module over an arbitrary ring such that $A = \bigcup_{n<\omega} A_n$ where $0 = A_0 \subseteq A_1 \subseteq \ldots \subseteq A_n \subseteq \ldots$ is a chain of finitely presented pure submodules of A. Then A is projective.

Moreover, if the hypothesis of flatness is dropped, then this theorem still holds with projectivity replaced by pure-projectivity. Also, it will turn out that this result can be extended to certain non-countably generated modules A where A is the direct limit of finitely presented pure submodules.

Pervenuto alla Redazione il 15 Gennaio 1972.

This paper was written while the first author held a National Science Foundation grant, number GP 29437. The second author gratefully acknowledges support by the South African Council for Scientific and Industrial Research.

2. All modules considered will be unital left modules over an arbitrary but fixed ring R, unless otherwise specified. We follow Cohn [3] in calling a submodule N of a module M *pure* if, for every right R-module P, the map $1 \otimes \iota : P \otimes_R N \longrightarrow P \otimes_R M$ induced by the injection $\iota : N \longrightarrow M$ is monic. A module F is *flat* if $\alpha \otimes 1 : P \otimes_R F \longrightarrow Q \otimes_R F$ is monic for every monomorphism $\alpha : P \longrightarrow Q$ of right R-modules. An exact sequence

$$0 \longrightarrow A \overset{\alpha}{\longrightarrow} B \overset{\beta}{\longrightarrow} C \longrightarrow 0 \tag{1}$$

is called *pure-exact* if Im α is pure in B. If C is flat, then (1) is pure exact. It is straightforward to prove:

LEMMA 1. *If* (1) *is pure-exact and B is flat, then both A and C are flat.*

Following Maranda [7], we shall call a module S *pure projective* if it has the projective property with respect to every pure-exact sequence (1), or, equivalently, if the map $\mathrm{Hom}_R(S, B) \longrightarrow \mathrm{Hom}_R(S, C)$ induced by β is epic. Direct summands of pure-projectives are likewise pure-projective, and all finitely presented modules are pure-projective.

3. We start our discussion with

THEOREM 1. *Let $0 = A_0 \subseteq A_1 \subseteq \dots \subseteq A_n \subseteq \dots$ be an ascending chain of submodules of A such that $A = \bigcup_{n<\omega} A_n$. If each A_n is finitely presented and pure in A, then A is pure-projective.*

PROOF. For every n, A_{n+1}/A_n is finitely presented, since quotients of finitely presented modules mod finitely generated submodules are again finitely presented. Hence A_{n+1}/A_n is pure-projective, and thus the pure-exact sequence

$$0 \longrightarrow A_n \longrightarrow A_{n+1} \longrightarrow A_{n+1}/A_n \longrightarrow 0 \tag{2}$$

splits, for every n. Consequently, $A_{n+1} = A_n \oplus B_n$ for some pure-projective submodule B_n of A_{n+1}. We conclude: $A_{n+1} = \bigoplus_{k=0}^{n} B_k$ and $A = \bigoplus_{k<\omega} B_k$. Therefore, A is pure-projective, in fact.

It is routine to check that the following result is equivalent to Theorem 1:

THEOREM 1′. *Let A be a countably generated module and suppose $A = \varinjlim_I A_i$ where the A_i $(i \in I)$ are finitely presented and pure submodules of A. Then A is pure projective.*

4. The following lemma is needed in the proof of our next theorem.

LEMMA 2. *A finitely generated pure submodule of a projective module is projective.*

PROOF. Without loss of generality, we may assume that we have a finitely generated pure submodule A of a finitely generated free module F. Then $0 \to A \to F \to F/A \to 0$ is a pure-exact sequence where F/A is finitely presented and hence pure-projective.

THEOREM 2. *Let* $A = \bigcup_{n<\omega} A_n$ *be a flat module where* $0 = A_0 \subseteq A_1 \subseteq \dots \dots \subseteq A_n \subseteq \dots$ *is a chain of finitely generated pure submodules of* A. *Then* A *is projective if and only if each* A_n *is finitely presented.*

PROOF. Sufficiency follows at once from Theorem 1 and from the following remark (due to Dr. W. R. Nico): a pure projective flat module A is projective. If $0 \to H \to F \to A \to 0$ is an exact sequence with F free, then the flatness of A implies that it is pure-exact, so it must split because A is pure-projective. Conversely, if A is projective, then every A_n is finitely presented in view of Lemma 2.

Recall that a module is said to be *pseudo coherent* (Bourbaki [2]) if all its finitely generated submodules are finitely presented. (Evidently, modules over Noetherian rings are pseudo-coherent.) If the module A in Theorem 1 or 2 is pseudo-coherent, then the hypothesis that the A_n are finitely generated (rather than finitely presented) suffices to guarantee that A is pure-projective and projective, respectively.

Under certain conditions on R, we can even conclude that A is free in Theorem 2. This is trivially the case if all projective R-modules are free (e. g. if R is local or P. I. D.). Less trivially, if R is left or right Noetherian and A/IA is not finitely generated for any two-sided ideal $I \neq R$ of R, then A must be free (cf. Bass [1]) (1).

It is readily checked that Theorem 2 is equivalent to

THEOREM 2′. *Let* A *be a countably generated flat module, and suppose* $A = \lim_I A_i$ *where* $A_i \, (i \in I)$ *are finitely generated pure submodules of* A. *Then* A *is projective exactly if each* A_i *is finitely presented.*

---

(1) KULKARNI [6] proved that a torsion-free module of countably infinite rank over a Dedekind domain is free whenever its finite rank submodules are finitely generated.

5. If we drop the hypothesis that A is countably generated, then A in Theorem 2′ need not be projective. This can be illustrated by the direct product of a countable set of infinite cyclic groups. In this group all pure subgroups of finite rank are free, hence the group is the direct limit of finitely generated, pure free subgroups, but the group itself fails to be free (see e. g. [4]). Therefore, in order to retain the projectivity of A in Theorem 2, while passing from countably generated modules A to larger modules, it is necessary to impose additional conditions on A. A suitable condition is the existence of a certain well-ordered ascending chain of submodules with A as union which can be chosen to be the trivial chain $0 = A_0 \subset \subset A_1 = A$ whenever A is countably generated.

We now prove our main result (2).

Theorem 3. *Let A be a module such that*

$$0 = C_0 \subseteq C_1 \subseteq \dots \subseteq C_\alpha \subseteq \dots \subseteq C_\mu = A$$

is a well-ordered ascending chain of submodules C_α in A where $C_\beta = \bigcup_{\alpha<\beta} C_\alpha$ for limit ordinals β. Furthermore suppose that

$$A = \varinjlim_I A_i$$

where A_i are submodules of A. If, for all α and i,

(i) *$C_{\alpha+1}/C_\alpha$ is countably generated;*

(ii) *A_i is finitely presented;*

(iii) *$A_i + C_\alpha$ is pure in A;*

(iv) *$A_i \cap C_\alpha$ is finitely generated and pure in A,*

then A is pure-projective.

Proof. It is readily verified that the purity of $A_i + C_\alpha$ and $A_i \cap C_\alpha$ implies that A_i and C_α are pure in A. To establish the pure-projectivity of $C_{\alpha+1}/C_\alpha$, observe that

$$0 \to A_i \cap C_{\alpha+1} \to A_i \to A_i/(A_i \cap C_{\alpha+1}) \to 0$$

(2) Let us note that Hill [5] proved a different generalization of Pontryagin's theorem to abelian groups.

is pure-exact with $A_i/(A_i \cap C_{\alpha+1})$ finitely presented, thus the sequence splits: $A_i = (A_i \cap C_{\alpha+1}) \oplus D$ where D is finitely presented. We claim:

$$A_i + C_\alpha = [(A_i \cap C_{\alpha+1}) + C_\alpha] \oplus D.$$

It suffices to show that the sum is direct. This follows from

$$[(A_i \cap C_{\alpha+1}) + C_\alpha] \cap D = [(A_i \cap C_{\alpha+1}) + C_\alpha] \cap A_i \cap D =$$

$$= [(A_i \cap C_{\alpha+1}) + (C_\alpha \cap A_i)] \cap D = (A_i \cap C_{\alpha+1}) \cap D = 0.$$

Setting

$$B_{\alpha i} = (A_i \cap C_{\alpha+1}) + C_\alpha ,$$

we see that $B_{\alpha i}$ is a summand of the pure submodule $A_i + C_\alpha$, and hence itself pure. Therefore $B_{\alpha i}/C_\alpha$ is a pure submodule of $C_{\alpha+1}/C_\alpha$. Clearly, $B_{\alpha i}/C_\alpha$ is a summand of $(A_i + C_\alpha)/C_\alpha \cong A_i/(A_i \cap C_\alpha)$ which is finitely presented in view of (ii) and (iv), thus $B_{\alpha i}/C_\alpha$ is finitely presented for every $i \in I$. We obviously have $C_{\alpha+1}/C_\alpha = \lim_I B_{\alpha i}/C_\alpha$, and so $C_{\alpha+1}/C_\alpha$ is pure-projective in view of (i) and Theorem 1′. Proceeding as in the proof of Theorem 1, now transfinitely up to μ, we are led to the pure-projectivity of A.

6. Specializing to flat modules, we obtain:

COROLLARY. *Let the module A in Theorem 3 be flat. Then A is projective.*

In fact, from the proof of Theorem 2 we can infer that a pure-projective flat module is projective.

Let us note that we can slightly improve on Corollary by dropping purity in (iv), but assuming that 0 is one of the A_i's. In this case, namely, (iii) implies that all A_i and C_α are pure in A, all $A_i + C_\alpha$ are flat in A (cf. Lemma 1), and the purity of $A_i \cap C_\alpha$ will then be a consequence of the following lemma.

LEMMA 3. *If A and C are pure submodules of a module M such that $A + C$ is flat, then $A \cap C$ is pnre in M.*

The sequence $0 \to A \to A + C \to (A + C)/A \to 0$ is pure-exact, so $(A + C)/A$ is flat by Lemma 1. Thus $C/(A \cap C)$ is flat. Therefore $A \cap C$ is pure in C and hence in M.

Tulane University, New Orleans, La. and
Rand Afrikaans University, Johannesburg, South Africa

REFERENCES

[1] H. BASS : *Big projective modules are free,* Illinois J. Math. 7 (1963), 24-31.

[2] N. BOURBAKI : *Modules plats, Algèbre commutatif,* Chap. 1 (Paris, 1961).

[3] P. M. COHN : *On the free product of associative rings,* Math. Z. 71 (1959), 380-398.

[4] L. FUCHS : *Infinite abelian groups,* Vol. 1 (New York, 1970).

[5] P. HILL : *On the freeness of abelian groups : A generalization of Pontryagin's theorem,* Bull. Amer. Math. Soc. 76 (1970), 1118-1120.

[6] K. S. KULKARNI : *On a theorem of Jensen,* Amer. Math. Monthly 74 (1967), 960-961.

[7] J. M. MARANDA : *On pure subgroups of abelian groups,* Arch. Math. 11 (1960), 1-13.