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INVARIANT MEANS ON VECTOR VALUED FUNCTIONS I

By T. HUSAIN and JAMES C. S. WONG (1)

1. Introduction.

Invariant means on spaces of vector-valued functions on a semigroup were first considered by Dixmier in [2, § 3]. Let S be a semigroup, E a real Banach space with continuous dual E^* and let $l_\infty(S, E^*)$ be the Banach space of all functions $f: S \to E^*$ such that $\|f\|_\infty = \sup_{s \in S} |f(s)| < \infty$. A linear mapping $M: l_\infty(S, E^*) \to E^*$ is called a *mean* if for any f, $M(f)$ belongs to the weak* closure of the convex hull of $\{f(s): s \in S\}$ in E^*. This definition of a mean clearly reduces to the usual one introduced by Day [1, § 3] when E is the space of real numbers (in which case, we write $l_\infty(S)$ for $l_\infty(S, E^*)$. A mean M is *left invariant* if $M(l_a f) = M(f)$ for any $a \in S$, $f \in l_\infty(S, E^*)$ where $l_a f(s) = f(as)$. Dixmier has shown in [2] that if m is a left invariant mean on $l_\infty(S)$, then m induces a left invariant mean M on $l_\infty(S, E^*)$ such that $M(f)x = m(f(\cdot)x)$ for any $x \in E$. Here $f(\cdot)x$ denotes the functions $s \to f(s)x$. (This is only a discrete version of Dixmier's results in [2]).

The purpose of this paper is to extend the theory of invariant means to vector valued functions on a semigroup. Indeed we shall consider more general topological vector spaces than Banach spaces.

2. Notations and terminologies.

All spaces and functions considered here will be real unless otherwise specified. For general terms in topological vector spaces, we follow Robertson

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and Robertson [9]. From now on, E will always denote a separated locally convex space with continuous dual E^*. We shall assume E is *quasi barrelled* (i. e. strongly bounded subsets of E^* are equicontinuous).

By definition, the strong topology $\beta(E^*, E)$ in E^* is determined by the family of seminorms p_A where A is any weakly bounded subset of E and $p_A(x^*) = \sup\{|x^*(x)| : x \in A\}$. Let $l_\infty(S, E^*)$ be the linear space (pointwise operations) of all functions $f: S \longrightarrow E^*$ which are $\beta(E^*, E)$ bounded, that is, $p_A(f(S))$ is a bounded subset of real numbers for each bounded subset A of E. (Bounded sets in E are the same in any topology of the dual pair (E, E^*). See Robertson [9, Theorem 1, p. 67]). For each such A, define a seminorm q_A on $l_\infty(S, E^*)$ by

$$q_A(f) = \sup_{s \in S} p_A(f(s)) = \sup_{s \in S} \sup_{x \in A} |f(s)\,x|,$$

then $l_\infty(S, E^*)$ becomes a separated locally convex space. Since $l_a f(S) \subset f(S)$, it is clear that $l_\infty(S, E^*)$ is invariant under left translation ($l_a f$ is defined as usual by $l_a f(s) = f(as)$). Moreover the left translation operator l_a: $l_\infty(S, E^*) \longrightarrow l_\infty(S, E^*)$ is continuous since $q_A(l_a f) \leq q_A(f)$ for each A.

DEFINITION 2.1: Let X be a linear subspace of $l_\infty(S, E^*)$. A map $M: X \longrightarrow E^*$ is called a *mean* on X if

(1) M is linear

(2) for each $f \in X$, $M(f)$ belongs to the weak* closure of the convex hull of $\{f(s) : s \in S\}$ in E^* (In symbols, $M(f) \in w^*\ CLCO\{f(s) : s \in S\}$).

If X is also left invariant ($l_a f \in X$ for any $a \in S$, $f \in X$), then M is called *left invariant* if $M(l_a f) = M(f)$ for any $a \in S$, $f \in X$.

As remarked in the introduction, it is easy to see that this definition agrees with the usual one when E is the real number field (and X contains the constants of course). Naturally, many important properties of the set of means in the real case are expected to carry over. In the next section, we shall gather some useful information about the set of means on an arbitrary linear subspace X of $l_\infty(S, E^*)$.

3. Elementary properties of the set of means.

Suppose A is a bounded subset of E and $r \geq 0$. Let $B_A(r) = \{x^* \in E^* : p_A(x^*) \leq r\}$. Clearly $B_A(r)$ is convex. We first record here the following well-known fact.

LEMMA 3.1. $B_A(r)$ is weak* closed in E^*.

PROOF: Let x_α^* be a net in $B_A(r)$ and $x_\alpha^* \rightarrow x^*$ weak* in E^*. Then $|x^*(x)| = \lim_\alpha |x_\alpha^*(x)| \leq r$ for any $x \in A$. Hence $p_A(x^*) \leq r$ or $x^* \in B_A(r)$.

LEMMA 3.2. If $f \in X$, $A \subset E$ is bounded and $r = q_A(f)$, then $w^*\ CLCO\{f(s): s \in S\} \subset B_A(r)$. Hence if M is a mean on X, $p_A(M(f)) \leq q_A(f)$ for any $f \in X$ and any bounded subset A in E.

Consequently any mean $M: X \rightarrow E^*$ on X is continuous when E^* has the strong topology $\beta(E^*, E)$ and X has the topology determined by the semi norms q_A, $A \subset E$ bounded.

PROOF: By definition, $r = q_A(f) = \sup_s p_A(f(s))$. Hence $\{f(s): s \in S\} \subset B_A(r)$ which is weak* closed (Lemma 3.1) and convex. Therefore $w^*\ CLCO\{f(s): s \in S\} \subset B_A(r)$. Now if M is a mean on X, then $M(f) \in w^*\ CLCO\{f(s): s \in S\}$.

It follows that $p_A(M(f)) \leq q_A(f)$ for any $f \in X$ and any bounded subset A of E and this implies that M is continuous when E^* and X have the topologies described in the lemma.

DEFINITION 3.3. Consider the product space $\Pi\{E^*: f \in X\}$. The product of the weak* topologies is called the weak* operator topology, with respect to which the set of all means on X is compact. The proof of this fact parallels that for the real case and is contained in the next two lemmas.

LEMMA 3.4. Let $\mathcal{S}$ be the set of all linear mappings $M: X \rightarrow E^*$ such that $p_A(M(f)) \leq q_A(f)$ for all $f \in X$ and A bounded in E ($\mathcal{S}$ is the closed unit ball of X^* in the real case), then $\mathcal{S}$ is compact in the weak* operator topology.

PROOF: For each $f \in X$, define $B(f) = \{x^* \in E^*: p_A(x^*) \leq q_A(f)$ for any bounded subset A of $E\} = \cap\{B_A(r_A): A \subset E$ bounded$\}$ with $r_A = q_A(f)$. Then $B(f)$ is weak* closed by Lemma 3.1, and is strongly bounded (by definition, for each A, $p_A(B(f)) \leq q_A(f) < \infty$). Since E is quasibarrelled (The first place where this assumption comes in), $B(f)$ is equicontinuous. Since $B(f)$ is weak* closed, $B(f)$ is weak* compact in E^* by [9, Corollary 1, p. 62].

Now the product $\Pi\{B(f): f \in X\}$ is compact in the product (weak*) topology by Tychonoff theorem. Define a map $T: \mathcal{S} \rightarrow \Pi\{B(f): f \in X\}$ by $T(M)(f) = M(f)$ far any $f \in X$ and $M \in \mathcal{S}$. Evidently, T is a bijection onto its image. Since $M_\alpha \longrightarrow M$ in weak* operator topology in $\mathcal{S}$ iff $M_\alpha(f) \longrightarrow M(f)$ weak* in E^* for any $f \in X$, or equivalently, iff $T(M_\alpha) \longrightarrow T(M)$ in the product topology of $\Pi\{B(f) : f \in X\}$. It follows that $T : \longrightarrow T(\mathcal{S})$ is a homeomorphism. We need only show that $T(\mathcal{S})$ is closed in the product. Suppose $M_\alpha \in \mathcal{S}$, and $T(M_\alpha) \longrightarrow N \in \Pi\{B(f) : f \in X\}$. Then for any $f \in X$, $N(f) = w^* \lim_\alpha M_\alpha(f)$. Define $M : X \longrightarrow E^*$ by $M(f) = N(f), f \in X$. It is obvious that M is linear. Since each $M_\alpha \in \mathcal{S}$, $M_\alpha(f) \in B_A(r)$ with $r = q_A(f)$ for any $f \in X$, $A \subset E$ bounded and any α, and since $B_A(r)$ is weak* closed (Lemma 3.1), we have $M(f) \in B_A(r)$ for any such f and A. In other words, $M \in \mathcal{S}$. It is easy to see that $T(M) = N$ Consequently $T(\mathcal{S})$ is closed and hence compact in $\Pi\{B(f) : f \in X\}$. This completes the proof.

LEMMA 3.5. **The set of all means is compact and convex in the weak* operator topology.**

PROOF: Clearly they form a convex set. Let $M_\alpha \longrightarrow M$ in the weak* operator topology where M_α are means on X. Since M_α is linear on X, so is M. But $M_\alpha(f) \in w^*\ CLCO\ \{f(s) : s \in S\}$ which is weak* closed. Hence $M(f) \in w^*\ CLCO\ \{f(s) : s \in S\}$. Therefore M is also a mean on X. In other words, the set of means is closed in $\mathcal{S}$ in weak* operator topology. The result now follows from the preceeding lemma.

DEFINITION 3.6. Let X be a left invariant linear subspace of $l_\infty(S, E^*)$, M a mean on X ane $f \in X$. Define a mapping $M_l(f) : S \longrightarrow E^*$ by $M_l(f)(s) = M(l_s f)$, $s \in S$. $M_l(f)$ is called the *left introversion* of f by M. Since $p_A(M_l(f)(s)) = p_A(M(l_s f)) \leq q_A(l_s f) \leq q_A(f)$ (Use Lemma 3.2), $q_A(M_l(f)) < \infty$, for any $A \subset E$, bounded. Hence $M_l(f) \in l_\infty(S, E^*)$.

X is called *left introverted* if $M_l(X) \subset X$ for any mean M on X, in which case, $M_l : X \longrightarrow X$ is a continuous linear operator in X, when X has the topology induced by the seminorms q_A, $A \subset E$, bounded. In fact, $q_A(M_l(f)) \leq q_A(f)$ for any A and f.

Next, we define the *Arens product* $M \odot N$ (or convolution) of two means M, N on X by $M \odot N(f) = M(N_l(f)), f \in X$. Obviously, $M \odot N$ is linear on X. Moreover, for any $s \in S, N(l_s f) \in w^*\ CLCO\ \{l_s f(t) : t \in S\} \subset w^*\ CLCO\ \{f(t) : t \in S\}$. Hence $M \odot N(f) = M(N_l(f)) \in w^*\ CLCO\ \{N_l(f)(s) : s \in S\} \subset w^*\ CLCO\ \{f(t) : t \in S\}$. Therefore $M \odot N$ is also a mean on X.

Some other properties of left introversion and Arens product are listed in the folllowing lemma. We shall omit the proof since it is the same as in the real case. (See Day [1], Namioka [6] and Wong [10]).

LEMMA 3.7. Let $s \in S$ and M, N, L be means on X, then

(*a*) The operators l_s and M_l commute

(*b*) $M_l \circ N_l = (M \odot N)_l$

(*c*) $(M \odot N) \odot L = M \odot (N \odot L)$ ($\odot$ is associative)

(*d*) For fixed N, the map $M \to M \odot N$ is affine and weak* operator continuous on the set of means into itself.

REMARK 3.8. Left introverted spaces were first introduced by Day in [1] and studied by Rao [8] and Mitchel [5].

4. Localisation theorems.

THEOREM 4.1. (Localisation). Let X be a left introverted and left invariant linear subspace of $l_\infty(S, E^*)$, then X has a left invariant mean iff for each $f \in X$, there is a mean M_f (depending on f) such that $M_f(l_a f) = M_f(f)$ for any $a \in S$.

PROOF: The proof, as expected will be adapted from the real case. (see Granirer and Lau [3, Theorem 1]). However, there is one important difference between the two situations in that we do not assume that X contains the constant functions (i. e. functions whose ranges are singletons). Thus our result is new even in the case when E is the real field.

One part (necessity) of the theorem is clear. Conversely, assume that for each $f \in X'$ there is a mean (depending on f) which is left invariant on f. Define $K_f = \{M : M$ is a mean on X and $M(l_a f) = M(f)$ for any $a \in S\}$. By assumption, $K_f \neq \emptyset$. Indeed, K_f is a weak* operator closed subset of the set of means on X. For if $M_\alpha \to M$ in weak* operator topology, $M_\alpha(f) \xrightarrow{w^*} M(f)$, $M_\alpha(l_a f) \xrightarrow{w^*} M(l_a f)$ in E^*. But $M_\alpha \in K_f$ implies $M_\alpha(l_a f) = M_\alpha(f)$. Hence $M(f) = M(l_a f)$ or $M \in K_f$.

Next, we show that the family $\{K_f : f \in X\}$ has the finite intersection property. Let $f_1, f_2, \dots, f_n \in X$. If $n = 1$, this is clear. Assume $\bigcap_{i=1}^{n-1} K_{f_i} \neq \emptyset$ and take $N \in \bigcap_{i=1}^{n-1} K_{f_i}$. Define $f = N_l(f_n) \in X$ (which is left introverted) and let $M \in K_f$. Consider the mean $M \odot N$. We claim that $M \odot N \in \bigcap_{i=1}^{n} K_{f_i}$.

For $1 \leq i \leq n-1, N_l(f_i)(s) = N(l_s f_i)$ for any $s \in S$. (i. e. $N_l(f_i)$ is a constant function). Hence for any $a \in S$, $(M \odot N)(l_a f_i) = M(N_l(l_a f_i)) = M(l_a(N_l(f_i)) = M(N_l(f_i)) = (M \odot N)(f_i)$ (Recall that N_l and l_a commute). Thus $M \odot N \in \bigcap_{i=1}^{n-1} K_{f_i}$. Also $(M \odot N)(l_a f_n) = M(N_l(l_a f_n)) = M(l_a(N_l(f_n)) = M(l_a f) = M(f)$ (Since $M \in K_f) = M(N_l(f_n)) = M \odot N(f_n)$ for any $a \in S$. Therefore $M \odot N \in K_{f_n}$ and so $M \odot N \in \bigcap_{i=1}^{n} K_{f_i}$.

By weak* operator compactness of the set of means on $X, \cap\{K_f : f \in X\} \neq \varnothing$. Any mean in this intersection is a left invariant mean on X. This completes the proof.

REMARK 4.2. (*a*) Incidentally, we have shown that if X is left introverted, left invariant and has a left invariant mean, then X must contain the constant functions $M(f)$ for any $f \in X$ and $M \in K_f$, since $M_l(f)(s) = M(f)$ for any $s \in S$ and $M_l(f) \in X$.

(*b*) This theorem is a generalisation of a result in Granirer and Lau [3]. Our proof is based on the idea of their work.

The direction of development of our theory is now clear. We have found a separated locally covex space ($\Pi\{E^* : f \in X\}$ with weak* operator topology) in which the set of means on X is compact. By the famous KREIN-MILMAN Theorem the set of means on X is the weak* operator closed convex hull of its extreme points. Following the ideas in Granirer and Lau [3], it is expected that a theorem of similar type as [4, Theorem 1] (which characterises the existence of a left invariant mean on X in terms of constant functions) could be obtained.

In the case when E is the real field and $X = l_\infty(S)$, the extreme points of the set of means on $l_\infty(S)$ are precisely the multiplicative means on $l_\infty(S)$ (See Phelps [7]) which coincide with the weak* closure in $l_\infty(S)^*$ of the evaluation functionals $\{\mu_a : a \in S\}$ where $\mu_a(f) = f(a)$, $f \in l_\infty(S)$ (See Mitchell [5, p. 119]). Thus the set of means on $l_\infty(S)$ is precisely $w^*\, CLCO\{\mu_a : a \in S\}$ in $l_\infty(S)^*$ (This can also be proved directly by using Hahn-Banach Theorem, of course).

However, in the general case, although the evaluation functionals are also available, it is not known if the extreme points of the set of means on X could have such a nice identification, nor do we know whether the means coincide exactly with the weak* operator closed convex hull of the evaluation functionals. The situation is still more complicated if X is a proper linear subspace of $l_\infty(S, E^*)$, in that unlike the real case, we might not be able to extend a mean on X to $l_\infty(S, E^*)$.

Despite this unpleasant set back, we can turn our attention to the set $\mathcal{K}(X)$ which is the weak* operator, closed convex hull of $\{\delta_a : a \in S\}$ (where δ_a is defined as usnal by $\delta_a(f) = f(a)$, $f \in X$). $\mathcal{K}(X)$ is again weak* operator compact. This compactness property of $\mathcal{K}(X)$ enables us to prove an analogue (as well as a generalisation) of [4, Theorem 1], namely Theorem 4.5 below, which characterises the existence of a left invariant mean in $\mathcal{K}(X)$ in terms of constant functions. For this purpose, we need the following « Localisation Theorem » on left invariant means in $\mathcal{K}(X)$.

THEOREM 4.3 (Localisation) Let X be a left introverted and left invariant linear subspace of $l_\infty(S, E^*)$ and $\mathcal{K}(X)$ as above, then there is a left invariant mean in $\mathcal{K}(X)$ iff for each $f \in X$, there is a mean $M_f \in \mathcal{K}(X)$ such that $M_f(l_a f) = M(f)$ for any $a \in S$.

PROOF: As remarked above, $\mathcal{K}(X)$ is weak* operator compact. Hence we can repeat the arguments in the proof of Theorem 4.1, considering means in $\mathcal{K}(X)$ instead of means on X, except we have to show that $\mathcal{K}(X)$ is closed under Arens product. To show this, observe that $\delta_a \odot N = N o\, l_a$ for any mean N and $a \in S$. Hence the map $N \longrightarrow \delta_a \odot N$ is affine and weak* operator continuous. It follows that the set $\{N : N$ is a mean and $\delta_v \odot N \in \mathcal{K}(X)\}$ is weak* operator closed and convex. Since it contains any δ_b (notice that $\delta_a \odot \delta_b = \delta_{ab}$), it must also contain $\mathcal{K}(X)$. In other words, $\delta_a \odot N \in \mathcal{K}(X)$ for any $N \in \mathcal{K}(X)$, $a \in S$. Again the map $M \longrightarrow M \odot N$ is affine and weak* operator continuous by Lemma 3.7. Hence the set $\{M : M$ is a mean and $M \odot N \in \mathcal{K}(X)\}$ is weak* operator closed and convex. The same argument shows that it must contain $\mathcal{K}(X)$. That is, $M \odot N \in \mathcal{K}(X)$ for any $M, N \in \mathcal{K}(X)$. This completes the proof.

DEFINITION 4.4. Consider lhe product space $\Pi\{E^* : s \in S\}$ where each E^* has the weak* topology, the product topology is called the pointwise weak* topology and is denoted by pw^*. For each $f \in l_\infty(S, E^*)$, define $Z_R(f)$ as the pw^* closed convex hull of $\{r_a f : a \in S\}$ (where $r_a f(s) = f(sa)$, $s \in S$).

Recall that a constant function is a function whose range is a singleton. For brevity, we make no distiction between the element $x^* \in E^*$ and the constant function on S which is identically equal to x^*.

We say that X is *right stationary* iff for any $f \in X$, $Z_R(f)$ contains a constant function. This concept is first introduced by Mitchel in [4] for scalar valued functions. It is clear that our definitions agrees with that given in [4].

We now prove

THEOREM 4.5. Lex X be a left introverted linear subspace of $l_\infty(S, E^*)$, then there is a left invariant mean in $\mathcal{K}(X)$ iff X is right stationary. In this case, if $f \in X$, then $x^* \in Z_R(f)$ iff there is a left invariant mean M in $\mathcal{K}(X)$ such that $M(f) = x^*$.

PROOF: The proof is more or less the same as the real case (see Granirrer and Lau [3]). We present it here for completeness. Notice we do not assume that X contains the constant functions.

Suppose there is a left invariant mean M in $\mathcal{K}(X)$, and let $M_a = \sum_{i=1}^{n_a} \lambda_i^a \delta_{a_i^a}$ be a net of convex combinations of evaluation functionals such that $M_a \to M$ in weak* operator topology. Take any $f \in X$ and consider the net $f_a = \sum_{i=1}^{n_a} \lambda_i^a r_{a_i^a} f$ associated with M_a. Clearly $f_a \in Z_R(f)$. For each $s \in S$, we have

$$w^* \lim_a f_a(s) = w^* \lim_a \sum_{i=1}^{n_a} \lambda_i^a f(sa_i^a) = w^* \lim_a \sum_{i=1}^{n} \lambda_i^a \delta_{a_i^a}(l_s f)$$
$$= w^* \lim_a M_a(l_s f) = M(l_s f) = M(f),$$

which means that f_a converges to the constant function $M(f)$ in pw^* topology. Hence $M(f) \in Z_R(f)$.

Conversely, assume that for each $f \in X$, $Z_R(f)$ contains a constant function. Let $x^* \in Z_R(f)$ and let $f_a \to x^*$ in pw^* topology with $f_a = \sum_{i=1}^{n_a} \lambda_i^a r_{a_i^a}(f)$ as above. Consider the means $M_a = \sum_{i=1}^{n_a} \lambda_i^a \delta_{a_i^a}$ associated with f_a. Passing to a subnet if necessary, we can assume $M_a \to N \in \mathcal{K}(X)$ in weak* operator topology by compactness of $\mathcal{K}(X)$. Define $K_f = \{M : M$ is a mean in $\mathcal{K}(X)$ and $M(l_a f) = M(f)\ \forall\, a \in S\}$. We claim tuat $N \odot N \in K_f$. Since $\mathcal{K}(X)$ is closed under Arens product (see proof of 4.3), $N \odot N \in \mathcal{K}(X)$. Now $N_l(f)(s) =$
$= N(l_s f) = w^* \lim_a M_a(l_s f) = w^* \lim_a \sum_{i=1}^{n_a} \lambda_i^a f(sa_i^a) = w^* \lim f_a(s) = x^*$.

Hence $N_l(f)$ is the constant function x^*. It follows that $(N \odot N)(l_a f) = (N \odot N)(f)$. (Recall that N_l and l_a commute). Therefore $N \odot N \in K_f$. By Theorem 4.3 $\cap \{K_g : g \in X\} \neq \varnothing$. Take $L \in \cap \{K_g : g \in X\}$. Then L is a left invariant mean in $\mathcal{K}(X)$. Consider $M = L \odot N \in \mathcal{K}(X)$. For any $a \in S$, $g \in X$, $M(l_a g) = L(N_l(l_a g)) = L(l_a N_l(g)) = L(N_l(g)) = L \odot N(g) = M(g)$. Therefore M is a left invariant mean in $\mathcal{K}(X)$. Moreover $M(f) = L(N_l(f)) =$
$= L(x^*) = x^*$, since $L(x^*) \in w^*\, CLCO\, \{x^*\} = \{x^*\}$. This completes the proof.

5. Connection between the means on vector valued and scalar valued Functions.

Let X be a left invariant linear subspace of $l_\infty(S, E^*), f \in X$ and $x \in E$. Denote by $f(\cdot)x$ the real-valued function $s \to f(s)x$. It is clear that $f(\cdot)x$ is a function in $l_\infty(S)$. In general, the family of all such functions need not be a linear subspace of $l_\infty(S)$ nor does it contain the constants. This is true when $X = l_\infty(S, E^*)$. In fact every function $g \in l_\infty(S)$ has the form $f(\cdot)x$ with $x \neq 0$, $x^* \in E^*$, $x^*(x) = 1$ and $f(s) = g(s)x^*$. Nevertheless there is still a close connection between the means in $\mathcal{K}(X)$ and the means on $l_\infty(S)$, as shown in the following theorem.

THEOREM 5.1. Let m be a meam on $l_\infty(S)$ and define $M: X \to E^*$ by $M(f)x = m(f(\cdot)x), f \in X, x \in E$. Then M is a mean in $\mathcal{K}(X)$. Moreover, M is left invariant on f (i. e. $M(l_a f) = M(f)$ for any $a \in S$) iff m is left invariant on $f(\cdot)x$ for any $x \in E$.
Conversely, any mean M is $\mathcal{K}(X)$ is of this form.

PROOF: The first part of the theorem is essentially due to Dixmier ([2, § 3], where E is assumed to be a Banach space).

In general, let m be a mean on $l_\infty(S)$. It is obvious that $M(f)x = m(f(\cdot))$ is linear in x. Also since $\|m\| = 1$, we have $|M(f)x| = |m(f(\cdot)x)| \leq \|m\| \sup_s |f(s)x| = \sup_s |f(s)x|$. Now the set $\{f(s) : s \in S\}$ is strongly bounded, hence equicontinuous (since E is quasi barrelled). Therefore by [9, Proposition 3 p. 48], $p(x) = \sup_s |f(s)x|$ is a continuous semi-norm on E. Since $M(f)$ is dominated by $p(x)$, $M(f)$ is continuous. Hence $M(f) \in E^*$. It is also obvious that $M(f)$ is linear in f. Thus $M: X \to E^*$ is well-defined and linear. To show that M is a mean in $\mathcal{K}(X)$, let $m_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \mu_{a_i^\alpha}$ be a net of convex combinations of evaluation functionals on $l_\infty(S)$ (here $\mu_a(f) = f(a)$, $f \in l_\infty(S)$) such that $m_\alpha \to m$ in weak* topology of $l_\infty(S)^*$. Define $M_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{a_i^\alpha}$. Then it is easy to see that $M_\alpha \to M$ in weak* operator topology. Hence $M \in \mathcal{K}(X)$. Also, since $(l_s f(\cdot))x = l_s(f(\cdot)x)$, it is clear that M is left invariant on f iff m is left invariant on $f(\cdot)x$ for any $x \in E$.

Conversely, let $M \in \mathcal{K}(X)$ and let $M_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{n_i^\alpha}$ be a net such that $M_\alpha \to M$ in weak* operator topology. Define $m_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \mu_{a_i^\alpha}$ then m_α converges weak* to some mean m in $l_\infty(S)^*$ (passing to a subnet if necessary). Evidently $M(f)x = m(f(\cdot)x)$, $f \in X$, $x \in E$. This completes the proof.

REMARK 5.2. *a*) The above theorem says nothing about the uniqueness of the mean m which « represents » the mean M in $\mathcal{K}(X)$. In general, it is of course not unique since if m and n are means of $l_\infty(S)$ which agree on functions of ths form $f(\cdot)x$ with $f \in X$ and $x \in E$ they « induce » the same mean in $\mathcal{K}(X)$.

(*b*) Because of the preceeding theorem, we can also prove (the more difficult) part of Theorem 4.5 for the case when $X = l_\infty(S, E^*)$, namely that if $l_\infty(S, E^*)$ is right stationary, then there is a left invariant mean in $\mathcal{K}(l_\infty(S, E^*))$. Moreover $x^* \in Z_R(f)$ implies there is a left invariant mean M in $\mathcal{K}(l_\infty(S, E^*))$ such that $M(f) = x^*$ by appealing to [4, Theorem 1] (real case of Theorem 4.5). The argument proceeds like this. By assumption, for each $f \in l_\infty(S, E^*)$, there is a net $f_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha r_{a_i^\alpha} f$ of convex combinations of right translates of f such that f_α coverges in pw^* topology to some constant function x^*. Then for any $x \in E$, $\sum_{i=1}^{n_\alpha} \lambda_i^\alpha r_{a_i^\alpha}(f(\cdot)x) \longrightarrow x^*(x)$ pointwise on S. Since the functions of the form $f(\cdot)x$ with $f \in l_\infty(S, E^*)$, $x \in E$ fill out the whole of $l_\infty(S)$, this means tbat $l_\infty(S)\,(= m(S)$ in the notation of [4]) is right stationary. By [4, Theorem 1], $l_\infty(S)$ has a left invariant mean m. If M is the mean in $\mathcal{K}(l_\infty(S, E^*))$ induced by m (as in Theorem 5.1), then M is a left invariant mean in $\mathcal{K}(l_\infty(S, E^*))$. Also if $x^* \in Z_R(f)$, then (with notations as above $\sum_{=1}^{n_\alpha} \lambda_i^\alpha r_{a_i^\alpha}(f(\cdot)x) \longrightarrow x^*(x)$ pointwise on S. By [4 Theorem 1] again, there is a left invariant mean n on $l_\infty(S)$ such that $n(f(\cdot)x) = x^*(x)$. Since the net f_α depends only on f and not on x, one can assume n to be independent of x(2). Now let N be the left invariant mean

(2) One has to trace the proof of [4, Theorem 1] given in Granirer and Lau [3]. According to their proof (traslated into the present situation), the required mean n is given by $n = \mu \odot \nu$ where μ is any left invariant mean on $l_\infty(S)$ and ν is a cluster point of the net $\nu_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \mu_{a_i^\alpha}$ ($\mu_a(f) = f(a)$, $f \in l_\infty(S)$) of finite means associated with the net of convex combinations of right translates $\sum_{i=1}^{n_\alpha} \lambda_i^\alpha r_{a_i^\alpha}(f(.)x)$ of the function $f(.)x$. The important thing is to observe that the net ν_α is Independent of x. To give the complete detail would amount to proving our Localisation Theorem (Theorem 4.1 or 4.3) again.

in $\mathcal{K}(l_\infty(S, E^*))$ induced by n, then $N(f)x = n(f(\cdot)x) = x^*(x)$ for any $x \in E$. Hence $N(f) = x^*$.

Othe results on invariant means, especially those for locally compact groups, can be extended to vector valued functions. They will be discussed in a forth coming paper by the same authors.

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REFERENCES

[1] M. M. Day, *Amenable semigroups,* Illinois J. Math. 1 (1957) 509-544.

[2] J. Dixmier, *Les moyennes invariantes dans les semi-groupes et leur applications,* Acta Sci. Math. (Szeged) 12 (1950) 213-227.

[3] E. Granirer and Anthony Lau, *A characterisation of locally compact amenable groups* , (to appear in Illinois J. Math.).

[4] T. Mitchell, *Constant functions and left invariant means on semi-groups,* Trans. Amer. Math. Soc. 119 (1965) 244-261.

[5] T. Mitchell, *Function algebras, means and fixed points,* Trans. Amer. Math. Soc. 130 (1968) 117-126.

[6] I. Namioka, *On certain actions of semigroups on L-spaces,* Studia Math. Soc. 29 (1967) 63-77.

[7] R. R. Phelps, *Extreme positive operators and homomorphism,* Trans. Amer. Math. Soc. 108 (1963) 265-274.

[8] C. R. Rao, *Invariant means on spaces of continuous or measurable functions,* Trans. Amer. Math. Soc. 114 (1965) 187-196.

[9] A. P. Robertson and W. J. Robertson, *Topological vector spaces,* Cambridge Tracts in Mathematics and Mathematical Physics, No. 53, Cambridge University Press, 1964.

[10] James C. S. Wong, *Topologically stationary lacally compact groups and amenability,* Trans. Amer. Math. Soc. 144 (1969) 351-363.