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On the Rectifiability of Domains with Finite Perimeter.

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In the study of free boundaries in variational inequalities it arises naturally the question of regularity of domains with finite perimeter in the sense of Caccioppoli [1] and De Giorgi [3]. In two dimensions, for instance, with the rectifiability of the topological boundary of such domains; which we prove in the context of this paper; the tools developed by H. Lewy [6], and H. Lewy and G. Stampacchia [7] can be utilized to show further regularity of the free boundaries, see [7], [5], [2]. The domains in question usually verify the following two conditions:

(A) The interior of its complement is composed of finitely many connected components. (B) The points in the boundary of the domain are of uniform positive density with respect to the interior of its complement, see [2]. In the two dimensional case either condition yields the rectifiability of the domain. In the notation of geometric measure theory (1) our results prove that the irreducible currents associated with $\tilde{\Gamma}\chi_E$ are the boundary curves of the connected components of E. In the n-dimensional case condition (B) implies the additivity of the perimeter of Ω in terms of its connected components.

Domains with finite perimeter.

Following the notation of De Giorgi [3], set

$$W_\lambda f(x) = (\pi\lambda)^{-n/2}\int_{R^n} \exp\left(-|\xi|^2/\lambda\right) f(x-\xi)\, d\xi .$$

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(1) H. FEDERER, *Geometric Measure Theory*, p. 421, Springer Verlag, 1969.

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If χ_E denotes the characteristic function of E, we define its perimeter as

$$\mathcal{T}(E) = \lim_{\lambda \to 0} \int_{R^n} |\text{grad}\,(W_\lambda \chi_E)(x)|\, dx\,.$$

In [3] it is shown that if $\{\pi_n\}$ is a sequence of polygonal sets converging in measure to E, then

$$\mathcal{T}(E) \leqslant \underline{\lim}\, \mathcal{T}(\pi_n)\,.$$

Moreover when E has finite perimeter there is a sequence of polygonal domains $\{\pi_n\}$, such that π_n converges in measure to E and

$$\mathcal{T}(E) = \lim_{n\to\infty} \mathcal{T}(\pi_n)\,.$$

Denote with $\complement E$ the complement of E and with E_0 its interior. $m(E)$ will denote the Lebesgue measure, for a measurable set E.

LEMMA I. *Let Ω be an open set of finite perimeter. The sequence, $\{\pi_n\}$, of polygonal domains converging to Ω in measure and approximating its perimeter can be chosen so that:*

(i) *If K_1 and K_2 are compact sets, $K_1 \subset \Omega$ and $K_2 \subset (\complement\Omega)^0$; then, for $n \geqslant n_0$, $K_1 \subset \pi_n$ and $K_2 \subset (\complement\pi_n)^0$.*

(ii) *If $m(\{x, |x - x_0| < \varrho\} \cap (C\Omega)^0) \geqslant \alpha\varrho^n$, when $x_0 \in \partial\Omega$ ($\partial\Omega$ denotes the boundary of Ω), $\varrho \leqslant \varrho_0$, and $\alpha > 0$ independently of ϱ and x_0; then $\pi_n \subset \pi_{n+1} \subset \Omega$.*

PROOF. We follow the construction of De Giorgi [3] of a polygonal sequence of functions, $\{g_k\}$, defined from R^n into R such that if

$$\pi_k(\theta) = \{x, g_k(x) \geqslant \theta\} \qquad \text{and} \qquad \varrho_k(\theta) = \mathcal{T}(\pi_k(\theta))\,,$$

then

$$(a) \qquad \int_{1/k}^{\infty} \varrho_k(\theta)\, d\theta \leqslant \mathcal{T}(\Omega) + 1/k\,.$$

(b) When $1/k \leqslant \theta \leqslant 1 - 1/k$, $m(\pi_k(\theta)\,\Delta\,\Omega) \leqslant 1/k$. Here Δ denotes the symmetric difference; $A \,\Delta\, B = (A \cap B') \cup (A' \cap B)$.

(c) $0 \leqslant g_k - W_{\lambda_k}\chi_\Omega \leqslant 1/k$, where $\lambda_k \to 0$ as $k \to \infty$.

Observe that, in virtue of (b) and a previous remark, $\mathcal{T}(\Omega) \leqslant \lim \varrho_k(\theta)$, for $0 < \theta < 1$.

On the other hand, by Fatou's lemma and (a),

$$\int_{\varepsilon}^{1-\varepsilon} \underline{\lim}\, \varrho_k(\theta)\, d\theta \leqslant \underline{\lim} \int_{\varepsilon}^{1-\varepsilon} \varrho_k(\theta)\, d\theta \leqslant (1-2\varepsilon)\,\sigma(\Omega)\ .$$

Therefore $\mathcal{T}(\Omega) = \underline{\lim}\, \varrho_k(\theta)$ in a set E, $m(E \cap (0, 1)) = 1$.

Given η, choose λ_k so small that $W_{\lambda_k}\chi_\Omega(x) \leqslant \eta$ when $x \in K_2$ and $W_{\lambda_k}\chi_\Omega(x) \geqslant 1-\eta$ when $x \in K_1$. Then if we choose $\pi_k(\theta)$ such that $\theta \in (2\eta, 1-\eta) \cap E$ and $1/k < \eta$, part (i) follows.

To prove part (ii), observe first that when $x \in \partial\Omega$, $0 \leqslant W_\lambda \chi_\Omega(x) \leqslant \alpha' < 1$, uniformly for $x \in \partial\Omega$, $\lambda \leqslant \lambda_0$. Observe also that $W_\lambda \chi_\Omega(x)$ satisfies the heat equation in the (x, λ) variables, when $x \in (\complement\Omega)^0$; moreover for such x, $W_\lambda \chi_\Omega(x) \to 0^+$ as $\lambda \to 0^+$. Therefore, by virtue of the maximum principle $W_\lambda \chi_\Omega(x)$, $x \in \complement\Omega$, must reach its maximum at a point (χ_0, λ_1), with $x_0 \in \partial(\complement\Omega)^0 \subset \partial\Omega$, $\lambda_1 \leqslant \lambda_0$: That is $W_\lambda \chi_\Omega(x) \leqslant \alpha' < 1$ when $(x, \lambda) \in \complement\Omega \times (0, \lambda_0]$. Choose $\theta \in E \cap (\alpha', 1)$; in virtue of the observations above, for n_1 large enough $\pi_{n_1}(\theta) \subset \Omega$. Using part (i) of the lemma we can choose n_2; $\pi_{n_1}(\theta) \subset \pi_{n_2}(\theta) \subset \Omega$; by induction part (ii) follows.

LEMMA II. *Let $\Omega \subset R^2$, be a bounded open set such that (a) $\partial\Omega = \partial[\complement(\Omega)^0]$; (b) $\complement(\Omega)^0 = \bigcup_{j\leqslant m} D_j$, D_j connected. If further $\mathcal{T}(\Omega) < \infty$, then the boundary of each of its connected components is accessible. Moreover, if C is a connected component of Ω, given $\{x_n\} \subset C$ converging to x, $x \in \partial C$, there exists a Jordan arc, Γ: $[0, 1) \to C$, and a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\Gamma(t_k) = x_{n_k}$ and $\Gamma(t) \to x$ as $t \to 1^-$.*

PROOF. Let $x_0 \in \partial C$, set $B_k(x_0) = \{x, |x - x_0| < 1/k\}$. Take $x_{n_1} \in B_1(x_0) \cap C$ and let S_1 be the collection of polygonal curves joning x_{n_1} with $x_n \in B_2(x_0) \cap C$.

Let $\alpha = \inf\{d(\Gamma), \Gamma \in S_1\}$, $d(\cdot)$ denotes the diameter of the set. Choose Γ_1, joining x_{n_1} with x_{n_2}, such that $d(\Gamma_1) < 2\alpha$, $x_{n_2} \in B_2(x_0) \cap C$.

Repeating the above argument inductively we obtain a sequence of polygonal curves, $\{\Gamma_k\}$, joining x_{n_k} with $x_{n_{k+1}}$ such that, $x_{n_k} \in B_k(x_0) \cap C$ and $\alpha_k \leqslant d(\Gamma_k) \leqslant 2\alpha_k$. The lemma follows if we show that $\alpha_k \to 0$ as $k \to \infty$:

We argue by contradiction; suppose there exists a sequence, $\{\alpha_k\}$, such that $\alpha_k > \varepsilon$, $x_{n_k} \in B_k(x_0) \cap C$. Consider k_0 so large that $1/k_0 \leqslant \varepsilon/4$, then x_{n_k}, $k \geqslant k_0$, belong to different components of $C \cap \{x, |x - x_0| < \varepsilon/2\}$. If otherwise, x_{n_k} and x_{n_l} belong to the same connected component, $n_k < n_l$, then $d(\Gamma_k) < \varepsilon$, contradicting the fact that $\alpha_k > \varepsilon$.

Call E_k the connected component of $C \cap \{x, |x - x_0| < \varepsilon\}$ containing x_{n_k}, $k > k_0$. Again by virtue of the fact that $d(\Gamma_k) > \varepsilon$, $E_k \cap \{x, |x - x_0| = \varepsilon/4\} \neq \emptyset$.

Let $\tilde{\Gamma}_k$ be a polygonal arc joining χ_{n_k} with a point of $\{x, |x-x_0| = \varepsilon/4\}$, $\tilde{\Gamma}_k \subset E_k$.

Set $S_r = \{x, |x-x_0| = r\}$, and φ_r from $[0, 2\pi)$ onto S_r the standard transformation into polar coordinates. Observe that when $\varphi_r(\theta_1) \in E_{k_1}$, $\varphi_r(\theta_2) \in E_{k_2}$, $r < \varepsilon/2$, there exists θ; $\theta_1 < \theta < \theta_2$; $\varphi_r(\theta) \in \partial C$. On the other hand if $S_{r_1} \cap (\complement\Omega)^0 = \emptyset$ and $S_{r_2} \cap (\complement\Omega)^0 = \emptyset$, $r_i < \varepsilon/2$; since $\partial\Omega = \partial(\complement\Omega)^0$ there exists a connected component of $(\complement\Omega)^0$, D_j, $D_j \subset \{x; r_2 < |x-x_0| < r_1\}$. By hypothesis there are at most $m+1$ radii such that $S_r \cap (\complement\Omega)_0 = \emptyset$.

Finally we restrict our attention to a finite number of polygonal paths $\tilde{\Gamma}_l$, $l_0 \leqslant l \leqslant l_0 + N$, dividing the set $\{x; \varepsilon/8 < |x-x_0| < \varepsilon/2\}$ into N different regions. In each region we can construct a compact set K_l composed of radial segments $K_l \subset (\complement\Omega)^0$, such that the perimeter of the circular projection of K_l into a given radius exceeds $\varepsilon/16$. By Lemma I we may choose π_n; such that $\pi_n \subset CK_l$ and $\pi_n \supset \tilde{\Gamma}_l \cap \{x, \varepsilon/8 \leqslant |x-x_0| \leqslant \varepsilon/2\}$. As a consequence, on each region the perimeter of π_n must exceed $\varepsilon/16$ and therefore $\mathcal{T}(\pi_n) > N\cdot\varepsilon/16$. This contradicts our hypothesis on the finiteness of the perimeter of Ω and the lemma follows.

REMARK I. *Let Ω be as in Lemma* II, *then $\bar{C}_j \cap \bar{C}_k$ consists of at most $m+1$ points.*

PROOF. If two points belong to the common boundary of C_j and C_k there are two Jordan arcs, by virtue of Lemma II, the first joining them in C_j the second in C_k. Together they form a Jordan curve whose interior must contain a component of $\complement(\Omega)^0$, since $\partial\Omega = \partial\complement(\Omega)^0$.

REMARK II. *Under the assumptions of Lemma II the boundary of each of the connected components, C_j, of Ω is composed of at most m Jordan curves.*

PROOF. Clearly $\complement(C_j)^0$ is composed of at most m connected components. Let E_1 be the unbounded component of $\complement(C_j)^0$ and $C^* = \complement(E_1)^0$, then (i) $C_j \subset C^*$; (ii) $\partial C^* \subset \partial C_j$; (iii) C^* is connected and simply connected.

In virtue of Lemma II every « prime end » of C^* consists of exactly one point which is a simple point, since $\partial C^* = \partial E_1$. As a consequence ∂C^* is a Jordan curve (see [8], Secs. 7 and 8). The other components, $\{E_j\}_{j<1}$, reduce to the above case by an inversion mapping.

Next we will « separate » two connected components, C_1 and C_2, of Ω by a Jordan curve contained in $\complement(\Omega)^0$ and a finite number of small balls.

By Remark II, given $\varepsilon > 0$, there are at most $m+1$ balls, $\{B_k\}_{k\leqslant m+1}$, of radius ε such that if

$$C_2^* = C_2 \cap \left(\complement\Big(\bigcup_k (B_k)\Big)^0\right), \quad \text{then } \overline{C_2^*} \cap \bar{C}_1 = \emptyset .$$

LEMMA III. *With the above notation, let E be a connected component of C_2^*. There exists a finite nmber of balls $\{B_k\}$, at most $m+2$, of radius less than ε such that* (i) $B_k \cap E = \emptyset$, (ii) *There exists a Jordan curve, Γ, contained in $\complement(\Omega)^0 \cup \left(\bigcup_k B_k\right)$ separating $\overline{C_1 \cap \left(\complement\left(\bigcup_k (B_k)\right)^0\right)}$ from E.*

PROOF. In virtue of Remark I we may assume that C_2 is contained in the exterior of the exterior Jordan arc of C_1 (applying an inversion if needed). We may cover the finite holes of C_1 and assume that C_1 is connected and simply connected. Since $\overline{C}_1 \cap \overline{E} = \emptyset$, we may choose δ, such that if $x \subset \overline{C}_1$, $y \in \overline{E}$, $|x-y| > 2\delta$.

Consider $x_0 \in \partial C_1$, since $\partial\Omega = \partial\complement(\Omega)^0$, then $x_0 \in \partial D_{j_1}$. Observe that if $z \in \partial C_1 \cap \partial D_{j_1}$, then for any $\delta > 0$ there exists a polygonal arc, γ, contained in D_{j_1} and connecting $\{x, |x-x_0| < \delta\}$ with $\{x, |x-z| < \delta\}$. On the other hand the points x_0 and z divide the boundary of C_1 into two Jordan arcs, Γ' and Γ''. Note that either γ together with Γ' and both balls separates Γ'' from E or else γ together with Γ'' and both balls separate Γ' from E.

Parametrize ∂C_1 with $\{z, |z| = 1\} = S^1$, $F(S^1) = \partial C_2$, $f(\xi_0) = x_0$. Using the above argument let u_1 be the end point of the longest arc, to the left of ξ_0, such that for z in the open arc, (i) $F(z) \in \overline{D}_{j_1}$; (ii) $F(\Gamma_{z,\xi_0})$ is separated from E by the corresponding arc γ, the complementary arc of $F(\Gamma_{z,\xi_0})$ in ∂C_1 and the balls $\{x, |x-x_0| < \delta\{$, $\{x, |x-z| < \delta\}$. As a consequence $F(\Gamma_{u_1,\xi_0})$ is separated from E by an arc $\Gamma_1 \subset \complement(\Omega)^0$, the complementary arc of $F(\Gamma_{u_1,\xi_0})$ and two balls of radius 2δ centered at x_0 and $F(u_1)$.

It follows from the construction that either $u_1 = \xi_0$, and the lemma follows, or else $F(u_1) \in \overline{D}_{j_2}$, $j_2 \neq j_1$. If we repeat the construction induction inductively after at most $m+1$ steps, $u_{m+1} = \xi_0$ and the separation process is completed.

THEOREM I. *Let Ω be as in Lemma II, then if $\Omega = \bigcup_{j=1}^{\infty} C_j$, C_j connected components*

(i) $\mathcal{T}(\Omega) = \sum_{j=1}^{\infty} \mathcal{T}(C^0)$;

(ii) *∂C_j is composed of a finite number, at most m, of rectifiable Jordan curves, Γ_{jk}, and $\mathcal{T}(C_j) = \sum_k$ length (Γ_{jk}).*

PROOF. To prove (i) we separate, recursively, each component from all the others using Lemma III.

Let $x_1 \in C_1$, $x_2 \in C_2$, in virtue of Lemma III we may select a family of balls, $\{B_k^1\}_{k \leqslant 2m+3}$, of radius less than ε_1 and a Jordan curve, Γ, such that

(i) $\Gamma \subset \complement(\Omega)^0 \cup \left(\bigcup_k B_k^1\right)$; (ii) Γ separates $C_1^{(1)}$ from $C_2^{(1)}$, where $C_i^{(1)}$ denotes the connected component of $\Omega_1 = \Omega \cup \left(\complement \bigcup_k (B_k^1)^0\right)$ containing x_i.

As a consequence of Lemma I, we may choose a polygonal domain $\pi_{n_1}^{(1)}$ approximating Ω_1 such that; $\Gamma \subset \complement(\pi_{n_1}^{(1)})^0$ and $K_1 \subset \pi$:: where K_1 is a compact set of connected interior, $x_1 \in K_1 \subset C_1^{(1)}$.

Next we « separate » C_1 from C_2 and C_3 by the above process removing at most $4m + 6$ balls of radius less than ε_2; and so on. In this fashion we construct a sequence of polygonal domains, $\{\pi_n^*\}$, associated to the sequence ε_n, $\varepsilon_n \to 0$ as $n \to \infty$ and to the sequence $\{K_n\}$, $K_n \subset \pi_n^*$, where $m(C_1 \cap \complement(K_n)) \to 0$ as $n \to \infty$.

If B_n denotes the connected component of π_n^* containing K_n observe that, by construction, $m(B_n \cap C_j) \to 0$, as $n \to \infty$, if $j \neq 1$. On the other hand $m(C_1 \cap \complement(B_n)) \leqslant m(C_1 \cap \complement(K_n)) \to 0$, as $n \to \infty$. Hence, since $m(\pi_n^* \Delta \Omega) \to 0$, we have $m(B_n \Delta C_1) \to 0$ and $m\left(\left[\bigcup_{j\geqslant 2} C_j\right] \Delta [\pi_n^* \cap \complement(B_n)]\right) \to 0$, as $n \to \infty$. Therefore

$$\mathfrak{T}(C_1) \leqslant \underline{\lim}\, \mathfrak{T}(B_n) \qquad \text{and} \qquad \mathfrak{T}\Big(\bigcup_{j\geqslant 2} C_j\Big) \leqslant \underline{\lim}\, \mathfrak{T}(\pi_n^* \cap \complement(B_n)) ,$$

that is

$$\mathfrak{T}(C_1) + \mathfrak{T}\Big(\bigcup_{j\geqslant 2} C_j\Big) \leqslant \underline{\lim}\, \mathfrak{T}(B_n) + \underline{\lim}\, \mathfrak{T}(\pi_n^* \cap \complement(B_n)) \leqslant$$
$$\leqslant \underline{\lim}\, [\mathfrak{T}(B_n) + \mathfrak{T}(\pi_n^* \cap \complement(B_n))] + \underline{\lim}\, \mathfrak{T}(\pi_n^*) = \mathfrak{T}(\Omega) .$$

On the other hand the sublinearity of the perimeter yields $\mathfrak{T}(\Omega) = \mathfrak{T}(C_1) + \mathfrak{T}\Big(\bigcup_{j\geqslant 2} C_j\Big)$ and property (i) follows by induction.

To complete the proof we analize ∂C_1.

Note that ∂C_1 is composed of at most m Jordan curves, $\{\Gamma_j\}$, two of which intersect in at most one point. Adding to C_1 at most m balls, $\{B_k^1\}$, of radius ε_1 we may « separate » all the Jordan curves in the following sense: let A_j be a connected component of $\complement(C_1)^0$, $\Gamma_j = \partial A_j$; if $C_{11} = C_1 \cup \left(\bigcup_k B_k^1\right)$ and $x_j \in A_j$, we may select a polygonal approximation, π_1, of C_{11} such that each x_j belongs to different connected components of $\complement(\pi_1)^0$. We proceed inductively to construct a sequence $\{\pi_n\}$ associated to ε_n, $\varepsilon_n \to 0$ as $n \to \infty$.

Let π_n^* be the simply connected closure of π_n (i.e. the set of points that can not be joined to infinity without crossing π_n^0). Clearly $\mathfrak{T}(\pi_n^*) \leqslant \mathfrak{T}(\pi_n) \leqslant \leqslant \mathfrak{T}(C_1) + \delta_n$.

Let $D = \{z, |z| < 1\} \subset \mathbb{C}$, and f_n the conformal mapping from D onto π_n^* such that $f_n(0) = x_0$. If $H^1(D)$ denotes the space of analytic functions on D whose L^1-norm over circumferences of fixed radius remains uniformly bounded (see [9]), then

$$\left(\frac{\partial}{\partial z}\right) f_n \in H^1(D) \qquad \text{and} \qquad \left\| \frac{\partial}{\partial z} f_n \right\|_{H^1} \leqslant \mathcal{T}(\pi_n^*) \leqslant \mathcal{T}(C_1) + \delta_n \, .$$

Therefore $\|f_n\|_{L^\infty(D)} \leqslant \mathcal{T}(C_1) + \delta_n$; and, since $f_n(0) = x_0$, in virtue of Harnack's Theorem there exists a subsequence (which we readily rename $\{f_n\}$) such that f_n converges to a limit f, uniformly in $\bar{D}$. By Fatton's lemma

$$\frac{\partial f}{\partial z} \in H^1(D) \qquad \text{and} \qquad \left\| \frac{\partial}{\partial z} f \right\|_{H^1} \leqslant \mathcal{T}(C_1) \, .$$

At this point we make use of Caratheodory's Theorem, (see Theorem 2.1 of [8]) and choose our sequence $\{f_n\}$ so that f is a conformal mapping from D onto the kernel of the said sequence, G_{x_0}; see Def. on pg. 33 of [8].

In virtue of Lemma I, $G_{x_0} = \complement(A_1)^0$ where A_1 denotes the unbounded component of $\complement(C_1)^0$. Hence

$$\text{length} \quad (\Gamma_1) = \left\| \frac{\partial}{\partial z} f \right\|_{H^1(D)} \leqslant \underline{\lim}\, \mathcal{T}(\pi_n^*) \, .$$

To complete the argument, for each $x_j \in A_j$, consider the connected component of $\complement(\pi_n)^0$ containing x_j, T_{jn}, form its simply connected closure, T_{jn}^*, then repeating the argument above it follows that length $(\Gamma_j) \leqslant \lim \mathcal{T}(T_{jn}^*)$. Therefore, since $\mathcal{T}(\pi_n^*) + \sum \mathcal{T}(T_{jn}^n) \leqslant \mathcal{T}(\pi_n)$ and $\mathcal{T}(\pi_n) \to \mathcal{T}(C_1)$, as $n \to \infty$, we have $\mathcal{T}(C_1) \geqslant \sum \text{length } (\Gamma_j)$.

Finally C_1 is a connected domain bounded by at most m rectifiable Jordan curves and the converse inequality follows restricting the conformal mappings to subcircles of radius less than one.

REMARK. *Under the assumptions of Lemma II, if* $\mathcal{T}((\complement\Omega)^0) < \infty$ *instead of* $\mathcal{T}(\Omega)$ *the arguments of Lemmas II and III, together with Theorem I yield that* $\mathcal{T}(\Omega) \leqslant \mathcal{T}((\complement\Omega)^0)$ *and hence the conclusions of Theorem I remain valid.*

LEMMA IV. *Let* $\Omega \subset R^n$, *be a bounded region with connected components* C_j; $\Omega = \bigcup_{j=1}^{\infty} C_j$: *Assume that* $\mathcal{T}(\Omega) < \infty$ *and that there exists a sequence of polygonal regions,* $\{\pi_n\}$, *increasing* (i.e. $\pi_n \subset \pi_{n+1} \subset \Omega$) *such that* $m(\Omega \cap \complement\pi_n) \to 0$ *and* $\mathcal{T}(\pi_n) \to \mathcal{T}(\Omega)$ *as* $n \to \infty$. *Then*

$$\mathcal{T}(\Omega) = \sum_{n=1}^{\infty} \mathcal{T}(C_j) \, .$$

The sequence $\{\pi_n\}$ *can be constructed under condition* (ii) *of Lemma* I.

PROOF. Let B_n be the union of the connected components of π_n contained in C_1. Clearly $m(C_1 \cap \complement B_n) \to 0$ and $m\Big(\bigcup_{j\geqslant 2} C_j \cup (\complement\pi_n \cup B_n)\Big) \to 0$, as $n \to \infty$, hence

$$\mathcal{T}(C_1) \leqslant \underline{\lim}\, \mathcal{T}(B_n) \qquad \text{and} \qquad \mathcal{T}\Big(\bigcup_{j\geqslant 2} C_j\Big) \leqslant \underline{\lim}\, \mathcal{T}(\pi_n \cap \complement B_n)\,,$$

that is

$$\mathcal{T}(C_1) + \mathcal{T}\Big(\bigcup_{j\geqslant 2} C_j\Big) \leqslant \underline{\lim}\, \mathcal{T}(B_n) + \underline{\lim}\,(\pi_n \cap \complement B_n) \leqslant \leqslant \underline{\lim}\,(\mathcal{T}(B_n) + \mathcal{T}(\pi_n \cap \complement B_n)) \leqslant \mathcal{T}(\Omega)\,.$$

Inductively, it follows that $\sum_{j=1}^{\infty} \mathcal{T}(C_j) \leqslant \mathcal{T}(\Omega)$, the subadditivity of the perimeter proves the theorem.

THEOREM II. *Consider $\Omega \subset R^2$, open, such that if $x \in \partial\Omega$, $m\big((x + B_\varepsilon) \cap \cap (\complement\Omega)^0\big) \geqslant \alpha m(B_\varepsilon)$, where $0 < \alpha < 1$ is uniform for $x \in \partial\Omega$. Write $\Omega = \bigcup_{j=1}^{\infty} C_j$, C_j connected component; then $\mathcal{T}(\Omega) = \sum_{j=1}^{\infty} \mathcal{T}(C_j)$. Moreover $\partial C_j = \bigcup_{k=1}^{\infty} \Gamma_{jk} \cup E_j$ where*

(*a*) *Γ_{jk} is a rectifiable Jordan curve and $\mathcal{T}(C_j) = \sum_{k=1}^{\infty}$ length (Γ_{jk}).*

(*b*) *E_j is the set of limit points of $\{\Gamma_{jk}\}_k$* (i.e. *$x \in E_j$ iff $x + B_\varepsilon$ intersects infinitely many Γ_{jk} and $x \notin \bigcup_k \Gamma_{jk}$) and $H_1(E) = 0$, $H_1(\cdot)$ represents the one dimensional Hausdorff measure.*

PROOF. In virtue of (ii), Lemma I, the first part of the theorem follows from Lemma IV.

To study the boundary of C_j we set $(\complement C_j)^0 = \tilde{\Omega}$ then, by the assumption made on Ω, it follows that $(\complement\tilde{\Omega})^0 = C_j$: Observe that $m(\partial\Omega) = 0$, since $\partial\Omega \subset \bar{\Omega}$ and, if $x \in \partial\Omega$, $m\big((x + B_\varepsilon) \cap \bar{\Omega}\big) + m\big((x + B_\varepsilon) \cap (\complement\Omega)^0 = m(B_\varepsilon)$. $m\big((x + B_\varepsilon) \cap \bar{\Omega}\big) \leqslant (1 - \alpha) m(B_\varepsilon)$; that is $\partial\Omega$ is contained in the subset of $\bar{\Omega}$ of points of density less than one and hence of measure zero. Therefore $\partial C_j = \partial\tilde{\Omega}$ is a set of measure zero. In consequence $\mathcal{T}(C_j) = \mathcal{T}(\tilde{\Omega})$.

But $\tilde{\Omega}$ satisfies the conditions of Theorem II and hence if $\bar{\Omega} = \bigcup_k D_{jk}$, D_{jk} connected components; $\partial D_{jk} = \Gamma_{jk}$ is a rectifiable Jordan curve; $\mathcal{T}(C_j) = \mathcal{T}(\tilde{\Omega}) = \sum_k$ length (Γ_{jk}).

Finally if E_j is the set of limit points of the family $\{D_{jk}\}$ and $\{U_\beta\}$ is a covering by balls of radius less than ε, we may divide the family into four subfamilies $\{U_\beta\}_{\beta \in f_i}$ where all the balls are disjoint. By assumption $m\big(U_\beta \cap (\complement\Omega)^0\big) \geqslant \alpha m(U_\beta)$. On the other hand $\tilde{\Omega} \supset (\complement\Omega)^0$. Then, for each β, either:

(i) $\sum m(D_{jk}) \geqslant \alpha/2m(U_\beta)$ or $D_{jk} \subset U_\beta$, $k \geqslant N_\varepsilon$;

(ii) There exists D_{jk}, $k \geqslant N_\varepsilon$ such that the length of $\partial D_{jk} \cap U_\beta$ is greater than or equal to $\alpha/2r(U_\beta)$.

Observe that, if (i) holds, the isoperimetric inequality of de Giorgi; [3], Theorem VI; yields $\alpha/2r^2(U_\beta) \leqslant C\Big(\sum_{D_{jk}\subset U_\beta, k \geqslant N_\varepsilon} \mathcal{T}^2(D_{jk})\Big)$. Therefore in either case $\sum r(U_\beta) \leqslant C_\alpha \sum_{k \geqslant N_\varepsilon} \mathcal{T}(D_{jk})$ and, since $N_\varepsilon \to \infty$ as $\varepsilon \to 0$, the theorem follows.

Finally, we present a family of simple examples showing that the hypotheses of theorems I and II are essential either for the additivity of the perimeter or for the representation of the topological boundary of the set.

Consider two squares, C_1 and C_2, with a common side, I, and from such side remove countably many disjoint balls, $\{B_j\}$, the sum of whose perimeter is as small as we please and such that $\overline{\bigcup_j B_j} \supset I$.

Set

$$D_1 = C_1 \cap \Big(\bigcup_j \complement(B_j)\Big)^0; \qquad D_2 = C_2 \cap \Big(\bigcup_j \complement(B_j)\Big)^0,$$

then

$$\mathcal{T}(D_1 \cup D_2) < \mathcal{T}(D_1) + \mathcal{T}(D_2).$$

In the same fashion, placing countably many balls on the curves $y = x \sin(1/x)$ or $y = \sin(1/x)$ we can construct domains Ω such that Ω is a connected, simply connected open set of finite perimeter, $\partial\Omega = \partial(\complement\Omega)^0$ and, in the first example, its boundary is a non rectifiable curve. In the second example, the boundary is not even a curve.

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