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On Siegel's Zero.

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1. – Let d be fundamental discriminant, and let

$$\chi(n) = \left(\frac{d}{n}\right) \qquad \text{(Kronecker's symbol)} .$$

It is well known (see [1]) that $L(s, \chi)$ has at most one zero β in the interval $(1 - c_1/\log|d|, 1)$ where c_1 is an absolute positive constant. The main aim of this paper is to prove:

THEOREM 1. *Let d, χ and β have the meaning defined above. Then the following asymptotic relation holds*

$$1 - \beta = \frac{6}{\pi^2}\frac{L(1, \chi)}{\sum' 1/a}\left[1 + O\left(\frac{(\log\log|d|)^2}{\log|d|}\right) + O\big((1-\beta)\log|d|\big)\right] \tag{1}$$

where $\sum'$ is taken over all quadratic forms (a, b, c) of discriminant d such that

$$-a < b \leqslant a < \tfrac{1}{4}\sqrt{|d|}\,, \tag{2}$$

and the constants in the O-symbols are effectively computable.

In order to apply the above theorem we need some information about the size of the sum $\sum' 1/a$. This is supplied by the following.

THEOREM 2. *If (a, b, c) runs through a class C of properly equivalent primitive forms of discriminant d, supposed fundamental, then*

$$\sum_{\substack{\frac{1}{4}\sqrt{|d|} \geqslant |a| \geqslant b > -|a| \\ (a,b,c)\in C}} \frac{1}{|a|} \leqslant \begin{cases} 1/m_0 & \text{if } d < 0\,, \\ \dfrac{\log \varepsilon_0}{\log\left(\frac{1}{2}\sqrt{d} - 1\right)} + \dfrac{4}{\sqrt{d}} & \text{if } d > 676\,, \end{cases}$$

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where m_0 is the least positive integer represented by C and ε_0 is the least totally positive unit of the field $Q(\sqrt{d})$.

Theorems 1 and 2 together imply

COROLLARY. *For any $\eta > 0$ and $|d| > c(\eta)$ (d fundamental) we have*

$$1-\beta \geqslant \begin{cases} \left(\dfrac{6}{\pi}-\eta\right)\dfrac{1}{\sqrt{|d|}} & \text{if } d<0\,, \\[2ex] \left(\dfrac{6}{\pi^2}-\eta\right)\dfrac{\log d}{\sqrt{d}} & \text{if } d>0\,, \end{cases}$$

where $c(\eta)$ is an effectively computable constant.

REMARK. In the case $d < 0$, the constant $6/\pi$ could be improved by using the knowledge of all fields with class number $\leqslant 2$.

Similar inequalities with $6/\pi$ and $6/\pi^2$ replaced by unspecified positive constants have been claimed by Hanecke [3], however, as pointed out by Pintz [8], Hanecke's proof is defective and when corrected gives inequalities weaker by a factor $\log\log|d|$. Pintz himself has proved the first inequality of the corollary with the constant $6/\pi$ replaced by $12/\pi$ (see [8]).

For $d < 0$, the first named author [2] has obtained (1) with a better error term by an entirely different method. M. Huxley has also found a proof in the case $d < 0$ by a more elementary method different, however, from the method of the present paper.

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2. – The proofs of Theorems 1 and 2 are based on several lemmata.

LEMMA 1. *Let $f(d) = (\log|d|/\log\log|d|)^2$. Then*

$$\sum_{N\mathfrak{a}\leqslant\frac{1}{4}\sqrt{|d|}f(d)}\frac{1}{N\mathfrak{a}} = \frac{\pi^2}{6}\sum{}'\frac{1}{a}\left(1+O\left(\frac{(\log\log|d|)^2}{\log|d|}\right)\right),$$

where the left hand sum goes over all ideals $\mathfrak{a}\in Q(\sqrt{d})$ with norm $\leqslant \frac{1}{4}\sqrt{|d|}f(d)$ and the constant in the O-symbol is effectively computable.

PROOF. Every ideal $\mathfrak{a}$ of $\Omega(\sqrt{d})$ can be represented in the form

$$\mathfrak{a} = u\left[a, \frac{b+\sqrt{d}}{2}\right]$$

where u, a are positive integers and $b^2 \equiv d(\bmod 4a)$ (see [5], Theorem 59). If we impose the condition that

$$-a < b \leqslant a$$

then the representation becomes unique. Since $N\mathfrak{a} = u^2 a$, it follows that

$$\sum_{N\mathfrak{a} \leqslant \frac{1}{4}\sqrt{|d|}f(d)} \frac{1}{N\mathfrak{a}} = \sum{}' \frac{1}{a} \sum_{1 \leqslant u^2 \leqslant \frac{\sqrt{|d|}f(d)}{4a}} \frac{1}{u^2} + O\left(\sum_{\frac{1}{4}\sqrt{|d|} < a < \frac{1}{4}\sqrt{|d|}f(d)} \frac{1}{a}\right) = \tag{3}$$

$$= \sum{}' \frac{1}{a}\left(\frac{\pi^2}{6} + O\left(\left(f(d)\right)^{-\frac{1}{2}}\right)\right) + O(S).$$

To estimate the sum S, we divide it into two sums S_1 and S_2. In the sum S_1, we gather all the terms $1/a$ such that a has at least one prime power factor

$$p^\alpha > l(d) = d^{1/21 \log\log|d|},$$

$$p^\alpha | a,$$

and in S_2 all the other terms.

Let $\nu(a)$ be the number of representations of a as $N\mathfrak{a}$ where $\mathfrak{a}$ has no rational integer divisor > 1. Then $\nu(a)$ is a multiplicative function satisfying

$$\nu(p^\alpha) = \begin{cases} 1 + \left(\dfrac{d}{p^\alpha}\right) & \text{if } p \nmid d \text{ or } \alpha = 1, \\ 0 & \text{otherwise}. \end{cases}$$

Clearly

$$S_1 \leqslant \sum{}' \frac{1}{a} \sum{}'' \nu(p^\alpha) p^{-\alpha}$$

$$\leqslant \sum{}' \frac{1}{a} \sum{}'' 2p^{-\alpha}$$

where $\sum''$ goes over all prime powers p^α with

$$\max\left(l(d), \sqrt{|d|}/4a\right) < p^\alpha \leqslant \sqrt{|d|}\, f(d)/4a.$$

Now, by a well known result of Mertens

$$\sum_{p^\alpha < x} p^{-\alpha} = \log\log x + c + 0\left((\log x)^{-1}\right)$$

where c is a constant.

Hence

$$\sum_{x<p^\alpha<y} p^{-\alpha} = \log\left(\frac{\log y}{\log x}\right) + O\big((\log x)^{-1}\big) \leqslant$$
$$\leqslant \frac{\log y}{\log x} - 1 + O\big((\log x)^{-1}\big) =$$
$$= \frac{\log y/x + O(1)}{\log x}.$$

This gives

$$\sum{}'' p^{-\alpha} \leqslant \frac{\log f(d) + O(1)}{\log l(d)} \ll \frac{(\log\log|d|)^2}{\log d}$$

and we get

$$S_1 = O\left(\frac{(\log\log|d|)^2}{\log|d|}\right)\sum{}' \frac{1}{a}. \tag{4}$$

To estimate S_2, we notice that each a occuring in it must have at least

$$k_0 = \frac{\log\left(\frac{1}{4}\sqrt{|d|}\right)}{\log l(d)} \geqslant 10 \log\log|d|$$

distinct prime factors. Therefore

$$S_2 \leqslant \sum_{k\geqslant k_0} (1/k!)\Big(\sum_{p^\alpha<l(d)} \nu(p^\alpha)p^{-\alpha}\Big)^k$$
$$< (1/k_0!)\sigma^{k_0}e^\sigma$$

where

$$\sigma = \sum_{p^\alpha<l(d)} \nu(p^\alpha)p^{-\alpha} < 2\log\log l(d) + O(1)$$
$$= 2\log\log|d| + 0(1).$$

Now, Stirling's formula gives $k! > k_0^{k_0}\exp[-k_0]$. Hence

$$\log S_2 \leqslant -k_0\log k_0 + k_0\big((\log\sigma) + 1\big) + \sigma$$
$$\leqslant -k_0[\log 10 + \log\log\log|d| - \log 2 - \log\log\log|d| - 1] + \sigma$$
$$< -3\log\log|d| + 0(1)$$

and

$$S_2 = O\big((\log|d|)^{-3}\big). \tag{5}$$

The lemma now follows from equations (3), (4) and (5). The next lemma gives the growth conditions for the Riemann zeta-function and Dirichlet L-functions on the imaginary axis.

LEMMA 2. *For all real* t

$$|\zeta(it)| \ll \left(|t|^{\frac{1}{2}}+1\right)\log\left(|t|+2\right) \tag{6}$$

$$|L(it,\chi)| \ll \sqrt{|d|}\left(|t|^{\frac{1}{2}}+1\right)\log\left(|d|\left(|t|+2\right)\right). \tag{7}$$

PROOF. If $|t|>t_0$, the estimate

$$|\zeta(it)| \ll |t|^{\frac{1}{2}}\log|t|$$

holds (see [10], p. 19). Since $\zeta(s)$ has no pole on the imaginary axis, we have

$$|\zeta(it)| \ll 1 \qquad \text{for } |t| \leqslant t_0$$

and the inequality (6) now follows.

To prove (7), we note that

$$|L(1-it,\chi)| \ll \log\left(|d|(|t|+2)\right)$$

(see [1], p. 17, lemma 2 with $q=|d|$, $x=2|d|(|t|+2)$).

Now, by the fundamental equation for L-functions

$$|L(it,\chi)| = |L(1-it,\chi)|\,|d|^{\frac{1}{2}}|\Gamma(\tfrac{1}{2}it+A)\Gamma(\tfrac{1}{2}it+A)\Gamma^{-1}(\tfrac{1}{2}-\tfrac{1}{2}it+A)|$$

where

$$A=\tfrac{1}{4}(1-\chi(-1)).$$

Using the formula

$$|\Gamma(s)| = \sqrt{2\pi}|t|^{\sigma-\frac{1}{2}}\exp\left[-\tfrac{1}{2}\pi t\right]\left(1+O(|t|^{-1})\right)$$

valid for $s=\sigma+it$, $0\leqslant\sigma\leqslant\frac{1}{2}$, $|t|>1$ (see [9], p. 395), equation (7) follows, upon noting that

$$|\Gamma(\tfrac{1}{2}t+A)\Gamma^{-1}(\tfrac{1}{2}-\tfrac{1}{2}t+A)| \ll 1 \qquad \text{for } |t|<1.$$

PROOF OF THEOREM 1. By the standard argument ([4], p. 31)

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{y^s}{s(s+2)(s+3)}\,ds=\begin{cases}\dfrac{1}{6}-\dfrac{y^{-2}}{2}+\dfrac{y^{-3}}{3} & \text{if } y\geqslant 1\,,\\ 0 & \text{if } 0<y<1\,.\end{cases}$$

Since for $\operatorname{Re}(s)>1$

$$\zeta(s)L(s,\chi)=\sum(N\mathfrak{a})^{-s}\,,$$

it follows that for any $x>0$

$$\begin{aligned}I&=\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\zeta(s+\beta)L(s+\beta,\chi)\,\frac{x^s}{s(s+2)(s+3)}\,ds\\ &=\sum_{N\mathfrak{a}\leqslant x}(N\mathfrak{a})^{-\beta}\left|\frac{1}{6}-\frac{(N\mathfrak{a})^2}{2x^2}+\frac{(N\mathfrak{a})^3}{3x^3}\right|.\end{aligned}$$

Choose $x=\frac{1}{4}\sqrt{|d|}f(d)$ with $f(d)=\big(\log|d|/\log\log|d|\big)^2$.

If $N\mathfrak{a}\leqslant x$, we have

$$(N\mathfrak{a})^{-\beta}=(N\mathfrak{a})^{-1}\big(1+O((1-\beta)\log|d|)\big).$$

Hence

$$\begin{aligned}I&=\tfrac{1}{6}\sum_{N\mathfrak{a}\leqslant x}(N\mathfrak{a})^{-1}\big(1+O((1-\beta)\log|d|)\big)\\ &\quad+O\Big(\sum_{N\mathfrak{a}\leqslant x/f(d)}(N\mathfrak{a})^{-1}f(d)^{-2}\Big)+O\Big(\sum_{x/f(d)\leqslant N\mathfrak{a}\leqslant x}(N\mathfrak{a})^{-1}\Big)\,,\end{aligned}$$

and by lemma (1) (cf. formula (3))

$$I=\frac{1}{6}\sum{}'\frac{1}{a}\left|1+O\left(\frac{(\log\log|d|)^2}{\log|d|}\right)+O((1-\beta)\log|d|)\right|. \tag{8}$$

On the other hand, after shifting the line of integration to $\operatorname{Re}(s)=-\beta$

$$I=\frac{L(1,\chi)x^{1-\beta}}{(1-\beta)(3-\beta)(4-\beta)}+\frac{1}{2\pi i}\int_{-\beta-i\infty}^{-\beta+i\infty}\zeta(s+\beta)L(s+\beta,\chi)\frac{x^s}{s(s+2)(s+3)}\,ds\,. \tag{9}$$

By lemma (2), the integral on the right does not exceed

$$O(x^{-\beta}\sqrt{|d|}\log|d|)$$

and since

$$x^{1-\beta} = 1 + O\big((1-\beta)\log|d|\big)$$

$$(1-\beta)(3-\beta)(4-\beta) = 6 + O(1-\beta)$$

we get from (8) and (9)

$$1-\beta = \frac{6}{\pi^2}\frac{L(1,\chi)}{\sum' 1/a}\left|1 + O\left(\frac{(\log\log|d|)^2}{\log|d|}\right) + O\big((1-\beta)\log|d|\big)\right|.$$

3. – Proof of Theorem 2. For $d<0$ it is enough to prove that every class contains at most one form satisfying

$$(10)\qquad -|a| < b \leqslant |a| < \tfrac{1}{4}\sqrt{|d|}\,.$$

Now, since

$$|d| = 4ac - b^2$$

we infer from (10) that

$$a < \sqrt{|d|} < |d|/4a \leqslant c\,,$$

thus every form satisfying (10) is reduced, and it is well known that every class contains at most one such form.

For $d>0$, let us choose in the class C a form (*) (α,β,γ) reduced in the sense of Gauss, i.e. such that

$$(11)\qquad \beta + \sqrt{d} > 2|\alpha| > -\beta + \sqrt{d} > 0\,.$$

We can assume without loss of generality that $\alpha > 0$. Now, for any form $f\in C$, there exists a properly unimodular transformation

$$T = \begin{pmatrix} p & r\\ q & s\end{pmatrix}$$

taking (α,β,γ) into f. The first column of this transformation can be made to consist of positive rational integers by Theorem 79 of [5]. If f satisfies (10), we infer from

$$(12)\qquad \alpha p^2 + \beta pq + \gamma q^2 = a$$

(*) β is not to be confused with Siegel's zero.

that

$$\left|p+\frac{\beta-\sqrt{d}}{2\alpha}q\right|=a\left|\alpha p+\frac{\beta+\sqrt{d}}{2}q\right|^{-1}\leqslant$$

$$\leqslant\frac{1}{4}\sqrt{d}\cdot 2(\sqrt{d}\,q)^{-1}=\frac{1}{2}q^{-1}$$

and by lemma (16), p. 175 from [5], p/q is a convergent of the continued fraction expansion for

$$\omega=\frac{-\beta+\sqrt{d}}{2\alpha}.$$

From this point onwards, we shall use the notation of Perron's monograph [7]. Since by (11)

$$\omega^{-1}>1 \quad \text{and} \quad O>(\omega')^{-1}>-1,$$

ω^{-1} is a reduced quadratic surd and it has a pure periodic expansion into a continued fraction. Hence

$$\omega=[0,\overline{b_1, b_2 \dots b_k}]$$

where the bar denotes the primitive period. The corresponding complete quotients form again a periodic sequence

$$\omega_v=\frac{P_v+\sqrt{d}}{Q_v}, \qquad \omega_0=\omega$$

where for all $v\geqslant 1$, ω_v is reduced,

$$\omega_v=\omega_{v+k}, \tag{13}$$

and k is the least number with the said property.

Lemma 3. *Let* $[0,\overline{b_1, b_2, \dots, b_k}]$ *be the continued fraction for* ω *defined above. Then*

$$\sum_{(a,b,c)\in C}'' \frac{1}{|a|}\leqslant\frac{2}{\sqrt{d}}\sum_{\substack{v=2\\ \sqrt{d}\geqslant b_v\geqslant 2}}^{[k,2]}\min\left(\frac{\sqrt{d}}{2}, b_v+1\right)$$

where the sum on the left is taken over all (a, b, c) *in the class* C *satisfying* (10).

PROOF. If A_j/B_j is the j-th convergent of ω, we have by formula (18), § 20 of [7]

$$(A_{v-1}Q_0 - B_{v-1}P_0)^2 - d(B_{v-1})^2 = (-1)^v Q_0 Q_v$$

which gives on simplification

$$\alpha A_{v-1}^2 + \beta A_{v-1}B_{v-1} + \gamma B_{v-1}^2 = (-1)^v Q_v/2 \,. \tag{14}$$

Similarly, eliminating Q_v from formulae (16) and (17) in § 20 of [7], we get

$$2\alpha A_{v-1}A_{v-2} + \beta(A_{v-1}B_{v-2} + B_{v-2}A_{v-2}) + 2\alpha B_{v-1}B_{v-2} = (-1)^{v-1}P_v \,. \tag{15}$$

Let $p = A_{v-1}$, $q = B_{v-1}$ $(v \geqslant 1)$. By (12)

$$a = (-1)^v Q_v/2 \,.$$

Hence, by formula (1) of § 6 of [7]

$$\begin{vmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{vmatrix} = (-1)^v.$$

and since

$$\begin{vmatrix} A_{v-1} & r \\ B_{v-1} & s \end{vmatrix} = 1$$

it follows that

$$T = \begin{pmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix}, \qquad t \in Z \,.$$

Thus we find using (14) and (15)

$$\begin{aligned} f &= (\alpha, \beta, \gamma) \begin{pmatrix} A_{v-1} & A_{v-2} \\ B_{v-1} & B_{v-2} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix} = \\ &= \left((-1)^v \frac{Q_v}{2}, (-1)^{v-1} P_v, (-1)^v \frac{Q_{v-1}}{2} \right) \begin{pmatrix} 1 & t \\ 0 & (-1)^v \end{pmatrix} . \end{aligned}$$

In order to make f satisfy (10) we must choose

$$t = (-1)^v \left[\frac{P_v}{Q_v} + \frac{1}{2} \right] .$$

Thus f is uniquely determined by ω_v and in view of (13), we have

$$\sum''_{(a,b,c)\in C} \frac{1}{|a|} \leqslant \sum_{\substack{v=1 \\ Q_v<\frac{1}{2}\sqrt{d}}}^{[k,2]} 2(Q_v)^{-1} . \tag{16}$$

Since ω_v is reduced, we have further for v in question

$$\sqrt{d} \geqslant \frac{2\sqrt{d}}{Q_v} > \frac{P_v+\sqrt{d}}{Q_v} > \frac{\sqrt{d}}{Q_v} \geqslant 2 .$$

Hence for

$$b_v = [\omega_v] ,$$

we get the inequalities

$$\sqrt{d} > b_v \geqslant 2 , \qquad b_v + 1 > \sqrt{d}/Q_v ,$$

and by (16), lemma (3) follows.

Now, let ε_0 be the least totally positive unit $\varepsilon_0 > 1$ of the ring $Z(\sigma)$ where

$$\sigma = \begin{cases} \dfrac{1}{2}\sqrt{d} & \text{if } d \equiv 0 \pmod 4 , \\[2ex] \dfrac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4 . \end{cases}$$

By Theorem (7) of Chapter IV of [6]

$$\varepsilon_0 = \frac{u + v\sqrt{d}}{2} ,$$

where for $l = [k, 2]$,

$$v = (q_{l-1}, p_{l-1} - q_{l-2}, p_{l-2}) , \qquad u = p_{l-1} + q_{l-2}$$

and p_j, q_j are the numerator and denominator, respectively, of the j-th convergent for ω^{-1}. Moreover, since ω^{-1} satisfies the equation

$$-\gamma\omega^2 - \beta\omega^{-1} - \alpha = 0 , \qquad (-\gamma > 0)$$

we find from formula (1) of § 2 of Chapter IV of [6] that

$$q_{l-2} - p_{l-1} = -\beta v , \qquad -p_{l-2} = -\alpha v .$$

Hence

$$\varepsilon_0 = \frac{p_{l-1} + q_{l-2}}{2} + \frac{p_{l-2}\sqrt{d}}{2\alpha} = q_{l-2} + \frac{\beta + \sqrt{d}}{2\alpha} p_{l-2} .$$

Since $p_j = B_{j+1},\ q_j = A_{j+1}$, we get

$$\varepsilon_0 = B_{l-1}\left(\frac{A_{l-1}}{B_{l-1}} + \frac{\beta+\sqrt{d}}{2\alpha}\right) \geqslant B_{l-1}\left(\omega + \frac{\beta+\sqrt{d}}{2\alpha}\right) = \frac{\sqrt{d}}{\alpha} B_{l-1}\,. \tag{17}$$

Now,

$$\omega_l = b_l + \omega_{l+1}^{-1} = b_l + \omega_1^{-1} = b_l + \omega\,, \qquad \omega_l' = b_l + \omega'$$

and since ω_l is reduced $0 > b_l + \omega' > -1$

$$b_l = [-\omega'] = \left[\frac{\beta+\sqrt{d}}{2a\alpha}\right] < \frac{\sqrt{d}}{\alpha}\,.$$

Thus (17) gives

$$\varepsilon_0 > b_l B_{l-1} > \prod_{v=1}^{l} b_v\,,$$

and by (16)

$$\sum_{(a,b,c)\in C}'' \frac{1}{|a|} \leqslant \frac{2}{\sqrt{d}} \max \sum (x_i + 1) = \frac{2}{\sqrt{d}} M \tag{18}$$

where maximum is taken over all non-decreasing sequence of at most l numbers satisfying

$$2 \leqslant x_i \leqslant \tfrac{1}{2}\sqrt{d} - 1 = D\,, \qquad \prod x_i < \varepsilon_0\,.$$

Let $(x_1, x_2, \ldots, x_m)$ be a point in which the maximum is taken with the least number m. We assert that the sequence contains at most one term x with $2 < x < D$. Indeed, if we had $2 < x_i < x_{i+1} < D$, we could replace the numbers $x_i,\ x_{i+1}$ by

$$\frac{x_i}{\min(x_i/2,\ D/x_{i+1})}\,, \qquad x_{i+1}\min\left(\frac{x_i}{2}, \frac{D}{x_{i+1}}\right)$$

and the sum $\sum(x_i+1)$ would increase. Also, if we had $x_1 = x_2 = x_3 = 2$, we could replace them by $x_1 = 8$, and the sum $\sum(x_i+1)$ would remain the same while m would decrease.

Let

$$\frac{\varepsilon_0}{4} = D^e\theta\,, \qquad \text{where } e = \frac{\log(\varepsilon_0/4)}{\log D}\,.$$

Using $d > 676$, we get

$$M = \begin{cases} \tfrac{1}{2} e\sqrt{d} + \max(4\theta+1, 2\theta+4) & \text{if } 4\theta < D\,, \\ \tfrac{1}{2} e\sqrt{d} + 2\theta + 4 & \text{if } 2\theta < D \leqslant 4\theta\,, \\ \tfrac{1}{2} e\sqrt{d} + \theta + 7 & \text{if } D \leqslant 2\theta\,. \end{cases}$$

Now,

$$e=\frac{\log \varepsilon_0}{\log D}-\frac{\log 4\theta}{\log D}.$$

Since for $1\leqslant x\leqslant y$, $y(\log x/\log y)\geqslant x-1$, and for $d>676$, $D/\log D\geqslant$ $\geqslant 12/\log 12>4.8$, we obtain if $4\theta<D$.

$$M-\frac{1}{2}\sqrt{d}\frac{\log \varepsilon_0}{\log D}=\max(4\theta+1, 2\theta+4)-D\frac{\log 4\theta}{\log D}=\frac{\log 4\theta}{\log D}<$$
$$<\max(4\theta+1, 2\theta+4)-\max(4\theta-1, 6)\leqslant 2\,,$$

if $2\theta<D\leqslant 4\theta$

$$M-\frac{1}{2}\sqrt{d}\frac{\log \varepsilon_0}{\log D}=2\theta+4-D\frac{\log 2\theta}{\log D}-D\frac{\log 2}{\log D}-\frac{\log 4\theta}{\log D}<$$
$$<2\theta+4-2\theta+1-3-1=1\,,$$

if $D\leqslant 2\theta$

$$M-\frac{1}{2}\sqrt{d}\frac{\log \varepsilon_0}{\log D}=\theta+7-D\frac{\log \theta}{\log D}-D\frac{\log 4}{\log D}-\frac{\log 4\theta}{\log D}<$$
$$<\theta+7-\theta+1-6-1=1\,.$$

This together with (18) gives the theorem.

4. – Proof of Corollary. We can assume $1-\beta<(\log|d|)^{-2}$. It then by Theorem (1) that for every $\eta>0$, there exists $c(\eta)$ such that if $d>c(\eta)$

$$1-\beta\geqslant\frac{6}{\pi^2}\frac{L(1,\chi)}{\sum' 1/a}\left(1-\frac{\eta}{2}\right). \tag{19}$$

Let h_0 be the number of classes of forms in question. For $d<-4$, we have

$$L(1,\chi)=\frac{\pi h_0}{\sqrt{|d|}}\,,$$

and by Theorem (2)

$$\sum{}'\frac{1}{a}\leqslant h_0\,.$$

Hence by (19)

$$1-\beta\geqslant\frac{6}{\pi^2}\frac{h_0\pi}{h_0\sqrt{|d|}}\left(1-\frac{\eta}{2}\right)>\left(\frac{6}{\pi}-\eta\right)\frac{1}{\sqrt{|d|}}\,.$$

For $d > 0$, we have

$$L(1, \chi) = \frac{h_0 \log \varepsilon_0}{\sqrt{d}}.$$

Now, for any class C of forms

$$\sum_{(a,b,c)\in C}'' \frac{1}{|a|} = \sum_{\substack{(a,b,c)\in C \\ \frac{1}{4}\sqrt{d}\geqslant a \geqslant b > -a}} \frac{1}{a} + \sum_{\substack{(-a,b,-c)\in C \\ \sqrt{d}\geqslant -a \geqslant b > a}} \frac{1}{|a|}.$$

If (a, b, c) runs through C, $(-a, b, -c)$ runs through another class which we denote by $-C$ (It may happen that $-C = C$). If $C_1 \neq C_2$, then $-C_1 \neq -C_2$. Hence

$$\sum_C \sum_{\substack{(a,b,c)\in C \\ \frac{1}{4}\sqrt{d}\geqslant |a| \geqslant b > -|a}} \frac{1}{|a|} = 2 \sum' \frac{1}{a}$$

and by Theorem (2)

$$\sum' \frac{1}{a} \leqslant \frac{h_0}{2}\left(\frac{\log \varepsilon_0}{\log\left(\frac{1}{2}\sqrt{d} - 1\right)} + \frac{4}{\sqrt{d}}\right) < \frac{h_0 \log \varepsilon_0}{\log d}\left(1 + O\left(\frac{1}{\sqrt{d}}\right)\right),$$

where the constant in the O-symbol is effective. (Note that $\varepsilon_0 > \frac{1}{2}\sqrt{d}$). This together with (19) gives the corollary.

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