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On the Trace of Potentials.

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0. – Introduction.

Much work has been devoted to the study of the trace of various classes of differentiable functions in $\mathbb{R}^n$ on submanifolds of dimension $< n$. The trace of the Sobolev class W_p^1 on a hypersurface was completely determined by Aronszajn [4] and Babič-Slobodeckij [6] if $p = 2$ and by Gagliardo [12] if p general. The case of the space of (Bessel) potentials L_p^α was treated by Stein [20]. In a recent paper [1] Adams has initiated the study of the trace of L_p^α on an arbitrary closed set F in R^n. E.g. he showed that if F carries a b-dimensional measure ν (see def. 3.1) then for $u \in L_p^\alpha$ holds

$$(0.1) \qquad \Big(\int |u(x+h) - u(x)|^q \, d\nu(x)\Big)^{1/q} = O(|h|^\beta) \qquad \text{if } \beta = \alpha - n/p + b/q, \ q > p .$$

The same problem has been approached from a somewhat different angle by Jonsson [15] (see also Wallin [21], Sjödin [19], Blomquist-Jonsson [7] for earlier work along these lines). In particular Jonsson obtained the limiting case $p = q$ of (0.1) ([15], th. 2). In [2] Adams gives a simplified proof of estimates of the type (0.1).

In 1966 we wrote a paper [16] where we systematically applied the theory of interpolation spaces to some classical operators: Hilbert transform, potential transform etc. It is the purpose of this note to show how these methods can be used in connection with the results of Adams and Jonsson. The basic tool from interpolation space theory is the relatively simple th. 1.3. With some imagination one can say that it is implicit in [16]. Although no genuinely new results are obtained, we do hope that the present note will serve to clarify several points, in particular the precise interrelation between the results of Adams and Jonsson.

The organization of the note is as follows. Section 1 is devoted to a

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quick review of some abstract interpolation theorems, of which thus th. 1.3 is the result which will be used here. In Section 2 we prove estimates for general integral operators acting in L_p spaces. In Section 3 the latter are then applied in the special case of the (Riesz) potential. (Whether we use Riesz potentials or Bessel potentials here is to some extent immaterial.)

1. – Abstract interpolation theorems.

By a Banach couple we mean an entity $\boldsymbol{A} = \{A_0, A_1\}$ consisting of two Banach spaces A_0 and A_1 both continuously embedded in one and the same Hausdorff topological vector space, say, $\mathcal{A}$. With a Banach couple $\boldsymbol{A} = \{A_0, A_1\}$ one can associate certain spaces (« K-spaces ») $\boldsymbol{A}_{\theta q} = (A_0, A_1)_{\theta q}$, where θ and q are parameters, $0 < \theta < 1$, $1 \leqslant q \leqslant \infty$, which are also all continuously embedded in $\mathcal{A}$.

The basic property of K-spaces is the following interpolation theorem. (For more information see [16], Section 1; a complete treatment can be found e.g. in [8], chap. 3).

THEOREM 1.1. Let there be given two Banach couples $\boldsymbol{A} = \{A_0, A_1\}$ and $\boldsymbol{B} = \{B_0, B_1\}$ and denote by T a linear operator. Then holds:

$$T \in \mathfrak{L}(A_0, B_0) \cap \mathfrak{L}(A_1, B_1) \Rightarrow T \in \mathfrak{L}(\boldsymbol{A}_{\theta q}, \boldsymbol{B}_{\theta q}) \, (^1) \, .$$

Moreover for the operator norms involved holds the convexity inequality:

$$\|T\|_{\mathfrak{L}(A_{\theta q}, B_{\theta q})} \leqslant \|T\|^{1-\theta}_{\mathfrak{L}(A_0, B_0)} \|T\|^{\theta}_{\mathfrak{L}(A_1, B_1)} \, .$$

PROOF. Use the inequality

$$(1.1) \qquad K(t, Ta; B) \leqslant M_0 K(M_1 t / M_0, a; A) \qquad \big(\text{with } M_i = \|T\|_{\mathfrak{L}(A_i, B_i)} (i = 0, 1)\big)$$

and the definition of the K-spaces.

As has been pointed out in [17], considering the « operator couple » $\mathfrak{L}(\boldsymbol{A}, \boldsymbol{B}) = \{\mathfrak{L}(A_0, B_0), \mathfrak{L}(A_1, B_1)\}$, one can generalize th. 1.1 as follows:

(1) If A is a Banach space continuously embedded in $\mathcal{A}$ and B a Banach space continuously embedded in $\mathcal{B}$ (which is the Hausdorff topological vector space corresponding to B) $T \in \mathfrak{L}(A, B)$ means that the restriction of T to A maps A continuously into B.

THEOREM 1.2. We have:

$$T \in (\mathfrak{L}(A_0, B_0), \mathfrak{L}(A_1, B_1))_{\theta 1} \Rightarrow T \in \mathfrak{L}(\boldsymbol{A}_{\theta q}, \boldsymbol{B}_{\theta q}) .$$

In this note we shall however need the following variant of these results.

THEOREM 1.3. We have:

$$T \in (\mathfrak{L}(A_0, B_0), \mathfrak{L}(A_1, B_1))_{\theta \infty} \Rightarrow T \in \mathfrak{L}(\boldsymbol{A}_{\theta 1}, \boldsymbol{B}_{\theta \infty}) .$$

PROOF. Let ω be a fixed number $\neq 1$. That $T \in (\mathfrak{L}(A_0, B_0), \mathfrak{L}(A_1, B_1))_{\theta\infty}$ signifies that we have

$$T = \sum_{j=-\infty}^{\infty} T_j , \qquad \|T_j\|_{\mathfrak{L}(A_0, B_0)} \leqslant C\omega^{j\theta} , \qquad \|T_j\|_{\mathfrak{L}(A_1, B_1)} \leqslant C\omega^{j(\theta-1)} .$$

Let $a \in \boldsymbol{A}_{\theta 1}$. From (1.1), with $M_i = \omega^{j(\theta - i)}$ $(i = 0, 1)$, follows

$$K(t, T_j a; B) \leqslant C\omega^{j\theta} K(t/\omega^j, a; A) .$$

Therefore since $Ta = \sum_{j=-\infty}^{\infty} T_j a$ we get

$$K(t, Ta; B) \leqslant C \sum_{j=-\infty}^{\infty} \omega^{j\theta} K(t/\omega^j, a; A) . \tag{1.2}$$

It is readily seen that (1.2) implies $Ta \in \boldsymbol{B}_{\theta\infty}$.

2. – Estimates for integral operators.

Let X and Y be spaces equiped with positive measures μ and ν. Let $L_p = L_p^X$ and $L_q = L_q^Y$ be the corresponding Lebesgue spaces. More generally, let $L_p(E) = L_p^X(E)$ and $L_q(E) = L_q^Y(E)$ be the corresponding Lebesgue spaces in the vector valued case, E denoting any Banach space. Similarly L_{pr} and L_{qr} are the Lorentz spaces (see [16]). Consider the integral operator

$$Tf(y) = \int k(y, x) f(x) \, d\mu(x)$$

where the kernel k is assumed to be (μ, ν)-measurable. Our first result is presumably classical.

THEOREM 2.1. Assume that

$$k \in L_{\infty}^{Y}(L_{r}^{X}), \quad 1 \leqslant r \leqslant \infty \tag{2.1}$$

$$k \in L_{\infty}^{X}(L_{s}^{Y}), \quad 1 \leqslant s \leqslant \infty . \tag{2.2}$$

Then

$$T \in \mathfrak{L}(L_p, L_q) \quad \text{where } r/p' + s/q = 1,\ 1 \leqslant p \leqslant r',\ s \leqslant q \leqslant \infty \text{ (}^2\text{)} . \tag{2.3}$$

Also

$$\| T \|_{\mathfrak{L}(L_p, L_q)} \leqslant \| k \|_{L_{\infty}(L_r)}^{r/p'} \| k \|_{L_{\infty}(L_s)}^{s/q} . \tag{2.4}$$

PROOF. From (2.1) using Hölder's inequality we get $T \in \mathfrak{L}(L_r, L_\infty)$. Similarly using Minkowsky's inequality (conveniently viewed as a sort of dual of Hölder's inequality) (2.2) gives $T \in \mathfrak{L}(L_1, L_s)$. By the M. Riesz interpolation theorem interpolation between these endpoint results clearly yields (2.3) and (2.4). See Fig. 2.1 below

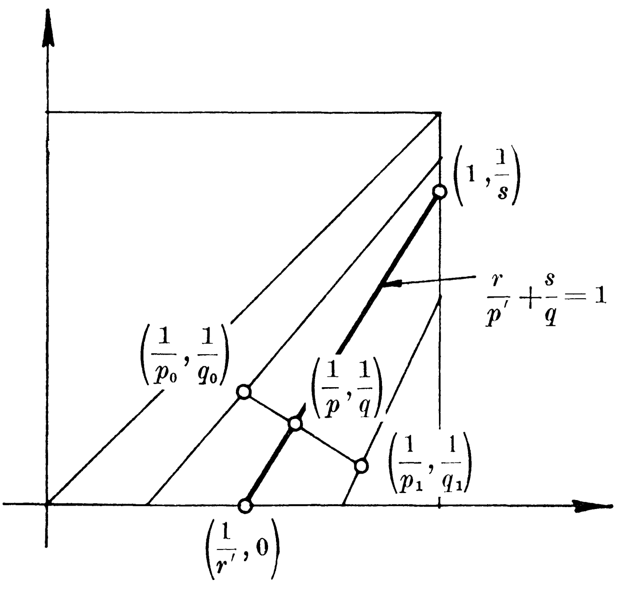

Figure 2.1

We now sharpen th. 2.1 as follows.

(2) Here and in what follows p' is the conjugate power of p, $1/p + 1/p' = 1$.

THEOREM 2.2. Assume that

(2.5) $$k \in L^Y_\infty(L^Y_{r\infty}), \quad 1 \leqslant r \leqslant \infty$$

(2.6) $$k \in L^X_\infty(L^Y_{s\infty}), \quad 1 \leqslant s \leqslant \infty$$

(2.7) not both $r = 1$ and $s = 1$.

Then

(2.8) $T \in \mathfrak{L}(L_p, L_q)$ where $r/p' + s/q = 1$, $1 < r < p'$, $s < q < \infty$.

PROOF. Consider the sets

$$e_j = \{(y, x) | 2^j \leqslant k(y, x) < 2^{j+1}\} \subset Y \times X \quad (j = 0, \pm 1, \pm 2, \ldots)$$

and define k_j and T_j by

$$k_j(y, x) = \begin{cases} k(y, x) & \text{if } (y, x) \in e_j \\ 0 & \text{elsewhere} \end{cases}$$

$$T_j f(y) = \int k_j(y, x) f(x) \, d\mu(x) .$$

Then clearly $T = \sum_{j=-\infty}^{\infty} T_j$. Moreover we also have for any $\bar{r}$ and $\bar{s}$

$$\|k_j\|_{L^Y_\infty(L)^X_{\bar{r}}} \leqslant C 2^{j(1-r/\bar{r})}$$

$$\|k_j\|_{L^X_\infty(L^Y_{\bar{s}})} \leqslant C 2^{j(1-s/\bar{s})}$$

for we have

$$\text{meas } e^y_j \leqslant C 2^{-jr}$$

$$\text{meas } e^x_j \leqslant C 2^{-js} .$$

Here, for $y \in Y$, e^y_j is the projection on X of $e_j \cap \{(y, x) | x \in X\}$ and similarly, for $x \in X$, e^x_j the projection on Y of $e_j \cap \{(y, x) | y \in Y\}$. Choose p_0, p_1, q_0, q_1, r_0, r_1, s_0, s_1 such that with some $0 < \theta < 1$ holds

$$1/p = (1-\theta)/p_0 + \theta/p_1 , \quad 1/q = (1-\theta)/q_0 + \theta/q_1 ,$$

$$r_i/p'_i + s_i/q_i = 1 (i = 0, 1) , \quad r(1/p_1 - 1_q p_0) + s(1/q_0 - 1/q_1) = 0 .$$

(This is possible in view of (2.6). The reader might want to contemplate fig. 2.1 again.) Then follows from (2.4) that (take $\bar{r} = r_i$, $\bar{s} = s_i$)

$$\|T_j\|_{\mathfrak{L}(L_{p_i}, L_{q_i})} \leqslant C\, 2^{j(1-r/p_i'-s/q_i)} \; (i = 0, 1)\,.$$

This we can rewrite, with $\omega = 2^{r(1/p_1-1/p_0)+s(1/q_0-1/q_1)}$, as

$$\|T_j\|_{\mathfrak{L}(L_{pi}, L_{qi})} \leqslant C\omega^{j(\theta-i)} \qquad (i = 0, 1)\,.$$

It is now plain that th. 1.3 (or rather its proof) can be applied. Since

$$(L_{p_0}, L_{p_1})_{\theta 1} = L_{p1}\,, \qquad (L_{q_0}, L_{q_1})_{\theta\infty} = L_{q\infty}$$

we get $T \in \mathfrak{L}(L_{p1}, L_{q\infty})$. Another interpolation now yields the desired conclusion (2.10).

REMARK 2.1. For convolution operators th. 2.1 and 2.2 are essentially the inequalities of Young and O'Neil (see [16]). Th. 2.2 is substantially [2], cor. of th. A. (It would not have been difficult to cover also [2], th. A itself in full.) An alternative proof of th. 2.2 (not using th. 1.3) could have been based on the Lorentz version of th. 2.1.

Now we consider the extremal case $p = q$. Since then $r/p' + s/p = 1$ we must have $r = s = 1$. (Note that in any case $r/p' + s/q = 1$ implies $p \leqslant q$.) In other words (2.7) is violated so anyhow th. 2.2 does not apply. Th. 2.1 gives us however the following classical result:

$$k \in L_\infty(L_1) \cap L_\infty(L_1) \Rightarrow T \in \mathfrak{L}(L_p, L_p)\,, \qquad 1 \leqslant p \leqslant \infty\,. \tag{2.9}$$

REMARK 2.2. Indeed, the matrix version of (2.9), with $p = 2$ at least, goes back to Frobenius [11] (1909) and Schur [18] (1911). This is related to Frobenius' theorem on positive matrices but also to Hilbert's double series theorem (see Hardy-Littlewood-Polya [14], chap. 9). A continuous version, also with $p = 2$, appears in Carleman [9] (1923), who refers to Holmgren. The same argument appears also in the proof of the Kolmogoror-Seliverstov-Plessner theorem in the theory of orthogonal series (1928) (see e.g. [3]).

Now we want to give an extension of (2.9).

THEOREM 2.3. Assume that

$$k \in L_\infty^Y(L_r^X)\,, \qquad 0 < r \leqslant \infty \tag{2.10}$$

$$k \in L_\infty^X(L_s^Y)\,, \qquad 0 < s \leqslant \infty\,. \tag{2.11}$$

Then

$$T \in \mathfrak{L}(L_p, L_p) \quad \text{where } r/p' + s/p = 1,\ 1 \leqslant p \leqslant \infty . \tag{2.12}$$

Also

$$\|T\|_{\mathfrak{L}(L_p, L_p)} \leqslant \|k\|^{r/p'}_{L_\infty(L_r)} \|k\|^{s/q}_{L_\infty(L_s)} . \tag{2.13}$$

PROOF. It suffices to estimate the integral

$$I = \int\int k(y, x)\, g(y)\, f(x)\, d\nu(y)\, d\mu(x)$$

where $\|f\|_{L_p} = \|g\|_{L_p} = 1$. We rewrite it as

$$I = \int\int \big(k(y, x)\big)^{s/p} f(x) \cdot \big(k(y, x)^{r/p'} g(y)\, d\nu(y)\, d\mu(x) .$$

Then Hölder's inequality clearly yields

$$|I| \leqslant \|k\|^{s/p}_{L_\infty(L_s)} \|k\|^{r/p'}_{L_\infty(L_r)} .$$

which is all we have to verify.

Using interpolation (the argument of the proof of th. 2.2) we also readily obtain.

THEOREM 2.4. Assume

$$k \in L^Y_\infty(L^X_{r\infty}), \quad 0 < r \leqslant \infty \tag{2.14}$$

$$k \in L^X_\infty(L^Y_{s\infty}), \quad 0 < s \leqslant \infty \tag{2.15}$$

and also (2.7). Then

$$T \in \mathfrak{L}(L_{p1}, L_{p\infty}) \quad \text{where } r/p' + s/p = 1,\ 1 < p < \infty . \tag{2.16}$$

Finally we also consider the case $q < p$. (This will not be used in Section 3.) The proof is by an easy adaptation of the argument of the proof of th. 2.3.

THEOREM 2.5. Assume

$$k \in L^Y_\infty(L^X_r), \quad 0 < r \leqslant \infty \tag{2.17}$$

$$k \in L^X_\infty(L^Y_s), \quad 0 < s \leqslant \infty \tag{2.18}$$

$$k \in L^{X \times Y}_\delta, \quad 0 < \delta \leqslant \infty . \tag{2.19}$$

Then we have

$$(2.20) \qquad T \in \mathfrak{L}(L_p, L_q) \quad \text{if } 1 - r/q' - s/p = \delta(1/q - 1/p)\,, \quad 1 \leqslant q \leqslant p \leqslant \infty\,.$$

Also

$$(2.21) \qquad \|T\|_{\mathfrak{L}(L_p, L_q)} \leqslant \|k\|_{L_\delta}^{\delta(1/q-1/p)} \|k\|_{L_\infty(L_r)}^{r/q'} \|k\|_{L_\infty(L_s)}^{s/p}\,.$$

REMARK 2.3. Condition (2.19) originates from Aronszajn (see Aronszajn-mulla-Szeptycki [5], see also Gagliardo [13]).

The interpolated version of th. 2.5 (if there is any) is left as an excercise for the reader.

3. – Application to potentials.

We want to apply the results of Section 2 with $X = Y = R^n$, $k(y, x) = |x - y|^{\alpha - n}$ $(0 < \alpha < n)$. Besides α (the order of differentiation) we also use $d = n - \alpha$ which has an obvious interpretation as dimension.

DEFINITION 3.1. A positive measure μ in R^n is termed a-dimensional $(0 \leqslant a \leqslant n)$ if we have for any ball K_r of radius r the estimate

$$(3.1) \qquad \mu(K_r) \leqslant Cr^a$$

with C depending only on μ.

EXAMPLE 3.1. If $n = 1$, thus $0 \leqslant a \leqslant 1$, $d\mu = |x|^{\alpha - 1}dx$ will do.

Given μ we also denote by $L_p^\alpha(\mu)$ $(0 < \alpha < n)$ the space of functions u which can be represented in the form

$$u(x) = \int |x - y|^{\alpha - n} f(y)\, d\mu(y)$$

with $f \in L_p(\mu)$. Then we have

THEOREM 3.1. Let μ be a-dimensional, ν b-dimensional. Then

$$(3.2) \qquad u \in L_p^\alpha(\mu) \Rightarrow u \in L_q(\nu) \quad \text{if } n - \alpha = a/p' + b/q,\ d \leqslant a,\ p > 1\,.$$

REMARK 3.1. This is Adams [1], th. 2, if $a = n$, $\mu =$ Lebesgue measure.

PROOF. Apply th. 2.1 in conjunction with the following

LEMMA 3.1. Let μ be a-dimensional. Then $k = |x|^{-d} \in L_{r\infty}(\mu)$ if $d = a/r$.

PROOF. It suffices to estimate the measure of the set where $\lambda \leqslant k < 2\lambda$, i.e. the annulus $(2\lambda)^{-1/d} < |x| \leqslant \lambda^{-1/d}$. By (3.1) this measure is $\leqslant \lambda^{-a/d}$. We get $|x|^{-d} \in L_{r\infty}(\mu)$.

We have also

THEOREM 3.2. Assume again μ a-dimensional, ν b-dimensional. Then

$$u \in L_p^\alpha(\mu) \Rightarrow \left\{ \int |u(x+h) - u(x)|^q d\nu(x) \right\}^{1/q} = O\left(|h|^\beta\right) \tag{3.3}$$

if $n - \alpha + \beta = a/p' + b/q$, $d + \beta \leqslant a$, $q > p > 1$, $0 < \beta \leqslant 1$.

REMARK 3.2. This is Adams [1], th. 3, if $a = n$, $\mu =$ Lebesgue measure.

PROOF. We now apply instead

LEMMA 3.2. Let μ be a-dimensional. Then $l = l_h = |x+h|^{-d} - |x|^{-d} \in L_{r\infty}(\mu)$ if $d + 1 \geqslant a/r \geqslant d$. Moreover $\|l_h\| = O\left(|h|^\beta\right)$ if $d + \beta = a/r$, $0 < \beta \leqslant 1$.

PROOF. It suffices to consider the case $|h| = 1$. That the restriction of l to $\{|x| \leqslant 2\}$ belongs to $L_{r\infty}(\mu)$ follows from lemma 3.1. That its restriction to $\{|x| > 2\}$ is in $L_{r\infty}(\mu)$ follows in the same way if we use the estimate $l = O\left(|x|^{-d-1}\right)$, $x \to \infty$.

More generally we have

THEOREM 3.3. Let again μ be a-dimensional, ν b-dimensional. Then

$$\begin{aligned} &u \in L_p^\alpha(\mu) \Rightarrow \\ &\left(\int |u(x+h) - u(x) - h \cdot Du(x) - \ldots - 1/N \cdot h^N \cdot D^N u(x)|^q d\nu(x) \right)^{1/q} \\ &= O\left(|h|^\beta\right) \quad \text{if } n - \alpha + \beta = a/p' + b/q,\ d + \beta \leqslant a,\ q > p > 1, \\ &N < \beta \leqslant N + 1 \ (^3). \end{aligned} \tag{3.4}$$

PROOF. Analogous to the case of th. 3.2.

Finally we consider the limiting case $q = p$.

(3) Here $D^N u$ denotes the N-th tensorial derivative of u and $h^N \cdot D^N$ its scalar product with the N-th tensorial product of h.

THEOREM 3.4. Let again μ be a-dimensional, ν b-dimensional. Then

$$(3.5)\qquad u \in L_p^\alpha(\mu) \Rightarrow \Big(\int |u(x+h)-u(x)|^p\, d\nu(x)\Big)^{1/p} = O(|h|^\beta)$$

if $\beta = \alpha + a - n + (a-b)/p,\ d+\beta \leqslant a,\ p>1,\ 0<\beta<1$.

REMARK 3.2. This is Jonsson [15], th. 2 if $a = n$, $\mu =$ Lebesgue measure.
PROOF. Same as for th. 3.2 but we use now th. 2.3.
We also discuss some variants of th. 3.4.

1) Let us introduce the «Besov spaces» $B_p^{\alpha r}(\mu)$ by setting

$$B_p^{\alpha r}(\mu) = \big(L_p^{\alpha_0}(\mu), L_p^{\alpha_1}(\mu)\big)_{\theta r} \qquad \text{with } \alpha = (1-\theta)\alpha_0 + \theta\alpha_1 .$$

(If $\mu =$ Lebesgue measure, these spaces are known to coincide with the usual Besov of Lipschitz spaces (cf. e.g. [15]).) A simple interpolation argument now shows that (3.5) can be sharpened into

$$(3.8)\qquad u \in B_p^{\alpha\infty}(\mu) \Rightarrow \Big(\int |u(x+h)-u(x)|^p\, d\nu(x)\Big)^{1/q} = O(|h|^\beta) .$$

2) By interpolation we can also prove that

$$(3.9)\qquad u \in B_p^{\alpha p}(\mu) \Rightarrow \left(\iint \frac{|u(x)-u(y)|^p}{|x-y|^{\beta p}} \frac{d\nu(x)\, d\nu(y)}{|x-y|^b}\right)^{1/p} < \infty .$$

Indeed from (3.1) follows (using (3.5))

$$\left(\iint_{2^j \leqslant |x-y| < 2^{j+1}} \frac{|(x)-u(y)|^p}{|x-y|^{\beta p}} \frac{d\nu(x)\, d\nu(y)}{|x-y|^b}\right)^{1/p} \leqslant C\, 2^{j(\beta-\bar\beta)} \qquad f \in L_p^{\bar\alpha}(\mu) .$$

It is now plain that th. 1.3 can be applied. We get (3.9) with $B_p^{\alpha 1}(\mu)$ in place of $B_p^{\alpha p}(\mu)$. Another interpolation leads to (3.9) in full.

REMARK 3.3. The same type of argument can be applied in a manifold of situations, e.g. in proving Flett [10], th. 1 (which result incidentally is used to prove certain theorems of Hardy-Littlewood).

If $a = n$, $\mu =$ Lebesgue measure, $b =$ integer $< n$, $\nu =$ the induced measure on a b-dimensional submanifold, we now recognize the Aronszajn-Babič-Slobodeckij-Gagliardo-Stein necessary and sufficient condition for the trace (cf. Introduction). It is tempting to conjecture that this is a necessary and sufficient condition also for more general (not too big) subsets with μ may be an equilibrium measure.

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