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ON THE HOLDER-CONTINUITY OF SOLUTIONS OF A NONLINEAR PARABOLIC VARIATIONAL INEQUALITY

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Parabolic variational inequalities have been studied extensively by Brezis [2], Browder [4], Lions [6], Lions-Stampacchia [7] and others. The existence of a weak solution is shown and when the elliptic operator involved is strongly monotone, the solution is unique.

Using the penalisation method, Lions [6] has shown the regularity of solutions of some linear parabolic inequalities. For nonlinear parabolic inequalities, the regularity of solutions with respect to time has been obtained by Brezis [2] and the regularity with respect to both space and time by the writer in [8].

The purpose of this paper is to show the Holder-continuity of solutions u of:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \sum\limits_{j=1}^{n} \dfrac{\partial}{\partial x_j} (a_j(x, u, Du)) = f \text{ on the region where } u(x, t) \geq 0, \\ u(x, t) = 0 \text{ elsewhere, } \ u(x, t) = 0 \text{ on } \partial G \times [0, T], \ u(x, 0) = 0 \text{ and} \\ \text{« continuity » of } u, \ \partial u / \partial x_j \text{ at the two regions.} \end{cases}$$

Moreover, if $a_j(x, u, Du) = a_j(x) D_j u$ for $j = 1, \ldots, n$, it will be shown that $u \in L^p(0, T; W^{2,p}(G))$ for any p, $2 \leq p < \infty$.

To prove the result, we use Lions' penalisation method, a time discretisation of the penalized equation and a nonlinear singular perturbution of the latter equation.

The notations and the main results of the paper are given in Section 1. Proofs are carried out in Section 2.

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SECTION 1: Let G be a bounded open subset of R_n with a smooth boundary ∂G. Set: $D_j = i^{-1}\,\partial/\partial x_j$, $j = 1, \dots, n$ and for each n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers, we write:

$$D^\alpha \prod_{j=1}^{n} D_j^\alpha j \quad \text{with } |\alpha| = \sum_{j=1}^{n} \alpha_j .$$

$W^{m,p}(G)$ is the real reflexive separable Banach space:

$$W^{m,p}(G) = \{u : u \text{ in } L^p(G),\ D^\alpha u \text{ in } L^p(G),\ |\alpha| \leq m\}$$

with the norm:

$$\| n \|_{m,p} = \{ \sum_{|\alpha| \leq m} \| D^\alpha u \|^p_{L^p(G)} \}^{1/p},\ 2 \leq p < \infty.$$

$W_0^{m,p}(G)$ is the completion of C_0^∞, the family of all infinitely differentiable functions with compact suport in G, in the $\|\cdot\|_{m,p}$ — norm. The pairing between $W_0^{m,p}(G)$ and its dual $W^{-m,q}(G)$ is denoted by $(.,.)$. Set: $H = L^2(G)$ and $\|u\|_s$ is the $L^s(G)$-norm of u for $1 < s < \infty$.

$C^\lambda(G)$ is the space of all Holder-continuous functions of any compact subset of G, with Holder-exponent λ, $0 < \lambda < 1$.

Let $[0, T]$ be a compact interval of the real line E. The derivative of u with respect to t will be denoted by u'.

$L^p(0, T; W^{m,p}(G))$ is the space of all equivalence classes of functions $n(t)$ from $[0, T]$ to $W^{m,p}(G)$ which are L^p-integrale. It is a real reflexive separable Banach space with the norm:

$$\| u \|_{L^p(0,T;\, W^{m,p}(G))} = \left\{ \int_0^T \| u(t) \|^p_{m,p}\, dt \right\}^{1/p}.$$

$C^\lambda(0, T; C^{2\lambda}(G))$ is the space of all Holder-continuous fonctions on any compact subsets of $G \times [0, T]$ with Holder-exponent λ with respect to t and with exponent 2λ with respect to x. $0 < 2\lambda < 1$.

We consider nonlinear partial differential operators on G of the form:

$$A(u) = \sum_{|\alpha| \leq 1} D^\alpha A_\alpha(x, x, Du).$$

ASSUMPTION (I): (i) *Let* $\zeta = \{\zeta_\alpha : |\alpha| \leq 1\}$, *then each* $A_\alpha(x, \zeta)$ *is continuously differentiable in* x *and in* ζ.

(ii) *There exists a positive constant C such that*:

$$| A_\alpha (x, \zeta) | + | D_x A_\alpha (x, \zeta) | + (1 + | \zeta |) \sum_{|\beta| \leq 1} | A_{\alpha\beta}(x, \zeta) | \leq C | \zeta |$$

where $A_{\alpha\beta} = \partial A_\alpha / \partial \zeta_\beta$.

(iii) $$\sum_{|\alpha|, |\beta| \leq 1} A_{\alpha\beta}(x, \zeta) \eta_\alpha \eta_\beta \geq c \sum_{|\alpha| \leq 1} \eta_\alpha^2 .$$

c is a positive constant.

(iv) $$\sum_{|\alpha| \leq 1} A_\alpha (x, \zeta) \zeta_\alpha \geq 0.$$

Let $K = \{u : u \text{ in } L^2(G),\ u \geq 0 \text{ a. e. on } G\}$. It is clear that K is a closed convex subset of both H and $W_0^{1,2}(G)$.

The main results of the paper are the following two theorems.

THEOREM 1: *Let A be an elliptic operator satisfying Assumption (I). Suppose that* $f \in L^\infty(0, T; L^\infty(G))$, $f' \in L^2(0, T; W^{-1,2}(G))$ *with* $f(0) = 0$. *Then there exists a unique solution u in* $L^2(0, T; W_0^{1,2}(G)) \cap L^2(0, T; W^{2,2}(G))$ *with u′ in* $L^2(0, T; W^{1,2}(G))$, $u(t)$ *in K* a. c. *and* $u(0) = 0$ *such that*:

$$\int_0^T (u' + Au - f, v - u)\, dt \leq 0$$

for all v in $L^2(0, T; W^{1,2}(G))$, *v′ in* $L^2(0, T; W^{-1,2}(G))$, $v(t)$ *in K and* $v(0) = 0$.

Moreover: $u \in C^\lambda([0, T]; C^{2\lambda}(\text{clG}))$, $D_j u \in C^\gamma(0, T; C^{2\gamma}(G))$, *for any j, with* $0 < 2\lambda,\ 2\gamma < 1$.

When A is a linear elliptic operator, we have a stronger result.

THEOREM 2: *Let* $Au = \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta}(x) D^\beta u)$ *be a positively strongly uniformly elliptic operator on G with coefficients* $a_{\alpha\beta}(x)$ *in* $C^1(\text{clG})$. *Suppose that* $f \in L^\infty(0, T; L^\infty(G))$, $f' \in L^2(0, T; W^{-1,2}(G))$ *and* $f(0) = 0$. *Then there exists a unique* $u \in L^2(0, T; W_0^{1,2}(G)) \cap L^p(0, T; W^{2,p}(G))$, $1 < p < \infty$, $u(t)$ *in K* a. e., $u' \in L^p(0, T; L^p(G)) \cap L^2(0, T; W^{1,2}(G))$ *and* $u(0) = 0$ *such that*:

$$\int_0^T (u' + Au - f, v - u)\, dt \geq 0$$

for all v in $L^2(0, T; W_0^{1,2}(G))$, *v′ in* $L^2(0, T; W^{-1,2}(G))$, $v(t)$ *in K and* $v(0) = 0$.

Moreover: $u \in C^{\lambda}([0, T]; C^{2\lambda}(\mathrm{cl}G))$, $D_j u \in C^{\gamma}([0, T]; C^{2\gamma}(G))$ *for any* j, *with* $0 < 2\lambda, 2\gamma < 1$.

Theorem 1 is a consequence of Theorem 3 which will be proved in Section 2.

THEOREM 3: *Suppose all the hypotheses of Theorem* 1 *are satisfied. Then for each* ε, $0 < \varepsilon < 1$, *there exists a unique solution* u_ε *of the equation*:

$$\varepsilon(u_\varepsilon' + Au_\varepsilon) - u_\varepsilon^- = \varepsilon f, \; u_\varepsilon(x, t) = 0 \text{ on } \partial G \times [0, T], u_\varepsilon(x, 0) = 0.$$

Moreover

$$\| u_\varepsilon \|_{L^\infty(0, T; L^\infty(G))} + \| u_\varepsilon \|_{L^2(0, T; W_0^{1,2}(G))} + \| u_\varepsilon' \|_{L^2(0, T; W^{1,2}(G))} \leq C.$$

C is a constant independent of ε.

PROOF OF THEOREM 1 USING THEOREM 3: We shall make use of the of the following crucial estimate of Theorem 3:

$$\| u_\varepsilon \|_{L^\infty(0, T; L^\infty(G))} \leq C.$$

1) Since $u_\varepsilon \in L^\infty(0, T; L^\infty(G)) \cap L^2(0, T; W_0^{1,2}(G))$, we have:

$$u_\varepsilon^- \in L^\infty(0, T; L^\infty(G)) \cap L^2(0, T; W_0^{1,2}(G)).$$

Thus: $| u_\varepsilon^- |^{2s-2} u_\varepsilon^- \in L^2(0, T; W_0^{1,2}(G))$ for any positive integer s. It follows from conditions (ii) and (iii) of Assumption (I) that:

$$\int_0^T (Au_\varepsilon, -| u_\varepsilon^- |^{2s-2} u_\varepsilon^-)\, dt = -\int_0^T Au_\varepsilon^-, | u_\varepsilon^- |^{2s-2} u_\varepsilon^-)\, dt \geq 0.$$

Since $u_\varepsilon' \in L^2(0, T; H)$ and $u_\varepsilon^- \in L^\infty(0, T; L^\infty(G))$ with $u_\varepsilon(0) = 0$, we have:

$$2s \int_0^T (u_\varepsilon', -| u_\varepsilon^- |^{2s-2} u_\varepsilon^-)_H\, dt = \int_0^T \int_{\substack{G \\ u_\varepsilon \geq 0}} \frac{d}{dt}(u_\varepsilon^{2s})\, dx\, dt = \int_{\substack{G \\ u_\varepsilon \gtrless 0}} | u_\varepsilon(x, T) |^{2s}\, dx \geq 0.$$

Hence:

$$\varepsilon^{-1} \| u_\varepsilon^- \|^{2s}_{L^{2s}(0, T; L^{2s}(G))} \leq C \| f \|_{L^\infty(0, T; L^\infty(G))} \| u_\varepsilon^- \|^{2s-1}_{L^{2s}(0, T; L^{2s}(G))}.$$

Therefore :

$$\varepsilon^{-1} \| u_\varepsilon^- \|_{L^{2s}(0, T; L^{2s}(G))} \leq C \| f \|_{L^\infty(0, T; L^\infty(G))} .$$

C is a constant independent of s and of ε.

Since u_ε^- lies in $L^\infty(0, T; L^\infty(G))$, we may let $s \to +\infty$ and the above inequality gives:

$$\varepsilon^{-1} \| u_\varepsilon^- \|_{L^\infty(0, T; L^\infty(G))} \leq C \| f \|_{L^\infty(0, T; L^\infty(G))} .$$

2) From the weak compactness oe the unit ball in a reflexive Banach space, we obtain by taking subsequences if necesary: $u_\varepsilon \to u$ weakly in $L^2(0, T; W_0^{1,2}(G))$, $u'_\varepsilon \to u'$ weakly in $L^2(0, T; W^{1,2}(G))$ $Au_\varepsilon \to h$ weakly in $L^2(0, T; W^{-1,2}(G))$, $\varepsilon^{-1} u_\varepsilon^- \to g$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$ and $u_\varepsilon^- \to 0$ in $L^2(0, T; H)$).

Thus: $u' + h + g = f$, $u(0) = 0$ and $u \in K$.

Condition (iii) of Assumption (I) implies that A is monotone. Moreover:

$$\int_0^T (Au_\varepsilon, u_\varepsilon - u)\, dt = \int_0^T (f + \varepsilon^{-1} u_\varepsilon^- - u'_\varepsilon, u_\varepsilon - u)_H\, dt.$$

Aubin's theorem [1] gives:

$$\lim\sup \int_0^T (Au_\varepsilon, u_\varepsilon - u)\, dt \leq 0.$$

By a standard argument of the theory of monotone operatcrs, we get $h = Au$ and

$$\int_0^T (Au, u)\, dt \leq \lim\inf \int_0^T (Au_\varepsilon, u_\varepsilon)\, dt.$$

We have:

$$\int_0^T (u'_\varepsilon + Au_\varepsilon - f - \varepsilon^{-1} u_\varepsilon^-, v - u_\varepsilon)\, dt = 0.$$

Let v be an element of $L^2(0, T; W_0^{1,2}(G))$, v' in $L^2(0, T; W^{-1,2}(G))$, with $v(t)$ in K and $v(0) = 0$. Then $v^- = 0$. and we have:

$$\int_0^T (u'_\varepsilon + Au_\varepsilon - f, v - u_\varepsilon)\, dt \geq 0.$$

Let $\varepsilon \rightarrow 0$ and we get:

$$\int_0^T (u' + Au - f, v - u)\, dt \geq 0.$$

3) So: $u' + Au = f - g \in L^\infty(0, T; L^\infty(G))$, $u(x, t) = 0$ on $\partial G\, x\, [0, T]$ and $u(x, 0) = 0$. It follows from Theorem 6.4 of Ladyzenskaya Solonnikov and Uralceva [5] (page 460), that $u \in C^\lambda[0, T]; C^{2\lambda}(\mathrm{cl}G))$, $D_j u \in C^\gamma(0, T; C^{2\gamma}(G))$. $0 < 2\lambda, 2\gamma < 1$.

4) All the other assertions of Theorem 1 have been proved in [8],

Proof of Theorem 2: From the proof of Theorem 1, we know that there exists a unique u in $L^2(0, T; W_0^{1,2}(G))$ with u' in $L^2(0, T; W^{1,2}(G))$, $u(t)$ in K and $u(0) = 0$ snch that:

$$\int_0^T (u' + Au - f, v - u)\, dt \geq 0$$

for all v in $L^2(0, T; W_0^{1,2}(G))$, v' in $L^2(0, T; W^{-1,2}(G))$, $v(t)$ in K and $v(0) = 0$.

Moreover u satisfies the equation: $u' + Au = f - g \in L^\infty(0, T; L^\infty(G))$. By a well-known result of the theory of linear parabolic equations of order 2 (Cf. e. g. Theorem 9.1 of [5], p. 341-342), we have:

$$u \in L^p(0, T; W^{2,p}(G)),\ u' \in L^p(0, T; L^p(G)) \quad \text{for any } p,\ 1 < p < \infty.$$

Section 2: The proof of Theorem 3 is long and will be carried out in this section. We shall give an outline of the proof before going into the details.

Consider the equation:

$$u_{h\varepsilon}^k - u_{\varepsilon h}^{k-1} + hAu_{h\varepsilon}^k - h\varepsilon^{-1}(u_{\varepsilon h}^k)^- = hf^k,\ u_{\varepsilon h}^0 = 0.$$

It is obtained by a discretisation of the time-variable of the equation of Theorem 3.

Let $A_2 v$ be the nonlinear elliptic operator:

$$A_2 v = \sum_{j=1}^{n} D_j(|D_j v|^{p-2} D_j v) \qquad \text{with } p > n.$$

1) First, we shall consider the equation:

$$u^k_{\varepsilon h\eta} - u^{k-1}_{\varepsilon h\eta} + \eta A_2 u^k_{\varepsilon h\eta} + hAu^k_{\varepsilon h\eta} - \varepsilon^{-1} h(u^k_{\varepsilon h\eta})^- = hf^k,\ u^0_{\varepsilon h\eta} = 0;\ \eta > 0.$$

It has a uniqne solution $u^k_{\varepsilon h\eta}$ in $W_0^{1,p}(G)$ and since $p > n$, $u^k_{\varepsilon h\eta}$ is in $C(\mathrm{clG})$.

2) $\| u^k_{\varepsilon h\eta} \|_{L^\infty(G)} \leq C$. C is a constant independent of ε, h, k and η. Then let $\eta \to 0$.

3) The final step is standard.

Set: $Bv = -v^-$. Denote by $f^k = h^{-1} \int_{kh}^{(k+1)h} f(t)\, dt$ with $h > 0$.

Lemma 1: *Let $h = T/N$ and suppose all the hypotheses of Theorem 3 are satisfied. Then for each k, $1 \leq k \leq N$, there exists a unique solution $u^k_{\varepsilon h\eta}$ in $W_0^{1,p}(G)$ of the equation:*

$$u^k_{\varepsilon h\eta} - u^{k-1}_{\varepsilon h\eta} + \eta h A_2 u^k_{\varepsilon h\eta} + hAu^k_{\varepsilon h\eta} + \varepsilon^{-1} hBu^k_{\varepsilon h\eta} = hf^k,\ u^0_{\varepsilon h\eta} = 0.$$

Moreover:

$$\| u^n_{\varepsilon h\eta} \|_{L^\infty(G)} + \eta h \sum_{k=1}^{n} \| u^k_{\varepsilon h\eta} \|^p_{1,p} + h \sum_{k=1}^{n} \| u^k_{\varepsilon h\eta} \|^2_{1,2} \leq C.$$

C is a constant independent of h, ε, η and n.

Proof: It is clear that $u + \eta hA_2 u + h\varepsilon^{-1} Bu$ is a monotone hemicontinuous, coercive operator mapping bounded sets of $W_0^{1,p}(G)$ into bounded sets of $W^{-1,q}(G)$. It follows from the theory of monotone operators that for each k, there exists a unique solution $u^k_{\varepsilon h\eta}$ of:

$$u^k_{\varepsilon h\eta} - u^{k-1}_{\varepsilon h\eta} + \eta h A_2 u^k_{\varepsilon h\eta} + hAu^k_{\varepsilon h\eta} + h\varepsilon^{-1} Bu^k_{\varepsilon h\eta} = hf^k,\ u^0_{\varepsilon h\eta} = 0, \quad k = 1, \dots, N.$$

So:

$$\frac{1}{2} \| u^k_{\varepsilon h\eta} \|^2_H + \eta h \| u^k_{\varepsilon h\eta} \|^p_{1,p} + h \| u^k_{\varepsilon h\eta} \|^2_{1,2} \leq Ch \| f^k \|_{L^\infty(G))} + \frac{1}{2} \| u^{k-1}_{\varepsilon h\eta} \|^2_H \,.$$

Taking the summation from $k=1$ to n, we obtain:

$$\sum_{k=1}^{n} \eta h \| u^k_{\varepsilon h\eta} \|^p_{1,p} + h \| u^k_{\varepsilon h\eta} \|^2_{1,2} \leq Ch \sum_{k=1}^{n} \| f^k \|_{L^\infty(G)} \leq C_2\, T.$$

C_2 is a constant independent of ε, η, h and n.

2) We show the crucial estimate: $\| u^n_{\varepsilon h\eta} \|_{L^\infty(G))}$.

Since $p > n$, the Sobolev imbedding theorem gives: $W_0^{1,p}(G) \subset C(\text{clG})$. Thus $| u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}$ is in $W_0^{1,p}(G)$ for any positive integer $s \geq 2$. Therefore:

$$\| u^k_{\varepsilon h\eta} \|^s_s + \eta h (A_2 u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}) + h (A u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta})$$

$$+ h\varepsilon^{-1} (B u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}) \leq Ch \| f^k \|_{L^\infty(G)} \| u^k_{\varepsilon h\eta} \|^{s-1}_s + \| u^{k-1}_{\varepsilon h\eta} \|_s \| u^k_{\varepsilon h\eta} \|^{s-1}_s.$$

Consider the second term of the left hand side of the inequality:

$$(A_2 u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}) = \sum_j (s-1) \int_G | u^k_{\varepsilon h\eta} |^{s-2} | D_j u^k_{\varepsilon h\eta} |^p \, dx \geq 0.$$

On the other hand:

$$(A u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}) = \sum_{|\alpha| \leq 1} (s-1) \int_G A_\alpha (x, u^k_{\varepsilon h\eta}, D u^k_{\varepsilon h\eta}) | u^k_{\varepsilon h\eta} |^{s-2} D^\alpha (u^k_{\varepsilon h\eta}) \, dx$$

It follows from condition (iv) of Assumption (I) that the above expression is positive.
It is trivial to check that: $(B u^k_{\varepsilon h\eta}, | u^k_{\varepsilon h\eta} |^{s-2} u^k_{\varepsilon h\eta}) \geq 0$.

Therefore:

$$\| u^k_{\varepsilon h\eta} \|_s \leq Ch \| f^k \|_{L^\infty(G)} + \| u^{k-1}_{\varepsilon h\eta} \|_s.$$

Taking the summation from $k=1$ to n, we obtain:

$$\| u^n_{\varepsilon h\eta} \|_s \leq Ch \sum_{k=1}^{n} \| f^k \|_{L^\infty(G))} \leq C_2\, T.$$

We know that $u^n_{\varepsilon h\eta}$ is in $C(\text{clG})$, thus letting $s \to +\infty$, we have:

$$\| u^n_{\varepsilon h\eta} \|_{L^\infty(G)} \leq C_2\, T.$$

The lemma is proved.

LEMMA 2: *Let $h = T/N$ and suppose all the hypotheses of Theorem 3 are satisfied. Then for each k, $1 \leq k \leq N$, there exists a unique solution $u^k_{\varepsilon h}$ in $W^{1,2}_0(G)$ of:*

$$u^k_{\varepsilon h} - u^{k-1}_{\varepsilon h} + hAu^k_{\varepsilon h} + h\varepsilon^{-1} Bu^k_{\varepsilon h} = hf^k, \; u^0_{\varepsilon h} = 0.$$

Moreover:

$$\| u^n_{\varepsilon k} \|_{L^\infty(G)} + \sum_{k=1}^{n} h \| u^k_{\varepsilon h} \|^2_{1,2} \leq C.$$

PROOF: From Lemma 1, we know that for each k, $k = 1, \dots, N$, there exists a unique solution $u^k_{\varepsilon h \eta}$ in $W^{1,p}_0(G)$ of the equation:

$$u^k_{\varepsilon h\eta} - u^{k-1}_{\varepsilon h\eta} + \eta h A_2 u^k_{\varepsilon h\eta} + hAu^k_{\varepsilon h\eta} + h\varepsilon^{-1} Bu^k_{\varepsilon h\eta} = hf^k, \; u^0_{\varepsilon h\eta} = 0.$$

Moreover:

$$\| u^n_{\varepsilon h\eta} \|_{L^\infty(G)} + \sum_{k=1}^{n} \eta h \| u^k_{\varepsilon h\eta} \|^p_{1,p} + h \sum_{k-1}^{n} \| u^k_{\varepsilon h\eta} \|^2_{1,2} \leq C.$$

C is a constant independent of ε, η, h and n.

Let $\eta \to 0$. The weak compactness of the unit ball in a reflexive Banach space gives : $h^{\frac{1}{2}} u^k_{\varepsilon h\eta} \to h^{\frac{1}{2}} u^k_{\varepsilon h}$ weakly in $W^{1,2}_0(G)$, $(h\eta)^{1/p} u^k_{\varepsilon h\eta} \to 0$ weakly in $W^{1,p}_0(G)$, $Au^k_{\varepsilon h\eta} \to g^k_{\varepsilon h}$ weakly in $W^{-1,2}(G)$ and $u^k_{\varepsilon k\eta} \to u^k_{\varepsilon h}$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$.

It follows from the Sobolev imbedding theorem that $u^k_{\varepsilon h\eta} \to u^k_{\varepsilon h}$ in $L^2(G)$ and thus $Bu^k_{\varepsilon h\eta} \to Bu^k_{\varepsilon h}$ weakly in $L^2(G)$.

We obtain:

$$u^k_{\varepsilon h} - u^{k-1}_{\varepsilon h} + h\, g^k_{\varepsilon h} + h\varepsilon^{-1} Bu^k_{\varepsilon h} = hf^k, \; u^0_{\varepsilon h} = 0.$$

Since A is monotone, it is easy to show that $g^k_{\varepsilon h} = Au^k_h$.

All the other assertions of the lemma follow trivially from the above arguments.

PROOF OF THEOREM 3: Let $u^k_{\varepsilon h}$, $k = 1, \dots, N$, be the solution of:

$$u^k_{\varepsilon h} - u^{k-1}_{\varepsilon h} + hAu^k_{\varepsilon h} + h\varepsilon^{-1} Bu^k_{\varepsilon h} = hf^k \text{ with } u^0_{\varepsilon h} = 0.$$

1) Set: $u_{\varepsilon h}(t) = u^k_{\varepsilon h}$ when $kh \leq t < (k+1)h$, $k = 0, \dots, N-1$ and $h = T/N$. Then from Lemma 2, we obtain:

$$\| u_{\varepsilon h} \|_{L^\infty(0, T; L^\infty(G))} + \| u_{\varepsilon h} \|_{L^2(0, T; W_0^{1,2}(G))} \leq C$$

C is a constant independent of both ε and h.

It is easy to show that:

$$\sum_{k=1}^{n} \| h^{-1}(u^k_{\varepsilon h} - u^{k-1}_{\varepsilon h}) \|^2_{W^{-1,2}(G)} \leq M(\varepsilon).$$

$M(\varepsilon)$ is independent of h and n.

2) From the weak compactness of the unit ball in a reflexive Banach space, we get by taking subsequences if necessary: $u_{\varepsilon k} \to u_\varepsilon$ weakly in $L^2(0, T; W_0^{1,2}(G))$, $u_{\varepsilon h} \to u_\varepsilon$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$, $Au_{\varepsilon h} \to g_\varepsilon$ weakly in $L^2(0, T; W^{-1,2}(G))$ and $h^{-1}(u_{\varepsilon h}(t) - u_{\varepsilon h}(t-h)) \to u'_\varepsilon$ weakly in $L^2(0, T; W^{-1,2}(G))$.

Since the injection mapping of $W^{1,2}(G)$ into $L^2(G)$ is compact, the discrete analogue of Aubin's theorem [1] gives: $u_{\varepsilon h} \to u_\varepsilon$ in $L^2(0, T; H)$. Hence: $Bu_{\varepsilon h} \to Bu_\varepsilon$ weakly in $L^2(0, T; H)$.

Thus:

$$u'_\varepsilon + g_\varepsilon = \varepsilon^{-1} Bu_\varepsilon = f.$$

3) We show thet $u_\varepsilon(0) = 0$.

Le $v \in W_0^{1,2}(G)$ and $\varphi \in C([0, T]$. Set: $\varphi_h(t) = \varphi(nh)$ with $nh \leq t\,(n+1)h$. Then:

$$h^{-1}(u^n_{\varepsilon h} - u^{n-1}_{\varepsilon h}, v)\,\varphi_h(t) + (Au^n_{\varepsilon h}, v)\,\varphi_h(t) + \varepsilon^{-1}(Bu^n_{\varepsilon h}, v)\,\varphi_h(t) = (f^n, v)\,\varphi_h(t)$$

Let $h \to 0$ and we get:

$$\int_0^T (u'_\varepsilon + g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, v)\,\varphi\, dt = 0$$

for all v in $W_0^{1,2}(G)$ and all φ in $C([0, T])$.

A standard argument gives:

$$\int_0^T (u'_\varepsilon + g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, \varphi)\, dt = 0$$

for all φ in $L^2(0, T; W_0^{1,2}(G))$.

On the other hand:

$$\sum_{n=1}^{N} -(u_{\varepsilon h}, v)_H(\varphi(nh)-\varphi(nh-h)) + h(Au_{\varepsilon h}, v)\varphi(uh-h) + h\varepsilon^{-1}(Bu_{\varepsilon h}, v)\varphi(nh-h)$$

$$- h(f^n, v)\varphi(nh-h) = -(u_{\varepsilon h}(T), v)_H \varphi(T).$$

Take $\varphi \in C([0, T])$ with $\varphi(T) = 0$ and let $h \to 0$. We obtain:

$$-\int_0^T (u_\varepsilon, v)\varphi' \, dt + \int_0^T (g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, v)\varphi \, dt = 0.$$

So:

$$-\int_0^T (u_\varepsilon, \varphi') \, dt + \int_0^T (g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, \varphi) \, dt = 0$$

for all φ in $L^2(0, T; W_0^{1,2}(G))$ with φ' in $L^2(0, T; W^{-1,2}(G))$ and $\varphi(T) = 0$. Therefore: $(u_\varepsilon(0), \varphi(0))_H = 0$ for all φ in $L^2(0, T; W^{1,2}(G))$ with φ' in $L^2(0, T; W^{-1,2}(G))$ and $\varphi(T) = 0$.

Since the set $\{\varphi(0) : \varphi$ in $L^2(0, T; W_0^{1,2}(G))$, φ' in $L^2(0, ; W^{-1,2}(G))$ and $\varphi(T) = 0\}$ is dense in H, we have: $u_\varepsilon(0) = 0$.

4) We show that $g_\varepsilon = Au_\varepsilon$.

An elementary computation gives:

$$\frac{1}{2} \| u_{\varepsilon h}^N(T) \|_H^2 + \int_0^T (Au_{\varepsilon h} + \varepsilon^{-1} Bu_{\varepsilon h} - f, u_{\varepsilon h}) \, dt \leq 0.$$

So:

$$\frac{1}{2} \| u_{\varepsilon h}(T) \|_H^2 + \limsup \int_0^T (Au_{\varepsilon h}, u_{\varepsilon h}) \, dt \leq \int_0^T (f - \varepsilon^{-1} Bu_\varepsilon, u_\varepsilon) \, dt.$$

On the other hand: $u_\varepsilon' + g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon = f$.

Thus:

$$\frac{1}{2} \| u_\varepsilon(T) \|_H^2 + \limsup \int_0^T (Au_{\varepsilon h}, u_{\varepsilon h}) \, dt \leq \int_0^T (u_\varepsilon' + g_\varepsilon, u_\varepsilon) \, dt.$$

$$\leq \frac{1}{2} \| u_\varepsilon(T) \|_H^2 + \int_0^T (g_\varepsilon, u_\varepsilon) \, dt.$$

Hence:

$$\limsup \int_0^T (Au_{\varepsilon h}, u_{\varepsilon h})\, dt \leq \int_0^T (g_\varepsilon, u_\varepsilon)\, dt.$$

Since A is monotone, the above inequality implies that $g_\varepsilon = Au_\varepsilon$. It is clear that the solution is unique.

5) It remains to show that $\| u'_\varepsilon \|_{L^2(0,T;W^{1,2}(G))} \leq C$. C is a constant independent of ε.

The proof has been carried out in [8]. To show it, we note that u_ε is the restriction to $[0, T]$ of v_ε where v_ε is the unique solution of a global boundary-value problem:

$$v'_\varepsilon + Av_\varepsilon + \varepsilon^{-1} Bv_\varepsilon = \widehat{f} \qquad \text{on } E \times G, \quad v_\varepsilon = 0 \text{ on } E \times \partial G.$$

$\widehat{f} = \zeta(t) f$ where $\zeta \in C_0^1(E)$, $\zeta(t) = 1$ for t in $[0, T]$, $\zeta(t) = 0$ for $t \leq -1$ and $t \geq 2T$. $f(t)$ is extended to E with $f(t) = 0$ for $t \leq 0$ and $f(t) = f(T)$ for $t \geq T$.

The method of difference quotients applied to v_ε gives the desired estimate. Since:

$$Au_\varepsilon + \varepsilon^{-1} Bu_\varepsilon = f - u'_\varepsilon \quad \text{is now in} \quad L^2(0, T; L^2(G)),$$

by using again the method of difference quotients and some standard results of the theory of elliptic operators, it is not difficult to show that:

$$\| u_\varepsilon \|_{L^2(0,T;W^{2,2}(G))} \leq C.$$

C is independent of ε. Cf. [8].

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Added in proof:

H. BREZIS: *Problèmes unilatéraux.* J. Math. Pures et Appl. 51 (1972), 1-68.