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Approximation of Smooth Functions and Covering Properties of Sets.

ROBERT KAUFMAN (*)

1. – Let Q be a closed cube in Euclidean space $E^{n+1} (n \geqslant 1)$; using various spaces of differentiable functions on Q, one obtains corresponding classes of massive or negligible sets $F \subseteq Q$. The space $C^1(Q)$ is well known; for a number $1 < \alpha < 2$ we define the space $C^\alpha(Q)$ to contain functions whose partial derivatives satisfy a uniform Lipschitz condition with exponent $\alpha - 1$. A closed set $F \subseteq Q$ is called N_1 if $f(F)$ has linear Lebesgue measure 0 for all f in $C^1(Q)$ except at most a set of first category. Let $C^{\alpha n}$ be the Banach space of mappings into E^n, whose co-ordinates are of class C^α.

THEOREM 1. There is a closed N_1-set F in Q, and an open set U in $C^{\alpha n}(Q)$, such that $f(F)$ contains a ball in E^n, for every f in U.

Let us say that a subset S in C^{1n} has *uniform rank n* if all tangent (Jacobian) mappings $J(f, x)$ $(f \in S, x \in Q)$ transform the unit ball in E^{n+1} onto the unit ball in E^n (or a larger set). If a ball $B(r, x_0)$ is contained in Q, then $f(B) \supseteq B(r, f(x_0))$; this can be seen by a variant of the Cauchy-Peano method in ordinary differential equations [1, pp. 1-7].

THEOREM 1'. Let Q_0 be a compact set interior to Q and S a bounded subset of the space $C^{\alpha n}$, of uniform rank n. Then there is an N_1-set $F \subseteq Q$, such that $f(F) \supseteq f(Q_0)$ for all f in S.

2. – Let T be a bounded subset of the Banach space $C^\alpha[0, 1]$, defined similarly to $C^\alpha(Q)$. For small numbers $r > 0$, T is contained in exp $[Ar^{-1/\alpha}]$ ball of radius r in the *uniform metric*—a theorem of Kolmogorov [3, p. 153]. It is essential that the domain of the functions be a linear set, but the same bound holds for bounded subsets of $C^{\alpha n}[0, 1]$.

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3. – Let K be a compact subset of Q, and let W be a neighborhood of K, interior to Q. For some $\varepsilon > 0$ every point of K has distance $> 2\varepsilon$ from the boundary of W. Let T be the set of all level curves $\Gamma(t)$, $0 \leqslant t \leqslant \varepsilon$, of functions f in S, meeting K. We choose arc-length as the parameter of each curve Γ, that is, $\|\Gamma'(t)\| = 1$. This condition on Γ, together with the boundedness of S in $C^{\alpha n}(Q)$ and the uniform rank n of S, implies that the set T is bounded in $C^{\alpha, n+1}[0, \varepsilon]$. Therefore Kolmogorov's estimate is valid for small $r > 0$.

The curves $\Gamma(t)$ have length ε and have equicontinuous tangent vectors $\Gamma'(t)$, so their diameters exceed some $c_1 > 0$. Let $r\Lambda$ be the set of vectors $(ru_1, \ldots, ru_{n+1})$, where each u_i is an integer. Since a curve Γ has diameter $> c_1$, some co-ordinate, say x_1, increases $> c_1 n^{-1}$ along Γ; when r is small x_1 then assumes $> c_2 r^{-1}$ values ru along Γ. Taking a point $(ru_1^0, x_2, \ldots, x_{n+1})$ on Γ, we observe that some element $(ru_1^0, ru_2, \ldots, ru_{n+1})$ of $r\Lambda$ has distance $< nr$ from Γ. Thus at least $c_2 r^{-1}$ elements λ of $r\Lambda$ have distance $< nr$ from the curve Γ. For small r, all these elements λ belong to Q.

Now we form a random selection Λ^* from the set $r\Lambda \cap Q$. To define the distribution of this selection, we fix once and for all a number τ in the interval $0 < \tau < 1 - \alpha^{-1}$. Then we select or reject the elements of $r\Lambda \cap Q$ independently of each other, with the probability of each selection exactly r^τ. The probability that Λ^* contains none of the $c_2 r^{-1}$ elements λ, found above, is $< \exp - c_2 r^{\tau - 1}$ (because $1 - y < \exp - y$ for $y > 0$).

By Kolmogorov's estimate, we can select at most $\exp A r^{-1/\alpha}$ curves $\Gamma_1 \in T$, so that every curve Γ is within r of some Γ_1 in the uniform metric on $[0, \varepsilon]$. The probability that *every* curve Γ_1 has distance $< nr$ from *some* element of Λ^*, exceeds $1 - \exp c_2 r^{\tau-1} \exp A r^{-1/\alpha} \to 1$, because $\tau < 1 - \alpha^{-1}$. But then the same is true for all curves Γ, and a distance $(n+1)r$.

Suppose now that $x \in K$ and $f \in S$. Then, considering the level curve Γ of f through x, we see that $\|f(\lambda) - f(x)\| < c_3 r$ for some element λ of Λ^*. If r is small enough, then the ball $B(\lambda, c_3 r)$—of center λ and radius $c_3 r$—is contained in W and $f(x) \in f(B)$.

4. – To construct N_1-sets by the random method, we require another property of Λ^*. We choose in succession an integer $k > 1$ and a number π in $(0, 1)$ so that $k\tau + (\pi - 1)k(n+1) > n + 1$. Consider the event M: Λ^* contains some k distinct elements $\lambda_1, \ldots, \lambda_k$ with all distances $\|\lambda_i - \lambda_j\| \leqslant 3r^\pi$. To estimate $P(M)$ we bound the number J of the k-tuples: $r\Lambda \cap Q$ has $\ll r^{-(n+1)}$ elements; and each ball of radius $3r^\pi$ contains $\ll r^{(\pi-1)(n+1)}$ elements of $r\Lambda$. Hence $J \ll r^{-a}$, where $a = -(n+1) + (\pi - 1)(n+1)(k-1)$. But $P(M) \leqslant J r^{k\tau} \to 0$ because $k\tau + a > 0$.

For small r we can choose Λ^* to have the covering property found in 3, while avoiding the event M. We write W_1 for the union of balls $B(\lambda, 2c_3 r)$, $\lambda \in \Lambda^*$, and V_1 for $\bigcup B(\lambda, c_3 r)$. Then $f(V_1) \supseteq f(K)$ for all f in S, and $W_1 \subseteq W$ for small r.

Now we can repeat this process, using V_1^- for K and $W_1 \cap W$ in place of W. Then we find a small r_2, and corresponding sets V_2 and W_2 so that $f(V_2) \supseteq f(V_1) \supseteq f(K)$, etc. Moreover, c_3 and π are uncharged in the successive applications of the basic construction. The set $F = \bigcap_1^\infty W_m^-$ then has the property $f(F) \supseteq f(K)$, and we prove finally that F is an N_1-set.

5. – To each g in $C^1(Q)$, and each $\varepsilon > 0$, we construct g_1 in $C^1(Q)$ so that $\|g - g_1\| < \varepsilon$ in $C^1(Q)$ and $g_1(F)$ has measure $< \varepsilon$. This shows that the elements of C^1, transforming F onto a null set, are a dense G_δ. We begin with a partition of the centers λ_q, that is, the elements of Λ^*, corresponding to a small value of the radius r. Let Y_1 be a maximal selection of centers λ_q, having distances at least r^π; let Y_2 be a maximal selection from the remaining centers, etc. If λ belongs to Y_k, then $\|\lambda - \lambda_q\| < r^\pi$ for $k-1$ centers $\lambda_q \neq \lambda$. But then we have k centers with distances $< 2r^\pi$ a contradiction. Therefore $Y_1 \cup \ldots \cup Y_{k-1}$ exhausts Λ^*.

Let $s^{k+1} = r^{1-\pi}$, and observe that every real number has distance $< rs^{-k}$ from some multiple urs^{-k} of rs^{-k}. Hence we can define h_1 in $C^1(Q)$ so that each number $g(\lambda) + h_1(\lambda)$, with λ in Y_1, is a multiple of rs^{-k}. In view of the distance r^π between the members of Y_1, we can take h_1 to have norm $s^{-k} r(1 + r^{-\pi})$, as in [2]. Then we construct h_2 so that each number $(g + h_1 + h_2)\lambda$, with λ in Y_2, is a multiple of rs^{1-k}. The norm of h_2 is again $\ll s^{-k} r^{1-\pi}$, and moreover $|h_2| < rs^{1-k}$. By this process we construct $g_1 = g + h_1 + \ldots + h_{k-1}$, and $\|g - g_1\|$ is small because $s^{-k} r^{1-\pi} \to 0$. Moreover, $|g + h_1 + \ldots + h_j - g_1| \ll \ll rs^{j-k}(1 \leqslant j < k)$, so $|g_1(\lambda) - urs^{j-k-1}| \ll rs^{j-k}$ for each λ in Y_j. When r is small, the partial derivatives of g_1 are bounded by some $B = B(g)$; thus $B(\lambda, c_3 r)$ is mapped inside a ball of radius $\ll r + rs^{j-k} \leqslant 2rs^{j-k}$, centered at urs^{j-k-1}. Since the set $g_1(Q)$ remains within some finite interval, the union $\bigcup B(\lambda, c_3 r)$ is mapped onto a set of measure $\ll s$, and this completes the proof.

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