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## Approximation of Smooth Functions and Covering Properties of Sets.

ROBERT KAUFMAN (\*)

**1.** – Let  $Q$  be a closed cube in Euclidean space  $E^{n+1}$  ( $n \geq 1$ ); using various spaces of differentiable functions on  $Q$ , one obtains corresponding classes of massive or negligible sets  $F \subseteq Q$ . The space  $C^1(Q)$  is well known; for a number  $1 < \alpha < 2$  we define the space  $C^\alpha(Q)$  to contain functions whose partial derivatives satisfy a uniform Lipschitz condition with exponent  $\alpha - 1$ . A closed set  $F \subseteq Q$  is called  $N_1$  if  $f(F)$  has linear Lebesgue measure 0 for all  $f$  in  $C^1(Q)$  except at most a set of first category. Let  $C^{\alpha n}$  be the Banach space of mappings into  $E^n$ , whose co-ordinates are of class  $C^\alpha$ .

**THEOREM 1.** There is a closed  $N_1$ -set  $F$  in  $Q$ , and an open set  $U$  in  $C^{\alpha n}(Q)$ , such that  $f(F)$  contains a ball in  $E^n$ , for every  $f$  in  $U$ .

Let us say that a subset  $S$  in  $C^{1n}$  has *uniform rank  $n$*  if all tangent (Jacobian) mappings  $J(f, x)$  ( $f \in S, x \in Q$ ) transform the unit ball in  $E^{n+1}$  onto the unit ball in  $E^n$  (or a larger set). If a ball  $B(r, x_0)$  is contained in  $Q$ , then  $f(B) \supseteq B(r, f(x_0))$ ; this can be seen by a variant of the Cauchy-Peano method in ordinary differential equations [1, pp. 1-7].

**THEOREM 1'.** Let  $Q_0$  be a compact set interior to  $Q$  and  $S$  a bounded subset of the space  $C^{\alpha n}$ , of uniform rank  $n$ . Then there is an  $N_1$ -set  $F \subseteq Q$ , such that  $f(F) \supseteq f(Q_0)$  for all  $f$  in  $S$ .

**2.** – Let  $T$  be a bounded subset of the Banach space  $C^\alpha[0, 1]$ , defined similarly to  $C^\alpha(Q)$ . For small numbers  $r > 0$ ,  $T$  is contained in  $\exp [Ar^{-1/\alpha}]$  ball of radius  $r$  in the *uniform metric*—a theorem of Kolmogorov [3, p. 153]. It is essential that the domain of the functions be a linear set, but the same bound holds for bounded subsets of  $C^{\alpha n}[0, 1]$ .

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3. — Let  $K$  be a compact subset of  $Q$ , and let  $W$  be a neighborhood of  $K$ , interior to  $Q$ . For some  $\varepsilon > 0$  every point of  $K$  has distance  $> 2\varepsilon$  from the boundary of  $W$ . Let  $T$  be the set of all level curves  $\Gamma(t)$ ,  $0 < t \leq \varepsilon$ , of functions  $f$  in  $S$ , meeting  $K$ . We choose arc-length as the parameter of each curve  $\Gamma$ , that is,  $\|\Gamma'(t)\| = 1$ . This condition on  $\Gamma$ , together with the boundedness of  $S$  in  $C^{\alpha n}(Q)$  and the uniform rank  $n$  of  $S$ , implies that the set  $T$  is bounded in  $C^{\alpha, n+1}[0, \varepsilon]$ . Therefore Kolmogorov's estimate is valid for small  $r > 0$ .

The curves  $\Gamma(t)$  have length  $\varepsilon$  and have equicontinuous tangent vectors  $\Gamma'(t)$ , so their diameters exceed some  $c_1 > 0$ . Let  $rA$  be the set of vectors  $(ru_1, \dots, ru_{n+1})$ , where each  $u_i$  is an integer. Since a curve  $\Gamma$  has diameter  $> c_1$ , some co-ordinate, say  $x_1$ , increases  $> c_1 n^{-1}$  along  $\Gamma$ ; when  $r$  is small  $x_1$  then assumes  $> c_2 r^{-1}$  values  $ru$  along  $\Gamma$ . Taking a point  $(ru_1^0, x_2, \dots, x_{n+1})$  on  $\Gamma$ , we observe that some element  $(ru_1^0, ru_2, \dots, ru_{n+1})$  of  $rA$  has distance  $< nr$  from  $\Gamma$ . Thus at least  $c_2 r^{-1}$  elements  $\lambda$  of  $rA$  have distance  $< nr$  from the curve  $\Gamma$ . For small  $r$ , all these elements  $\lambda$  belong to  $Q$ .

Now we form a random selection  $A^*$  from the set  $rA \cap Q$ . To define the distribution of this selection, we fix once and for all a number  $\tau$  in the interval  $0 < \tau < 1 - \alpha^{-1}$ . Then we select or reject the elements of  $rA \cap Q$  independently of each other, with the probability of each selection exactly  $r^\tau$ . The probability that  $A^*$  contains none of the  $c_2 r^{-1}$  elements  $\lambda$ , found above, is  $< \exp - c_2 r^{\tau-1}$  (because  $1 - y < \exp - y$  for  $y > 0$ ).

By Kolmogorov's estimate, we can select at most  $\exp Ar^{-1/\alpha}$  curves  $\Gamma_1 \in T$ , so that every curve  $\Gamma$  is within  $r$  of some  $\Gamma_1$  in the uniform metric on  $[0, \varepsilon]$ . The probability that every curve  $\Gamma_1$  has distance  $< nr$  from some element of  $A^*$ , exceeds  $1 - \exp c_2 r^{\tau-1} \exp Ar^{-1/\alpha} \rightarrow 1$ , because  $\tau < 1 - \alpha^{-1}$ . But then the same is true for all curves  $\Gamma$ , and a distance  $(n+1)r$ .

Suppose now that  $x \in K$  and  $f \in S$ . Then, considering the level curve  $\Gamma$  of  $f$  through  $x$ , we see that  $\|f(\lambda) - f(x)\| < c_3 r$  for some element  $\lambda$  of  $A^*$ . If  $r$  is small enough, then the ball  $B(\lambda, c_3 r)$ —of center  $\lambda$  and radius  $c_3 r$ —is contained in  $W$  and  $f(x) \in f(B)$ .

4. — To construct  $N_1$ -sets by the random method, we require another property of  $A^*$ . We choose in succession an integer  $k > 1$  and a number  $\pi$  in  $(0, 1)$  so that  $k\tau + (\pi - 1)k(n+1) > n+1$ . Consider the event  $M$ :  $A^*$  contains some  $k$  distinct elements  $\lambda_1, \dots, \lambda_k$  with all distances  $\|\lambda_i - \lambda_j\| \leq 3r^\pi$ . To estimate  $P(M)$  we bound the number  $J$  of the  $k$ -tuples:  $rA \cap Q$  has  $\ll r^{-(n+1)}$  elements; and each ball of radius  $3r^\pi$  contains  $\ll r^{(\pi-1)(n+1)}$  elements of  $rA$ . Hence  $J \ll r^{-a}$ , where  $a = -(n+1) + (\pi-1)(n+1)(k-1)$ . But  $P(M) \leq Jr^{k\tau} \rightarrow 0$  because  $k\tau + a > 0$ .

For small  $r$  we can choose  $A^*$  to have the covering property found in 3, while avoiding the event  $M$ . We write  $W_1$  for the union of balls  $B(\lambda, 2c_3r)$ ,  $\lambda \in A^*$ , and  $V_1$  for  $\cup B(\lambda, c_3r)$ . Then  $f(V_1) \supseteq f(K)$  for all  $f$  in  $S$ , and  $W_1 \subseteq W$  for small  $r$ .

Now we can repeat this process, using  $V_1^-$  for  $K$  and  $W_1 \cap W$  in place of  $W$ . Then we find a small  $r_2$ , and corresponding sets  $V_2$  and  $W_2$  so that  $f(V_2) \supseteq f(V_1) \supseteq f(K)$ , etc. Moreover,  $c_3$  and  $\pi$  are unchanged in the successive applications of the basic construction. The set  $F = \bigcap_1^\infty W_m^-$  then has the property  $f(F) \supseteq f(K)$ , and we prove finally that  $F$  is an  $N_1$ -set.

5. - To each  $g$  in  $C^1(Q)$ , and each  $\varepsilon > 0$ , we construct  $g_1$  in  $C^1(Q)$  so that  $\|g - g_1\| < \varepsilon$  in  $C^1(Q)$  and  $g_1(F)$  has measure  $< \varepsilon$ . This shows that the elements of  $C^1$ , transforming  $F$  onto a null set, are a dense  $G_\delta$ . We begin with a partition of the centers  $\lambda_q$ , that is, the elements of  $A^*$ , corresponding to a small value of the radius  $r$ . Let  $Y_1$  be a maximal selection of centers  $\lambda_q$ , having distances at least  $r^\pi$ ; let  $Y_2$  be a maximal selection from the remaining centers, etc. If  $\lambda$  belongs to  $Y_k$ , then  $\|\lambda - \lambda_q\| < r^\pi$  for  $k-1$  centers  $\lambda_q \neq \lambda$ . But then we have  $k$  centers with distances  $< 2r^\pi$  a contradiction. Therefore  $Y_1 \cup \dots \cup Y_{k-1}$  exhausts  $A^*$ .

Let  $s^{k+1} = r^{1-\pi}$ , and observe that every real number has distance  $< rs^{-k}$  from some multiple  $urs^{-k}$  of  $rs^{-k}$ . Hence we can define  $h_1$  in  $C^1(Q)$  so that each number  $g(\lambda) + h_1(\lambda)$ , with  $\lambda$  in  $Y_1$ , is a multiple of  $rs^{-k}$ . In view of the distance  $r^\pi$  between the members of  $Y_1$ , we can take  $h_1$  to have norm  $s^{-k}r(1+r^{-\pi})$ , as in [2]. Then we construct  $h_2$  so that each number  $(g+h_1+h_2)\lambda$ , with  $\lambda$  in  $Y_2$ , is a multiple of  $rs^{1-k}$ . The norm of  $h_2$  is again  $\ll s^{-k}r^{1-\pi}$ , and moreover  $|h_2| < rs^{1-k}$ . By this process we construct  $g_1 = g + h_1 + \dots + h_{k-1}$ , and  $\|g - g_1\|$  is small because  $s^{-k}r^{1-\pi} \rightarrow 0$ . Moreover,  $|g + h_1 + \dots + h_j - g_1| \ll \ll rs^{j-k} (1 < j < k)$ , so  $|g_1(\lambda) - urs^{j-k-1}| \ll rs^{j-k}$  for each  $\lambda$  in  $Y_j$ . When  $r$  is small, the partial derivatives of  $g_1$  are bounded by some  $B = B(g)$ ; thus  $B(\lambda, c_3r)$  is mapped inside a ball of radius  $\ll r + rs^{j-k} \leq 2rs^{j-k}$ , centered at  $urs^{j-k-1}$ . Since the set  $g_1(Q)$  remains within some finite interval, the union  $\cup B(\lambda, c_3r)$  is mapped onto a set of measure  $\ll s$ , and this completes the proof.

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