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On the Singular Support of Distributions and Fourier Transforms on Symmetric Spaces.

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0. – Introduction and Notation.

Let G be a non-compact connected semi-simple Lie group with finite center and K a maximal compact subgroup.

DEFINITION. *Let u be a distribution on $C^\infty(K\backslash G/K)$. Then the spherical singular support of u is the complement of the set of all x in G such that there exist a neighbourhood V_x of x with $KV_xK\subseteq V_x$ and a C^∞-spherical function f such that whenever φ is a C^∞-spherical function whose support lies in V_x, we have $(u,\varphi)=\int_G f(x)\varphi(x)\,dx$.* With standard notation we have

THEOREM 1. *Let u be a distribution on $C^\infty(K\backslash G/K)$. Then the spherical singular support of u is contained in a ball of radius R if the following condition $(*)$ is satisfied. There exist constant N and constants C_m for every positive integer m such that*

$$(*)\qquad |\tilde{u}(\lambda)|\leqslant C_m(1+\|\lambda\|)^N\exp(R\|\operatorname{Im}\lambda\|)$$

for all $\lambda\in\mathfrak{A}_C^*$ with

$$\|\operatorname{Im}\lambda\|\leqslant m\log(1+\|\lambda\|)\,.$$

Conversely if u is as above and u satisfies the condition $(*)$, then the spherical singular support of u is contained in the union of the ball of radius R and the walls of the Weyl chamber $\mathfrak{A}^+$.

The statement that the spherical singular support of u is contained in a ball of radius R means that for all x in the spherical singular support of u, we have $\|H(x)\|\leqslant R$.

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A similar partial result is given in Theorem 2 in the case of a distribution on the symmetric space G/K. See § 2 for a precise statement of Theorem 2.

NOTATION. Let G be a connected semi-simple Lie group with finite center, $\mathfrak{G}$ the Lie algebra of G, $\mathfrak{G}_C$ the complexification of $\mathfrak{G}$, $\mathfrak{U}(\mathfrak{G}_C)$ the universal enveloping algebra of $\mathfrak{G}_C$ and $B(\ ,\ )$ the Cartan Killing form of $\mathfrak{G}_C$. Let $\mathfrak{G} = k \oplus \mathfrak{P}$ be a Cartan decomposition, θ the corresponding Cartan involution $\mathfrak{A} = \mathfrak{A}_{\mathfrak{P}}$ a maximal abelian subspace of $\mathfrak{P}$, K, A the analytic subgroups of G with Lie algebras k and $\mathfrak{A}$ respectively. Let Σ denote the set of restricted roots of the pair $(\mathfrak{G}, \mathfrak{A})$. Let Σ^+ be a system of positive roots and $\mathfrak{A}^+$ the corresponding Weyl chamber. Let $A^+ = \exp \mathfrak{A}^+$. Then $G = K \operatorname{Cl} A^+ K$, where $\operatorname{Cl} A^+$ is the closure of A^+. Let $\mathfrak{A}^*$ be the real dual of $\mathfrak{A}$ and $\mathfrak{A}^*_C$ the complexification of $\mathfrak{A}^*$. Let $M^1 = \{k \in K : \operatorname{Ad} k\mathfrak{A} \subseteq \mathfrak{A}\}$

$$M = \{k \in K : \operatorname{Ad} k|_{\mathfrak{A}} = \text{Identity}\} .$$

Then M^1/M is the Weyl group of the pair $(\mathfrak{G}, \mathfrak{A})$. Let $\mathfrak{G} = k \oplus \mathfrak{A} \oplus \mathfrak{N}$ be an Iwasawa decomposition and $G = KAN$ the corresponding decomposition of the group. Let $\log : A \to \mathfrak{A}$ be the inverse of the diffeomorphisms $\exp : \mathfrak{A} \to A$. Then for $x \in G$, $x = kan$ we write $H(x) = \log a$. For $x \in G$, $X \in \mathfrak{A}$, we write

$$\|x\|^2 = B(H(x), H(x)) , \quad \|X\|^2 = B(X, X) .$$

Let $\mathcal{E}(K \diagdown G/K)$ denote the C^∞-functions which are spherical *i.e.* K bi-invariant. Let $D(K \diagdown G/K)$ denote the subspace of $\mathcal{E}(K \diagdown G/K)$ consisting of functions with compact support.

For $\lambda \in \mathfrak{A}^*_C$. Let φ_λ be the elementary spherical function defined by $\varphi_\lambda(x) = \int_K \exp [(i\lambda - \varrho) H(xk)]\, dk$ for $x \in G$. Then for any f in $D(K \diagdown G/K)$ and $\lambda \in \mathfrak{A}^*_C$, the spherical Fourier transform is the function $\tilde{f}$ on $\mathfrak{A}^*_C$ defined by

$$\tilde{f}(\lambda) = \int_G f(x) \varphi_{-\lambda}(x)\, dx .$$

Let $\mathcal{E}(K \diagdown G/K)$ be endowed with the standard topology (See Eguchi et al., [2], p. 113). Let $\mathcal{E}'(K \diagdown G/K)$ denote the distributions on $\mathcal{E}(K \diagdown G/K)$. Then they are compactly supported distributions. For any $u \in \mathcal{E}'(K \diagdown G/K)$ we define the Fourier transform of u to be the function $\tilde{u}$ on $\mathfrak{A}^*_C$ defined by

$$\tilde{u}(\lambda) = (u, \varphi_\lambda) .$$

Let $C(\lambda)$ be the Harish-Chandra's C-function, (see Warner 4, p. 338). In § 1, we prove a few lemmas which we need in the proof of Theorem 1. In § 2, we prove Theorem 1.

1. – Some basis lemmas.

LEMMA 1. *Let $\alpha > 0$, $\beta \geqslant 0$. Let $\varepsilon > 0$ be given. Then there exist a constant C_ε and a polynomial $P(t_1, t_2)$ such that*

$$\frac{|\Gamma(\alpha + t_1 - it_2)|}{|\Gamma(\beta + t_1 - it_2)|} \leqslant C_\varepsilon (1 + \varepsilon)^{t_1} P(t_1, t_2) \qquad \text{for all } t_1 \geqslant 0 ,$$

where Γ is the gamma function.

PROOF. Choose $M_1(\varepsilon)$ and $M_2(\varepsilon)$ such that

$$\left[\frac{\alpha}{(1 + t_2^2)^{\frac{1}{2}}} + 1\right] \left[\frac{\beta}{(1 + t_2^2)^{\frac{1}{2}}} + 1\right]^{-1} \leqslant 1 + \varepsilon$$

whenever $|t_2| > M_2(\varepsilon)$ and

$$\left|\tan^{-1}\left(\frac{t_2}{\alpha + t_1}\right)\right| < \frac{\pi}{4} \qquad \text{whenever } |t_2| \leqslant M_2(\varepsilon) \text{ and } t_1 > M_1(\varepsilon) .$$

We consider the following 3 cases.

1) $|t_2| > M_2(\varepsilon)$;

2) $|t_2| \leqslant M_2(\varepsilon)$ and $t_1 > M_1(\varepsilon)$.

3) $|t_1| \leqslant M_1(\varepsilon)$ and $|t_2| \leqslant M_2(\varepsilon)$.

Case (1). Let n be the non-negative integer such that $n \leqslant t_1 < n + 1$. Using $\Gamma(Z + 1) = Z\Gamma(Z)$, and $|\Gamma(x + iy)| \sim \sqrt{2\pi}|y|^{x-\frac{1}{2}} \exp(-\pi|y|/2)$ (Ref. Copson (1), p. 224) whenever x lies in a compact set, and y is sufficiently large, we get

$$\frac{|\Gamma(\alpha + t_1 - it_2)|}{|\Gamma(\beta + t_1 - it_2)|} \leqslant C'_\varepsilon \cdot (1 + \varepsilon)^{t_1} (1 + |t_2|^{\alpha-\beta}) .$$

Case (2). Since $|\tan^{-1}(t_2/\alpha + t_1)| \leqslant \pi/4$, using Stirling's asymptotic formula (Copson (1), p. 222) *viz.*, $\Gamma(Z) \sim \exp(-z) Z(1/\sqrt{Z}) O(1)$, we get

$$\frac{|\Gamma(\alpha + t_1 - it_2)|}{|\Gamma(\beta + t_1 - it_2)|} \leqslant C_\varepsilon^2 (1 + |t_1|^{\alpha-\beta}) .$$

Case (3). Since $(t_1, t_2) \to \Gamma(\alpha + t_1 - it_2)/\Gamma(\beta + t_1 - it_2)$ is continuous and therefore there exists a constant C_ε^3 such that

$$\frac{|\Gamma(\alpha + t_1 - it_2)|}{|\Gamma(\beta + t_1 - it_2)|} \leqslant C_\varepsilon^3 .$$

Combining the above three cases we get the result.

LEMMA 2. *Let Γ_μ be as in* (Helgason [3], § 5, p. 461). *Let $\alpha_1, \ldots, \alpha_r$ in Σ^+ be a system of simple roots. Then for every $H \in \mathfrak{A}^+$ there exists a constant K_H such that*

$$|\Gamma_\mu(\lambda)| \leqslant K_H \exp\big(\mu(H)\big)$$

whenever $\operatorname{Im}(\lambda, \alpha_1) \geqslant 0$ *and* $\operatorname{Im}\langle\lambda, \alpha_j\rangle = 0$ *for* $j = 2, \ldots, r$.

PROOF. It is easily seen that for λ satisfying the above conditions $|\langle\mu, \mu\rangle - 2i\langle\lambda, \mu\rangle| \geqslant Cm(\mu)$ for some constant $c > 0$, where $\mu \in L$, $m(\mu) = \sum m_i$, with $\mu = \sum m_i \alpha_i$. Now the lemma can be proved as (Lemma 7.1, Helgason [3], p. 470).

2. – Proof of Theorem 1.

(i) Let u be a distribution on $\mathcal{E}'(K\diagdown G/K)$ whose singular support is contained in a ball of radius R. Let $m > 0$ be an integer given. Let φ in $D(K\diagdown G/K)$ be such that

$$\begin{aligned} \varphi(a) &= 1 \qquad \text{if } \|\log a\| \leqslant R + \frac{1}{2m} \\ &= 0 \qquad \text{if } \|\log a\| > R + \frac{1}{m} . \end{aligned}$$

We then have $u = \varphi u + (1 - \varphi) u = u_1 + u_2$, say.

Since the spherical singular support of u and the support of φ are disjoint, we have $u_2 = (1 - \varphi) u \in D(K\diagdown G/K)$. Hence by Paley-Wiener theorem for spherical functions with compact support we have the following estimate.

There exists a constant l such that for every positive integer N there exists a constant C_N such that

$$|\tilde{u}_2(\lambda)| \leqslant C_N (1 + \|\lambda\|)^{-N} \exp\big(l\, \|\operatorname{Im} \lambda\|\big) . \tag{1}$$

Now u_1 is compactly supported distribution whose support is contained in a ball of radius $R + 1/m$. Hence by Theorem 3 (Eguchi et al., [2], p. 113).

$$(2) \qquad |\tilde{u}_1(\lambda)| \leqslant C((1 + \|\lambda\|)^{M-1} \exp\left(R + \frac{1}{m}\right) \|\mathrm{Im}\,\lambda\| .$$

Combining (1) and (2) we get the results.

Proof of the second statement of Theorem 1. We first observe that by using the technique of regularisation as in (Eguchi et al. § 3) we can prove the following Plancheral type formula.

For any $\psi \in D(K\diagdown G/K)$, $u \in \mathcal{E}'(K\diagdown G/K)$,

$$(3) \qquad \langle u, \bar{\psi}\rangle = \frac{1}{|w|}\int_{\mathfrak{A}^*} \tilde{u}(\lambda)\,\tilde{\bar{\psi}}(-\lambda)\,|C(\lambda)|^{-2}\,d\lambda\,,$$

where $\bar{\psi}$ denotes the complex conjugate of ψ and $C(\lambda)$ is the Harish-Chandra's C-function. (See Warner, p. 338). But $\tilde{\bar{\psi}}(-\lambda) = \int_{A^+} \bar{\psi}(a)\varphi_\lambda(a)\Delta(a)\,da$ where $\Delta(a) = \Delta(\exp H) = \prod_{\alpha\in\Sigma^+} (\sin h\alpha(H))^{m_\alpha}$, where $a = \exp H$, $H \in \mathfrak{A}$, m_α the multiplicity of α. So (3) becomes

$$(4) \qquad \langle u, \bar{\psi}\rangle = \frac{1}{|W|}\int_{\mathfrak{A}^*}\int_{A^+} \tilde{u}(\lambda)\,\bar{\psi}(a)\Delta(a)\varphi_\lambda(a)\,|C(\lambda)|^{-2}\,da\,d\lambda\,.$$

Using Harish-Chandra's asymptotic expansion of φ_λ viz.

$$(5) \qquad \varphi_\lambda(a) = \exp\left(-\varrho(\log a)\right) \sum_{s\in W} \varphi(s\lambda : a)\,c(s\lambda)$$

where

$$(6) \qquad \varphi(\lambda : a) = \exp\left[i\lambda(\log a)\right] \sum_{\mu\in L} \Gamma_\mu(\lambda)\exp\left[-\mu(\log a)\right)$$

where L is the set of integral linear combinations of the simple restricted roots. Substituting (5) in (4) and changing $\lambda \to s^{-1}\lambda$, for $s \in W$. We get

$$(7) \qquad \langle u, \bar{\psi}\rangle = \int_{\mathfrak{A}^*}\int_{A^+} \tilde{u}(\lambda)\exp\left(-\varrho(\log a)\right)\varphi(\lambda : a)\,C^{-1}(-\lambda)\,\bar{\psi}(a)\,\Delta(a)\,da\,d\lambda\,.$$

Let $H_1, \ldots, H_r$ be an orthonormal basis for $\mathfrak{A}$. Define $T^{-1}\colon \mathfrak{A} \to \mathfrak{A}$ by setting $T^{-1}\xi = \sum \langle \xi, \alpha_j \rangle H_j$. Changing λ to $T\lambda$ in (7). We get

$$(8) \qquad \langle u, \bar{\psi} \rangle = |\det T| \int_{\mathfrak{A}^*} \int_{A^+} \tilde{u}(T\lambda) \exp\left(-\varrho(\log a)\right) \varphi(T\lambda\colon a)\, C^{-1}(-T\lambda) \cdot \cdot \bar{\psi}(a)\, \Delta(a)\, da\, d\lambda .$$

Let $\psi \in C_c^\infty(A^+)$. Identifying $\mathfrak{A}_C^*$ with C^r through the basis of $\mathfrak{A}$ and the Killing form, we define for every positive integer m

$$\begin{aligned} \Gamma_m = \{(\lambda_1, \lambda') = \lambda \in \mathfrak{A}_C^* = C^r;\ \lambda_1 \in C, \\ \lambda' = (\lambda_2, \ldots, \lambda_r) \in R^{r-1} \text{ such that} \\ \operatorname{Im} \lambda_1 = m \log\left(1 + [\|\operatorname{Re} \lambda_1\|^2 + \|\lambda^1\|^2]^{\frac{1}{2}}\right)\} . \end{aligned}$$

By using the explicit formula for $C(\lambda)$ and the estimate in Lemma 1, we get for every $\varepsilon > 0$, there exists a constant C_ε such that

$$(9) \qquad |C^{-1}(-T\lambda)| \leqslant C_\varepsilon (1+\varepsilon)^{l \operatorname{Im} \lambda_1} (1 + \|\lambda\|)^L$$

for some positive integers l and L, independent of ε.

Using Lemma (2), we have for a given $H \in \mathfrak{A}^+$ such that

$$\left| \sum \Gamma_\mu(T\lambda) \exp\left(-\mu(\log a)\right) \right| \leqslant K_H \sum_{\mu \in L} \exp\left(-\mu(\log a - H)\right) .$$

Choosing H such that $\alpha(\log a - H) > 0$ for all $a \in$ Support of ψ, for all $\alpha \in \Sigma^+$, we see that $\sum_{\mu \in L} \Gamma_\mu(T\lambda) \exp\left(-\mu(\log a)\right)$ converges uniformly for $\lambda \in \Gamma_m$. These facts justify the shifting of the integral over $\mathfrak{A}^*$ in (8) to the integral over Γ_m.

Let now $\delta > 0$ be given. Choose $\psi \in C_c^\infty(A^+)$ such that support of

$$\psi \subseteq \{a\colon \langle TH_1, \log a \rangle > R\|TH_1\| + l \log(1+\varepsilon) + \delta\}.$$

For $a \in$ support of ψ, $\lambda \in \Gamma_m$ we have, using the given estimate for $\tilde{u}$,

$$\begin{aligned} (10) \qquad I &= |\hat{\psi}(T\lambda) \varphi(T\lambda\colon a)\, C^{-1}(-T\lambda)| \leqslant \\ &\leqslant C_\varepsilon (1 + \|\lambda\|)^{L+N} \exp\left(\operatorname{Im} \lambda_1 \{R\|TH_1\| + l \log(1+\varepsilon) - \langle TH_1, \log a \rangle\}\right) . \end{aligned}$$

For $a \in$ support of ψ, we have therefore

$$(11) \qquad I \leqslant (1 + \|\lambda\|)^{N+L-m\delta}$$

$\therefore$ (8) can be written as

$$\int_{A^+} \bar{\psi}(a)\Delta(a)\int_{\Gamma_m} G(T\lambda, a)\,d\lambda\,da\,.$$

Using (11) and choosing m sufficiently large and observing $\|\lambda\|/\|\mathrm{Re}\,\lambda\|$ and $\|d\lambda_1\|/\|d\,\mathrm{Re}\,\lambda_1\|$ remain bounded over Γ_m, we see that $F\colon a \to \int_{\Gamma_m} G(T\lambda, a)\,d\lambda$ is a C^∞ function in the support of ψ. As ε and δ are arbitrary, the above function F is C^∞ in

$$\{a \in A^+\colon \langle TH_1, \log a\rangle > R\|TH_1\|\}\,.$$

By changing $\xi \to T'\xi$ with $T' \in GL(r, \boldsymbol{R})$ and at least one of the rows of T' consists of non-negative entries and proceding as above we can show that F is C^∞ in A^+ outside the ball $B(0, R)$.

This completes the proof of Theorem 1.

THEOREM 2. 1) *Let $u \in \mathcal{E}'(G/K)$ be such that the singular support of u is contained in $B(eK, R)$, where the metric on G/K is the one induced by the Killing form. Then we have a constant N and a constant C_m for every non-negative integer m such that*

$$(*)\qquad |\tilde{u}(\lambda, kM)| \leqslant C_m(1 + \|\lambda\|)^N \exp\left(R\,\|\mathrm{Im}\,\lambda\|\right)$$

for all $\lambda \in \mathfrak{A}_C^*$ with $\|\mathrm{Im}\,\lambda\| \leqslant m \log\left(1 + \|\lambda\|\right)$; for any $kM \in K/M$.

2) *Let $u \in \mathcal{E}'(G/K)$ be such that u satisfies the estimate $(*)$ above. Then u is C^∞ in $Q = \{xK \in G/K\colon x \in KA^+K$ with $\|H(x)\| > R\}$.*

Part (1) can be proved as that of Theorem 1. To prove part (2), we decompose any $f \in C_c^\infty(G/K)$ as $f = \sum_{\delta\in\hat{K}}$ Trace f^δ (See, for all unexplained notations and definitions, Helgason [3]). Then part (2) is easily seen to be equivalent to showing that for every $\delta \in \mathrm{K}$, and for every $x_0 \in Q$ there exists a neighbourhood $U_{x_0} \subseteq Q$ and a C^∞-function g in U_{x_0}, g independent of δ such that

$$\langle u, f^\delta\rangle = \int_{A^+} g_1^\delta(a, 0) f^\delta(a, 0)\Delta(a)\,da\,,$$

where

$$g_1(a, 0) = \int_K g(ka)\;\delta(k)\;dk\,.$$

Using Harish-Chandra's asymptotic expansion for the Einstein integral and using Lemma (2) of § 1, we may proceed as in the proof of Part (2) of Theorem 1 and conclude that u is C^∞ in Q.

REMARK. When G is complex or the split rank of G is 1, one can show that the part (2) of Theorem (1) (resp. Th. 2) can be changed the converse of part (1) of Theorem (1) (resp. Th. 2).

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