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Exact Solutions of the Magneto-Fluid Dynamics: a Contribution.

VINCENZO MILLUCCI (*)

Summary. – *Applying the indirect method of hydrodynamics to the equations of Magneto-Fluid Dynamics (MFD) of an incompressible, viscous and electrically-conducting fluid, we give firstly the exact indefinite solutions for the steady-motion of revolution for parallel straight lines under certain hypotheses for the magnetic field. With such indefinite solutions we treat a particular boundary problem. Secondly we examine the same motion taking also in account the Hall effect. Finally we give a bibliographical appendix concerning papers dealing with the search of solutions of the MFD equations.*

1. – Introduction.

The purpose of this paper is the search of classes of exact solutions for the non-linear, stationary equations of the Magneto-Fuid Dynamics (MFD) describing an incompressible, viscous and electrically-conducting fluid.

We recall such equations in Sect. 2, and in Sect. 3 we discuss the meaning of *exact solution* and also a method which makes it possible to resolve the problem for certain classes of motions.

In Sect. 4 we first determine a class of exact indefinite solutions, relative to an MFD motion of revolution for parallel straight lines, under certain hypotheses for the magnetic field; then, with such indefinite solutions, we treat a specific boundary problem.

In Sect. 5 we study the MFD motion of Sect. 4, taking also in account the Hall effect. The result is that, in this case, the class of solutions given in Sect. 4 is reduced.

In Sect. 6, choosing for the magnetic field other hypotheses than those of Sect. 4, we briefly discuss the same MFD motion.

Finally we present a bibliographical appendix concerning papers dealing with the search of solutions of the MFD equations.

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2. – The MFD equations for the stationary case.

For the steady-motion of an homogeneous, viscous, incompressible, electrically-conducting fluid, the non-linear MFD equations, without Hall effect, are (Gaussian units):

$$\boldsymbol{v}\cdot\operatorname{grad}\boldsymbol{v}-\operatorname{grad}\left(U-\frac{p}{\varrho}\right)+\frac{1}{4\pi\mu\varrho}\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B}-\nu\nabla^2\boldsymbol{v}=0 \tag{2.1}$$

$$\operatorname{div}\boldsymbol{v}=0\,, \tag{2.2}$$

$$\operatorname{curl}(\boldsymbol{v}\times\boldsymbol{B})+\nu_m\nabla^2\boldsymbol{B}=0 \qquad \nu_m=\frac{c^2}{4\pi\mu\sigma}\,, \tag{2.3}$$

$$\operatorname{div}\boldsymbol{B}=0\,. \tag{2.4}$$

Where $\boldsymbol{v}$ is the velocity field, U the potential of non-electromagnetic body forces, p the pressure, ϱ the density (constant), μ the magnetic permeability (constant), $\boldsymbol{B}$ the magnetic induction vector, ν the kinematic viscosity (constant), σ the electrical conductivity (constant), c the speed of light in vacuum and ν_m is the magnetic diffusivity.

Then, the basic equations for the unknown $\boldsymbol{v}$, $\boldsymbol{B}$ and p are (2.1), (2.2) and (2.3) with the condition (2.4) for $\boldsymbol{B}$.

When $\boldsymbol{v}$ and $\boldsymbol{B}$ are known, we can have, if it is of interest, the current density $\boldsymbol{J}$ and the electric field $\boldsymbol{E}$ immediately from:

$$\boldsymbol{J}=\frac{c}{4\pi\mu}\operatorname{curl}\boldsymbol{B}\,, \tag{2.5}$$

$$\boldsymbol{E}=\frac{\boldsymbol{J}}{\sigma}-\frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\,. \tag{2.6}$$

3. – The indirect method.

Let us now examine the method by which we shall search the exact solutions of the equations written in Sect. 2.

Let us at first discuss the concept of exact solution. Practically we assume the definition given in [1] (p. 11), relatively to the hydrodynamical equations for an incompressible, stokesian, viscous fluid.

An exact indefinite solution must satisfy the complete system of equations of Sect. 2, containing therefore also the non-linear terms. Now, if we want the solution to satisfy also the boundary conditions of a real physical problem, the problem becomes, in general, very complicated. If we consider only hydrodynamical problems there are not many of such exact solutions and they are even fewer for MFD problems. Moreover it is necessary to note that an indefinite solution is useful also if at present we do not know a particular problem completely described by such a solution. For instance it could be useful, in the future, for new problems.

Therefore, in agreement with [1], we shall consider as exact solutions, the solutions, also indefinite, satisfying (2.1)-(2.4).

We shall use, in the search of such solutions, the so called *indirect method*, already known in hydrodynamics (see [1], p. 13).

Namely, we assume certain functional dependence on coordinates for the unknowns $\boldsymbol{v}$ and $\boldsymbol{B}$, in order these fields have given symmetries, without particular hypotheses on the boundary containing the fluid. After this, the complete determination of the unknown functions will can follows from (2.1) and (2.4) if the basic system of MFD equations allows such hypothetical solutions. Finally p will be given by integration of (2.1).

In this way we will have a class of exact indefinite solutions. They give some information on the interaction between the magnetic field and the fluid in motion with certain symmetries.

The next step is, clearly, to characterize, if possible, the indefinite solutions applying boundary conditions.

4. – MFD steady-motion of revolution for parallel straight lines.

(*a*) *Indefinite solutions.*

In a system T of orthogonal cylindrical coordinates z, r, φ and unit vectors $\boldsymbol{e}_z$, $\boldsymbol{e}_r$, $\boldsymbol{e}_\varphi$ let v_z, v_r, v_φ, B_z, B_r, B_φ be the physical components of $\boldsymbol{v}$ and $\boldsymbol{B}$. The steady-motion of revolution for parallel straight lines is characterized in T by the following velocity field (see [1], p. 47):

$$v_z = v(r, z) \qquad v_r = 0 \qquad v_\varphi = 0 \tag{4.1}$$

It follows then from (4.1) that:

$$\boldsymbol{a} \equiv \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot \operatorname{grad} \boldsymbol{v} = 0 \tag{4.2}$$

and from the continuity eq. (2.2):

$$v = v(r) \tag{4.3}$$

As for the vector $\boldsymbol{B}$, in agreement with the indirect method previously exposed, we assume:

$$\boldsymbol{B} = B_z(r)\boldsymbol{e}_z + B_r(r)\boldsymbol{e}_r\,. \tag{4.4}$$

From (2.4) it follows:

$$rB_r = A_1 \tag{4.5}$$

where A_1 is an arbitrary constant of integration.

We may note that such a magnetic field is practically obtainable; for instance it is used in electrodynamic loud-speakers as it has been observed in [2], p. 353.

Owing to the singularity of B_r on the z axis, our solution will be of interest only for problems concerning a fluid lying in the region outside a cylinder with axis z and given radius. Furthermore the boundary conditions shall determine the constant A_1, as we shall see in Sect. 4(b).

From the projection of (2.3) on T axis, an identity follows for the $\boldsymbol{e}_\varphi$ component whereas along $\boldsymbol{e}_z$ we have:

$$\nu_m \frac{1}{r}\frac{d}{dr}\left(r\frac{dB_z}{dr}\right) + \frac{1}{r}\frac{d}{dr}(rvB_r) = 0 \tag{4.6}$$

and along $\boldsymbol{e}_r$:

$$\nu_m\left[\frac{1}{r}\frac{d}{dr}\left(r\frac{dB_r}{dr}\right) - \frac{B_r}{r^2}\right] = 0\,.$$

On account of (4.5), this last equation is identically satisfied, whereas (4.6) becomes:

$$\nu_m \frac{1}{r}\frac{dB_z}{dr} + \nu_m\frac{d^2B_z}{dr^2} + \frac{A_1}{r}\frac{dv}{dr} = 0\,. \tag{4.7}$$

Let us now consider the equation obtained applying the curl operator to (2.1). In analogy with the similar hydrodynamical situation, (see [1], p. 3), we shall call this further equation *compatibility equation*.

This equation has only one component different from zero, that is the projection along $\boldsymbol{e}_\varphi$; hence we have:

$$\nu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) + \frac{1}{4\pi\mu\varrho}\frac{A_1}{r}\frac{dB_z}{dr} = A_2 \tag{4.8}$$

where A_2 is an arbitrary constant of integration.

The problem is now reduced to the determination of the unknown functions v and B_z, satisfying the system of differential eqs. (4.7) and (4.8). We can rewrite these in the form:

$$\left\{\begin{array}{l}\dfrac{d^2v}{dr^2}+\dfrac{1}{r}\dfrac{dv}{dr}+\dfrac{h}{r}\dfrac{dB_z}{dr}=k\\[2ex] \dfrac{d^2B_z}{dr^2}+\dfrac{1}{r}\dfrac{dB_z}{dr}+\dfrac{\gamma}{r}\dfrac{dv}{dr}=0\end{array}\right. \tag{4.9}$$

where:

$$h=\frac{A_1}{4\pi\mu\varrho\nu}\qquad k=\frac{A_2}{\nu}\qquad \gamma=\frac{A_1}{\nu_m} \tag{4.10}$$

Once v and B_z will be determined from (4.9), the pressure will be obtained integrating (2.1).

Eliminating dv/dr from the first and the second equation in (4.9) it follows:

$$\frac{d^3B_z}{dr^3}+\frac{3}{r}\frac{d^2B_z}{dr^2}+\frac{1}{r^2}(1-\gamma h)\frac{dB_z}{dr}=-\frac{\gamma k}{r} \tag{4.11}$$

whose solution, if $\gamma h\neq 4$, is:

$$B_z=Ar^{k_1}+Pr^{k_2}+C-\frac{\gamma k}{2(4-\gamma h)}r^2 \tag{4.12}$$

with A, P and C arbitrary constants and $K_2^1=\pm\sqrt{\gamma h}$.

Hence for v we have:

$$v=-\frac{A}{\gamma}k_1r^{k_1}-\frac{P}{\gamma}k_2r^{k_2}+\frac{k}{4-\gamma h}r^2+D \tag{4.13}$$

where D is an arbitrary constant.

Solutions similar to (4.12) and (4.13) are indicated also in the paper [2] devoted to the study of particular cylindrical waves in MFD.

For the pressure, from (2.1) it follows:

$$U-\frac{p}{\varrho}=-A_2z+\frac{B_z^2}{8\pi\mu\varrho}+\text{constant} \tag{4.14}$$

The term containing B_z is not present in the analogous expression for a purely hydrodynamical problem; it is an effect of the fluid-field interaction.

If $\gamma h = 4$ the solutions for v and B_z are:

$$B_z = A_* r^2 + B^* r^{-2} + C_* - \frac{\gamma k}{8}(r^2 + r^2 \lg r) \tag{4.15}$$

$$v = -\frac{2A_*}{\gamma} r^2 + \frac{2P_*}{\gamma} r^{-2} + \frac{3}{8} k r^2 + \frac{k}{4} r^2 \lg r + D_* \tag{4.16}$$

with A_*, P_*, C_* and D_* arbitrary constants. The pressure is given, also in this case, by (4.14) with B_z by (4.15).

(*b*) *Boundary conditions.*

Let the fluid be in the region delimitated by two indefinite cylindrical surfaces, of z axis, inner radius a and outer radius b. If the inner surface is fixed and the outer is moving with velocity:

$$\boldsymbol{u} = u\boldsymbol{e}_z \qquad (u \text{ given constant})$$

the boundary conditions for $\boldsymbol{v}$ are:

$$v(a) = 0 \qquad v(b) = u\,. \tag{4.17}$$

In the case $\gamma h \neq 4$ the indefinite solutions for v and B_z can be written:

$$v = c_1 r^{k_1} + c_2 r^{k_2} + c_3 r^2 + c_4 \tag{4.18}$$

$$B_z = -\frac{\gamma}{k_1} c_1 r^{k_1} - \frac{\gamma}{k_2} c_2 r^{k_2} - \frac{\gamma}{2} c_3 r^2 + c_5 \tag{4.19}$$

with c_1, c_2, c_3, c_4 and c_5 arbitrary constants.

Applying the condition (4.17) to (4.18) it follows:

$$v = c_1\left[r^{k_1} - a^{k_1} - \left(\frac{b^{k_1} - a^{k_1}}{b^{k_2} - a^{k_2}}\right)(r^{k_2} - a^{k_2})\right] + \frac{u}{b^{k_2} - a^{k_2}}(r^{k_2} - a^{k_2}) + \\ + c_3\left[r^2 - a^2 - \left(\frac{b^2 - a^2}{b^{k_2} - a^{k_2}}\right)(r^{k_2} - a^{k_2})\right].$$

Let us now consider the electromagnetic boundary-conditions.

Suppose that the applied magnetic field is:

$$\boldsymbol{B}_0 = \frac{\alpha}{r}\,\boldsymbol{e}_r \tag{4.21}$$

where α is a given constant (we have already see such a field in Sect. 4(a)).

The continuity condition for the normal component of $\boldsymbol{B}$ across the boundary containing the fluid, is then satisfied, by the solution given in Sect. 4(a), assuming $A_1 = \alpha$.

We note that, in this problem, the applied field is not modified by the interaction with the electrically-conducting fluid; that is, the induced magnetic field is only along the z axis.

For the axial component of $\boldsymbol{B}$ the boundary condition, in the present case, is (see [3], p. 352):

$$B_z(b) = 0\,. \tag{4.22}$$

Introducing (4.22) in (4.19) we can determine c_5:

$$c_5 = \frac{\gamma}{k_1}\,c_1 b^{k_1} + \frac{\gamma}{k_2}\,c_2 b^{k_2} + \frac{\gamma}{2}\,c_3 r^2\,. \tag{4.23}$$

Having no other conditions for $\boldsymbol{v}$ and $\boldsymbol{B}$, the constants c_1 and c_3 remain undetermined. On the other hand c_3 is connected to the pressure gradient; in effect from (4.14), (4.13) and (4.18) it follows:

$$\frac{\partial}{\partial z}\left(U - \frac{p}{\varrho}\right) = \nu(\gamma h - 4)\,c_3\,. \tag{4.24}$$

Finally the last arbitrary constant c_1 can be useful to satisfy the boundary conditions, if any, for the electric field.

Similar observations can be done for the case $\gamma h = 4$; for instance the expression of v which satisfies the (4.17) is:

$$\begin{aligned} v = {} & c_1^*\left[r^2 - a^2 - \left(\frac{b^2 - a^2}{b^{-2} - a^{-2}}\right)(r^{-2} - a^{-2})\right] + \frac{u}{b^{-2} - a^{-2}}\,(r^{-2} - a^{-2}) + \\ & + c_3^*\left[\frac{3}{8}\,r^2 + \frac{r^2}{4}\lg r - \left(\frac{3}{8}\,a^2 + \frac{a^2}{4}\lg r\right) - \right. \\ & \left. - \frac{(3/8)b^2 + (b^2/4)\lg b - (3/8)a^2 - (a^2/4)\lg b}{b^{-2} - a^{-2}}\,(r^{-2} - a^{-2})\right] \end{aligned}$$

with c_1^* and c_3^* arbitrary constants.

5. – MFD motion of revolution for parallel straight lines with Hall effect.

In Sect. 2 we have assumed the following form of the Ohm's Law (see (2.6)):

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right). \tag{5.1}$$

However it is well-known that in case $\omega\tau \gg 1$, where ω is the cyclotron frequency of the charged particles and τ is the mean time between particle collisions, the Hall effect is also important. This implies an anisotropic electrical conductivity and, in most MFD problems, (5.1) is suitably substituted by (see [4] and Bibliography there mentioned):

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c} - \beta_H \boldsymbol{J}\times\boldsymbol{B}\right) \tag{5.2}$$

where β_H is the Hall coefficient. It is easy to verify that only (2.3) must be replaced with the following (see [4], eq. (4.7)):

$$\nu_m \nabla^2 \boldsymbol{B} + \operatorname{curl} \boldsymbol{v}\times\boldsymbol{B} + \beta \operatorname{curl}(\boldsymbol{B}\times\operatorname{curl}\boldsymbol{B}) = 0 \tag{5.3}$$

where $\beta = \beta_H c^2/4\pi\mu$.

(4.5) remain unaltered and the components of (5.3) are:

$$\nu_m \nabla^2 B_z + \frac{1}{r}\frac{d}{dr}(rvB_r) = 0 \tag{5.4}$$

$$\nabla^2 B_r - \frac{B_r}{r^2} = 0 \tag{5.5}$$

$$\beta \frac{d}{dr}\left(\frac{dB_z}{dr} B_r\right) = 0 \tag{5.6}$$

and we will still have the compatibility equation:

$$\nu \frac{d}{dr}(\nabla^2 v) + \frac{1}{4\pi\mu\varrho}\frac{d}{dr}\left[\frac{dB_z}{dr} B_r\right] = 0. \tag{5.7}$$

It follows from (4.5) that (5.5) is identically satisfied, whereas (5.6) gives:

$$\frac{dB_z}{dr} = \frac{C_1}{A_1} r$$

with c_1 arbitrary constant and also:

$$B_z = \frac{C_1}{2A_1} r^2 + \text{constant}. \tag{5.8}$$

Therefore in the presence of the Hall effect, B_z will have solutions of parabolic type.

Using (4.5) and (5.8), from (5.4) it follows:

$$\frac{2C_1}{A_1} \nu_m + \frac{A_1}{r} \frac{dv}{dr} = 0$$

that is:

$$v = -\frac{\nu_m}{A_1^2} c_1 r^2 + \text{constant}. \tag{5.9}$$

Then, from (5.6) and (5.9), the compatibility eq. (5.7) is satisfied.

For the pressure, finally, in the usual way, we have:

$$U - \frac{p}{\varrho} = c_5 z + \frac{B_z^2}{8\pi\mu\varrho} + \text{constant} \tag{5.10}$$

where

$$c_5 = \frac{4\nu\nu_m}{A_1^2} c_1 + \frac{c_1}{4\pi\mu\varrho}. \tag{5.11}$$

In conclusion, the solution for the MFD steady-motion of revolution for parallel straight lines of an homogeneous, incompressible, viscous and electrically conducting fluid in the presence of the Hall effect are:

$$\left\{\begin{array}{lll} B_z = \dfrac{c_1}{2A_1} r^2 + \text{constant} & B_r = \dfrac{A_1}{r} & B_\varphi = 0 \\ v_z = -\dfrac{\nu_m}{A_1^2} c_1 r^2 + \text{constant} & \nu_r = 0 & v_\varphi = 0 \\ p = \varrho \left[U - c_5 z - \dfrac{B_z^2}{8\pi\mu\varrho} \right] + \text{constant} & & \end{array}\right. \tag{5.12}$$

where A_1 and c_1 are arbitrary constants and c_5 is given by (5.11).

We note that the solutions (5.12) are different from the solutions given in Sect. 4, in absence of the Hall effect.

Now we no longer have either the terms like r^{k_1} and r^{k_2} nor the distinction between the case $\gamma h = 4$ and the case $\gamma h \neq 4$.

The pressure dependence is like that founded in absence of the Hall effect.

We can now treat, using (5.12), the same boundary conditions given in Sect. 4(b).

Here it is $A_1 = \alpha$, with α arbitrary constant. From (4.17) it follows for v:

$$v = \frac{u}{b^2 - a^2}(r^2 - a^2)\,.$$

Therefore B_z will be:

$$B_z = -\frac{\alpha}{2\nu_m}\frac{u}{b^2 - a^2}r^2 + \text{constant}$$

and from (4.22):

$$B_z = \frac{\alpha}{2\nu_m}\frac{u}{b^2 - a^2}(b^2 - r^2)$$

for $a \leqslant r \leqslant b$.

The pressure, finally, is still given by (5.10) with

$$A_1 = \alpha \qquad \text{and} \qquad c_1 = -\frac{\alpha u^2}{\nu_m(b^2 - a^2)}$$

6. – Different hypotheses on B.

Without changing the velocity field (4.1), we can now assume:

$$\boldsymbol{B} = B_z(r)\boldsymbol{e}_z + B_r(z)\boldsymbol{e}_r\,. \tag{6.1}$$

In this case from (2.4) follows:

$$B_r = 0\,. \tag{6.2}$$

This implies $\boldsymbol{v}\times\boldsymbol{B} = 0$. Motions with such conditions have already been studied (see [5] and [6]), but without taking in account a viscous fluid with a finite conductivity. In this case and under our hypotheses, we have from the basic equations of Sect. 2:

$$v \;= c_1 \lg r + c_2 r^2 + c_3 \tag{6.3}$$

$$B_z = c_4 \lg r + c_5 \tag{6.4}$$

with c_1, c_2, c_3, c_4 and c_5 arbitrary constants. Moreover:

$$U - \frac{p}{\varrho} = -c_2 z + \frac{B_z^2}{8\pi\mu\varrho} + \text{constant}.$$

If we consider also the Hall effect, with the conditions (4.1) and (6.1), it follows from the equations of Sect. 5 that $\boldsymbol{v}$ and $\boldsymbol{B}$ are still given by (6.2), (6.3) and (6.4).

Therefore, in this case, solutions of the steady-motion are not modified by the Hall effect.

Even further hypotheses are possible for $\boldsymbol{B}$; for instance if we take

$$\boldsymbol{B} = B_z(z)\boldsymbol{e}_z + B_r(r)\boldsymbol{e}_r$$

or

$$\boldsymbol{B} = B_z(z)\boldsymbol{e}_z + B_r(z)\boldsymbol{e}_r$$

it is easy to verify that, considering or not the Hall effect, the solutions for $\boldsymbol{v}$ and $\boldsymbol{B}$ are of no interest; in effect either they do not exist, or they describe a purely hydrodynamical situation.

7. – Bibliographical appendix.

The papers from [7] to [45], as well as [2], [5] and [6], regard the search of solutions of the MFD equations.

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