ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

LUNG OCK CHUNG
LEO SARIO
CECILIA WANG

Quasiharmonic L^p-functions on riemannian manifolds

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 2, nº 3 (1975), p. 469-478
<http://www.numdam.org/item?id=ASNSP_1975_4_2_3_469_0>

Quasiharmonic L^p-Functions on Riemannian Manifolds (*).

LUNG OCK CHUNG - LEO SARIO - CECILIA WANG

Let Q be the class of quasiharmonic functions q, defined by $\Delta q = 1$, with $\Delta = d\delta + \delta d$ the Laplace-Beltrami operator. Denote by P, B, D, C the classes of functions which are positive, bounded, Dirichlet finite, or bounded Dirichlet finite, respectively. For $X = P, B, D, C$, let O^N_{QX} be the classes of Riemannian N-manifolds, $N \geqslant 2$, for which $QX = Q \cap X = \emptyset$. These classes are known to be related by the strict inclusion relations

$$\begin{array}{ccccc} & & O^N_{QB} & & \\ & \swarrow & & \nwarrow & \\ O^N_{QP} & & & & O^N_{QC} \\ & \nwarrow & & \swarrow & \\ & & O^N_{QD} & & \end{array}$$

for each N, whereas there is no inclusion between O^N_{QB} and O^N_{QD} [3, 5].

In the present paper we introduce the class QL^p of quasiharmonic functions in L^p, with $1 \leqslant p < \infty$; the value $p = \infty$ will be excluded since $O^N_{QL^\infty}$ is nothing but O^N_{QB}. If $\tilde{O}^N$ signifies the complement of O^N with respect to the totality of Riemannian N-manifolds, then we shall show that $O^N_{QL^p} \cap O^N_{QX} \neq \emptyset$, $\tilde{O}^N_{QL^p} \cap \tilde{O}^N_{QX} \neq \emptyset$, $\tilde{O}^N_{QL^p} \cap O^N_{QX} \neq \emptyset$ for $p \geqslant 1$, $X = P, B, D, C$; $O^N_{QL^p} \cap \tilde{O}^N_{QX} \neq \emptyset$ for $p \geqslant 1$, $X = P, B$; and $O^N_{QL^p} \cap \tilde{O}^N_{QD} \neq \emptyset$ for $p > 1$. In striking contrast with these noninclusions, we shall establish the strict inclusions $O^N_{QL^1} < O^N_{QD}$, and $O^N_{QL^p} < O^N_{QC}$ for $p \geqslant 1$.

1. – We first prove the existence of N-manifolds which carry neither QL^p nor QX functions:

(*) The work was sponsored by the U.S. Army Research Office, Grant DA-ARO-31-124-73-G39, University of California, Los Angeles.
MOS Classification: 31B30.
Pervenuto alla Redazione il 10 maggio 1974 e in forma definitiva il 9 luglio 1974.

THEOREM 1. *$O^N_{QL^p} \cap O^N_{QX} \neq \emptyset$ for $N \geqslant 2$, $1 \leqslant p < \infty$, $X = P, B, D, C$.*

PROOF. An example is simply the Euclidean N-space E^N. We first show that $QL^1 = \emptyset$. Every $q \in Q$ can be written as $q = q_0 + h$, where $q_0 = -r^2/2N \in Q$ and h belongs to the class H of harmonic functions. Set $h = h(0) + k$, with $h(0)$ the value of h at the origin. Since $\int_{E^N} k\,dV = ck(0) = 0$, we have

$$\|q\|_1 = \int_{E^N} |q_0 + h(0) + k|\,dV \geqslant \left|\int_{E^N} (q_0 + h(0))\,dV\right| = \infty\,;$$

in fact, $q_0 + h(0) \sim cr^2$ and $dV = r^{N-1}\lambda(\theta)\,dr\,d\theta_1 \ldots d\theta_{N-1}$, with λ a trigonometric function of $\theta = (\theta_1, \ldots, \theta_{N-1})$.

To see that $QL^p = \emptyset$ for $p > 1$, take a function $\varphi \in C^\infty(E^N)$ with $\varphi = r^\alpha$ on $\{r > 1\}$, α a real constant to be specified later. If p' is determined by $1/p + 1/p' = 1$, then

$$\|\varphi\|_{p'}^{p'} = a + b\int_1^\infty r^{\alpha p' + N - 1}\,dr < \infty\,,$$

that is, $\varphi \in L^{p'}$, for $\alpha < -N/p'$. Suppose there exists a function $q \in QL^p$. Then $|(q, \varphi)| < \infty$. For some $h \in H$,

$$|(q, \varphi)| = |(q_0 + h(0) + k, \varphi)| = |(q_0 + h(0), \varphi)| = \infty$$

if $\alpha \geqslant -N - 2$. Thus any $\alpha \in [-N-2, -N/p')$ gives a contradiction, and we have $QL^p = \emptyset$ for every $p \geqslant 1$.

We proceed to show that $E^N \in O^N_{QX}$ for all X. It suffices to establish $QP = \emptyset$. Write again an arbitrary $q \in Q$ as $q = q_0 + h$. Take an increasing sequence $\{r_n\}_1^\infty$ with $r_n \to \infty$. For every n there exists an $\theta^n = (\theta_1^n, \ldots, \theta_{N-1}^n)$ such that $h(r_n, \theta^n) = h(0)$. Therefore $q(r_n, \theta^n) = q_0(r_n) + h(0) \to -\infty$, and $q \notin QP$.

2. – The relation $\tilde{O}^N_{QL^p} \cap \tilde{O}^N_{QX} \neq \emptyset$ is trivial in view of the Euclidean N-ball. We proceed to give a Riemannian N-manifold which carries QL^p functions but no QX functions.

THEOREM 2. *$\tilde{O}^N_{QL^p} \cap O^N_{QX} \neq \emptyset$ for $N \geqslant 2$, $1 \leqslant p < \infty$, $X = P, B, D, C$.*

PROOF. Consider the manifold

$$T\colon \{(x, y_1, \ldots, y_{N-1}) \,\big|\, |x| < \infty,\ |y_i| \leqslant 1,\ i = 1, \ldots, N-1\}\,,$$

with the opposite faces $y_i = 1$ and $y_i = -1$ identified for each i by a parallel translation perpendicular to the x-axis. Endow T with the metric

$$ds^2 = e^{-x^2}dx^2 + e^{-x^2/(N-1)}\sum_{i=1}^{N-1} dy_i^2 .$$

For a quasiharmonic function $q_0(x)$ we have

$$\Delta q_0(x) = -e^{x^2}\big(e^{-x^2}e^{x^2}q_0'(x)\big)' = 1 ,$$

which is satisfied by

$$q_0(x) = -\int_0^x\int_0^t e^{-t^2}dt\,dx .$$

To estimate

$$\|q_0\|_p^p = c\int_{-\infty}^{\infty}\left|\int_0^x\int_0^t e^{-t^2}dt\,dx\right|^p e^{-x^2}dx$$

we set $a = \int_0^\infty e^{-t^2}dt$ and obtain

$$\|q_0\|_p^p < a^p c\int_{-\infty}^{\infty}|x|^p e^{-x^2}dx < \infty .$$

Therefore $T \in \tilde{O}^N_{QL^p}$ for all p.

A harmonic function $h_0(x)$ satisfies $\Delta h_0(x) = -e^{x^2}\big(e^{-x^2}e^{x^2}h_0'(x)\big)' = 0$, which gives $h_0(x) = ax + b$. The harmonic measure ω of the boundary component at $x = \infty$ on $\{x \geqslant 0\}$ is $\omega = \lim_n \omega_n$, with $\omega_n = x/n$ harmonic on $\{0 \leqslant x \leqslant n\}$. Thus $\omega \equiv 0$, the same is true of the harmonic measure of the boundary component at $x = -\infty$, and therefore T belongs to the class O^N_G of parabolic N-manifolds. In view of $O^N_G \subset O^N_{QP}$ (loc. cit.), we obtain $T \in O^N_{QX}$ for all X.

3. – Our next problem is to find an N-manifold which admits QX functions for $X = P, B$, but no QL^p functions.

THEOREM 3. *$O^N_{QL^p} \cap \tilde{O}^N_{QX} \neq \emptyset$ for $N \geqslant 2$, $1 \leqslant p < \infty$, $X = P, B$.*

PROOF. Let M be the N-space equipped with the metric

$$ds^2 = dr^2 + \psi(r)^{2/(N-1)}\sum_1^{N-1}\lambda_i(\theta)\,d\theta_i^2 ,$$

where $\psi \in C^{\infty}[0, \infty)$,

$$\psi(r) = \begin{cases} r^{N-1} & \text{for } r \leqslant \frac{1}{2}, \\ e^{e^r} e^r & \text{for } r \geqslant 1, \end{cases}$$

and the λ_i are trigonometric functions of θ such that the metric is Euclidean on $\{r \leqslant \frac{1}{2}\}$. The function

$$q_0(r) = -\int_0^r \psi^{-1}(t) \int_0^t \psi(s)\, ds\, dt$$

satisfies the quasiharmonic equation $\Delta q_0 = -\psi^{-1}(\psi q_0')' = 1$. For $r > 1$,

$$q_0(r) = q_0(1) - \int_1^r e^{-e^t} e^{-t}\left(a + \int_1^t e^{e^s} e^s\, ds\right) dt$$

$$= q_0(1) - \int_1^r (e^{-t} + a_1 e^{-e^t} e^{-t})\, dt = O(1)$$

as $r \to \infty$. Therefore $q_0 \in QB$, and $M \in \tilde{O}^N_{QB} < \tilde{O}^N_{QP}$.

We next prove that $M \in O^N_{QL^1}$. Suppose there exists a $q \in QL^1$. Then $|(q, e^{-r})| < \infty$. We may again write $q = q_0 + c + k$, $k \in H$, $k(0) = 0$, and

$$(q, e^{-r}) = (q_0 + c, e^{-r}) = a + b\int_1^{\infty} (q_0 + c) e^{-r} e^{e^r} e^r\, dr .$$

Set

$$c_0 = -q_0(\infty) = \int_0^{\infty} \psi^{-1} \int_0^t \psi\, ds\, dt .$$

If $c \neq c_0$, then $\lim\limits_{r\to\infty} (q_0 + c) = d \neq 0$, and $|(q, e^{-r})| = \infty$. If $c = c_0$, then for $r > 1$

$$q_0 + c_0 = \int_r^{\infty} (e^{-t} + a_1 e^{-e^t} e^{-t})\, dt$$

and

$$(q, e^{-r}) = a + b\int_1^{\infty} (q_0 + c_0) e^{e^r}\, dr$$

$$= a + b\int_1^{\infty}\left(e^{-r} + a_1 \int_r^{\infty} e^{-e^t} e^{-t} dt\right) e^{e^r}\, dr$$

$$= a + b\int_1^{\infty} e^{-r} e^{e^r}\, dr + a_1 b \int_1^{\infty}\left(\int_r^{\infty} e^{-e^t} e^{-t}\, dt\right) e^{e^r} e^r e^{-r}\, dr .$$

Here

$$\lim_{r\to\infty}\left(\int_r^\infty e^{-e^t}e^{-t}dt\right)e^{e^r}e^r=\lim_{r\to\infty}\frac{-e^{-e^r}e^{-r}}{-e^{-e^r}(1+e^{-r})}=0\,,$$

so that for some $R>0$,

$$(q, e^{-r})=c+b\int_1^\infty e^{-r}e^{e^r}dr+a_1b\int_R^\infty o(1)e^{-r}dr\,.$$

The last integral converges, the first diverges, and therefore $|(q, e^{-r})|=\infty$. This contradiction shows that $q\notin L^1$, that is, $M\in O^N_{QL^1}$.

To see that $M\in O^N_{QL^p}$ for $p>1$, let p' be determined by $1/p+1/p'=1$. For a function $\varphi\in C^\infty(M)$ with

$$\varphi|\{r\geqslant 1\}=(e^{-e^r}e^{-r}r^{-2})^{1/p'}$$

we have

$$\|\varphi\|_{p'}^{p'}=a+b\int_1^\infty e^{-e^r}e^{-r}r^{-2}e^{e^r}e^r dr<\infty\,,$$

hence $\varphi\in L^{p'}$. If there exists a $q\in QL^p$, then $|(q,\varphi)|<\infty$. But $(q,\varphi)=$ $=(q_0+c,\varphi)$, and if $c\neq c_0$, the integrand in (q_0+c,φ) is asymptotically

$$c_1(e^{-e^r}e^{-r}r^{-2})^{1/p'}e^{e^r}e^r=c_1e^{e^r/p}e^{r/p}r^{-2/p'}\,,$$

so that $|(q_0+c,\varphi)|=\infty$, a contradiction.

In the case $c=c_0$ we observe that for $r>1$

$$q_0+c_0=e^{-r}+a_1\int_r^\infty e^{-e^t}e^{-t}dt\,,$$

with

$$\int_r^\infty e^{-e^t}e^{-t}dt<e^{-e^r}\int_r^\infty e^{-t}dt=e^{-e^r}e^{-r}\,,$$

so that $q_0+c_0\sim e^{-r}$ as $r\to\infty$. It follows that the integrand in (q_0+c_0,φ) is asymptotically

$$e^{-r}(e^{-e^r}e^{-r}r^{-2})^{1/p'}e^{e^r}e^r=e^{e^r/p}e^{-r/p'}r^{-2/p'}\,,$$

and therefore $|(q_0+c_0, \varphi)|=\infty$. This contradiction shows that $M\in O^N_{QL^p}$, and the proof of Theorem 3 is complete.

In the proofs of Theorems 1-3 we have actually shown somewhat more:

$$\bigcap_p O^N_{QL^p}\cap O^N_{QX}\neq\emptyset,$$
$$\bigcap_p \tilde{O}^N_{QL^p}\cap O^N_{QX}\neq\emptyset$$

for $N\geqslant 2$, $1\leqslant p<\infty$, $X=P, B, D, C$; and

$$\bigcap_p O^N_{QL^p}\cap \tilde{O}^N_{QX}\neq\emptyset$$

for $N\geqslant 2$, $1\leqslant p<\infty$, $X=P, B$.

4. – Next we are to find an N-manifold which carries QD functions but no QL^p functions. First we consider the case $p>1$.

THEOREM 4. *$O^N_{QL^p}\cap \tilde{O}^N_{QD}\neq\emptyset$ for $N\geqslant 2$, $1<p<\infty$.*

PROOF. Take the manifold

$$T: \{(x, y_1, ..., y_{N-1})|0<x<1,\ |y_i|\leqslant\pi,\ i=1, ..., N-1\},$$

with the opposite faces $y_i=\pi$, $y_i=-\pi$ identified as in No. 2, and choose the metric

$$ds^2=x^{-2\alpha}dx^2+x^{2\beta/(N-1)}\sum_{i=1}^{N-1}dy_i^2,$$

where α, β are real constants to be specified later; they will depend on p, so that we shall not have a generalization of the theorem in the same manner as at the end of No. 3.

The function

$$q_0(x)=-(\beta-\alpha+1)^{-1}(-2\alpha+2)^{-1}x^{-2\alpha+2}$$

satisfies the quasiharmonic equation

$$\Delta q_0=-x^{\alpha-\beta}(x^{\beta-\alpha}x^{2\alpha}q_0')'=1$$

provided

$$\beta-\alpha+1\neq 0,\quad \alpha\neq 1.$$

The Dirichlet integral is

$$D(q_0) = c\int_0^1 q_0'^2 x^{2\alpha} x^{\beta-\alpha}\,dx = c_1\int_0^1 x^{\beta-3\alpha+2}\,dx < \infty$$

if

$$\beta - 3\alpha + 3 > 0 .$$

For the L^p norm we have

$$\|q_0\|_p^p = c\int_0^1 x^{(-2\alpha+2)p} x^{\beta-\alpha}\,dx = \infty$$

if

$$\beta - (2p+1)\alpha + 2p + 1 \leqslant 0 .$$

An inspection of the last two inequalities shows that $p = 1$ is ruled out. For

$$p > 1 , \quad \alpha > \tfrac{3}{2} , \quad \beta \in \big(3(\alpha-1),\ (2p+1)(\alpha-1)\big] ,$$

all four inequalities are satisfied. In particular, $T \in \tilde{O}^N_{QD}$.

The exponent $\beta - \alpha$ in the volume element is positive, and the constant function 1 belongs to $L^{p'}$ for our $p' > 1$. Suppose there exists a $q \in QL^p$. Then $|(q, 1)| < \infty$.

Every $h \in H$ can be written

$$h(x, y) = h_0(x) + \sum_n{}' f_n(x) G_n(y) ,$$

where $h_0 \in H$, $f_n G_n \in H$, $n = (n_1, \ldots, n_{N-1})$, the n_i integers $\geqslant 0$, the G_n products of the form

$$G_n(y) = {\cos \atop \sin} n_1 y_1 {\cos \atop \sin} n_2 y_2 \cdots {\cos \atop \sin} n_{N-1} y_{N-1} ,$$

and the prime in $\sum'$ indicates that in each term at least one n_i does not vanish. The harmonic equation $\Delta h_0(x) = -x^{\alpha-\beta}(x^{\beta-\alpha}x^{2\alpha}h_0')' = 0$ is satisfied by

$$h_0(x) = a x^{-\alpha-\beta+1} + b .$$

Suppose first $a \neq 0$. Since $-2\alpha + 2 > -\alpha - \beta + 1$,

$$q_0(x) + h_0(x) \sim h_0(x) \quad \text{as } x \to 0 .$$

It follows that the integrand in $(q,1)=(q_0+h_0,1)$ is asymptotically $x^{-\alpha-\beta+1+\beta-\alpha}=x^{-2\alpha+1}$. A fortiori, $|(q,1)|=\infty$, a contradiction.

Now let $a=0$, $h_0=b$. Since

$$\varphi(x)=x^{(-2\alpha+2)/p'}\in L^{p'},$$

$|(q,\varphi)|<\infty$. On the other hand,

$$|(q,\varphi)|=|(q_0+b,\varphi)|=a_1+b_1\int_0^1 x^{-2\alpha+2}x^{(-2\alpha+2)/p'}x^{\beta-\alpha}dx=\infty$$

if

$$-2(\alpha-1)\left(1+\frac{1}{p'}\right)+\beta-(\alpha-1)\leqslant 0\,,$$

i.e.,

$$\beta\leqslant\left(3+\frac{2}{p'}\right)(\alpha-1)\,.$$

Since $2p+1>3+2/p'$ for $p>1$, the choice

$$\beta\in\left(3(\alpha-1),\left(3+\frac{2}{p'}\right)(\alpha-1)\right)$$

gives the contradiction $|(q,\varphi)|=\infty$ while preserving the earlier inequalities. We conclude that $O^N_{QL^p}\cap\tilde{O}^N_{QD}\neq\emptyset$ for all $p>1$.

5. – For $p=1$, $O^N_{QL^p}\cap\tilde{O}^N_{QD}\neq\emptyset$ is no longer true. In fact, we even have a strict inclusion:

THEOREM 5. *$O^N_{QL^1}<O^N_{QD}$ for $N\geqslant 2$.*

PROOF. To prove the inclusion relation $O^N_{QL^1}\subset O^N_{QD}$, suppose $u\in QD$. For any regular subregion Ω, the Riesz decomposition yields (cf. e.g. Nakai-Sario [3])

$$u(x)=h_\Omega(x)+\int_\Omega g_\Omega(x,y)\,dy$$

on Ω, where $h_\Omega(x)$ is the harmonic function on Ω with $h_\Omega=u$ on $\partial\Omega$, and $g_\Omega(x,y)$ is the Green's function on Ω with pole y. By Stokes' formula,

$$\int_\Omega\int_\Omega g_\Omega(x,y)\,dy\,dx\leqslant D_\Omega(u)\,.$$

On letting $\Omega \to R$ we obtain

$$\int_R \int_R g(x, y)\, dy\, dx \leqslant D(u) < \infty ,$$

where g is the Green function on R. Since $\Delta_x \int_R g(x, y)\, dy = 1$, we have $\int_R g(x, y)\, dy \in QL^1$, and therefore $O^N_{QL^1} \subset O^N_{QD}$. By Theorem 2, $\tilde{O}^N_{QL^1} \cap O^N_{QD} \neq \emptyset$, hence $O^N_{QL^1} < O^N_{QD}$.

6. – It remains to consider the class QC. Here we have the most elegant case, as there is strict inclusion for all p:

THEOREM 6. *$O^N_{QL^p} < O^N_{QC}$ for $N \geqslant 2$, $p \geqslant 1$.*

PROOF. In view of Theorem 2, it suffices to show that $O^N_{QL^p} \subset O^N_{QC}$. Suppose $R \notin O^N_{QC}$, and take a $u \in QC$. The Riesz decomposition of u on Ω implies

$$\int_\Omega g_\Omega(x, y)\, dy \leqslant |u(x)| + |h_\Omega(x)| \leqslant 2 \sup_\Omega |u(x)| .$$

On letting $\Omega \to R$ we obtain $\int_R g(x, y)\, dy \in B$. From the proof of Theorem 5 we conclude that $\int_R g(x, y)\, dy \in C$. Let

$$R_1 = \{x \in R \mid \int_R g(x, y)\, dy > 1\} .$$

Then

$$V(R_1) = \int_{R_1} dx < \int_R \int_R g(x, y)\, dy\, dx < \infty .$$

For $p \geqslant 1$,

$$\int_R \left(\int_R g(x, y)\, dy \right)^p dx = \int_{R_1} \left(\int_R g(x, y)\, dy \right)^p dx + \int_{R-R_1} \left(\int_R g(x, y)\, dy \right)^p dx$$
$$\leqslant M V(R_1) + \int_R \int_R g(x, y)\, dy\, dx < \infty ,$$

and therefore $\int_R g(x, y)\, dy \in QL^p$.

BIBLIOGRAPHY

[1] L. Chung - L. Sario, *Harmonic L^p-functions and quasiharmonic degeneracy,* J. Indian Math. Soc. (to appear).

[2] Y. K. Kwon - L. Sario - B. Walsh, *Behavior of biharmonic functions on Wiener's and Royden's compactifications,* Ann. Inst. Fourier (Grenoble), **21** (1971), pp. 217-226.

[3] M. Nakai - L. Sario, *Quasiharmonic classification of Riemannian manifolds,* Proc. Amer. Math. Soc., **31** (1972), pp. 165-169.

[4] L. Sario, *Biharmonic and quasiharmonic functions on Riemannian manifolds,* Duplicated lecture notes 1968-70, University of California, Los Angeles.

[5] L. Sario, *Quasiharmonic degeneracy of Riemannian N-manifolds,* Kōdai Math. Sem. Rep., **6** (1973), pp. 1-14.

[6] L. Sario - C. Wang, *Quasiharmonic functions on the Poincaré N-ball,* Rend. Mat., **6** (1973), pp. 1-14.

[7] L. Sario - C. Wang, *Negative quasiharmonic functions,* Tôhoku Math. J., **26** (1974), pp. 85-93.

[8] L. Sario - C. Wang, *Radial quasiharmonic functions,* Pacific J. Math., **46** (1973), pp. 515-522.

[9] L. Sario - C. Wang, *Harmonic L^p-functions on Riemannian manifolds,* Kōdai Math. Sem. Rep., **26** (1975), pp. 204-209.