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On Resolving Singularities and Relating Bordism to Homology.

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Introduction.

A well-known theorem of Thom ([8]) asserts that, given an n-dimensional (integral) homology class z, there exists a positive integer N, depending only on n, such that Nz is representable by a smooth oriented manifold (briefly, z is *N-representable*).

Wall has proved that N can be taken to be odd (see [9], or [2], 15.3). Consequently, any 2-torsion homology class is representable by a smooth oriented manifold.

We obtain the following results (see § 0 for notation):

(4.3) there exists an explicit upper bound $\alpha(n-3, \Omega)$ for N, which is odd and contains as prime divisors precisely the odd primes $\pi < n/2$.

Consequently:

(4.5) let z be an n-dimensional homology class such that $\beta z = 0$. Then z is ϱ-representable, with $\varrho = \text{G.C.D.}\ (\alpha(n-3, \Omega); \beta)$. In particular, if z belongs to the π-torsion with π prime and $\pi = 2$ or $\pi \geqslant n/2$, then z is representable by a smooth oriented manifold.

(4.7*b*) an obvious generalisation of Thom's example ([8], p. 62) implies that the above result is the best possible as regards representability of π-torsion homology classes: in fact, for any odd prime π, there exists a class $z \in H_{2\pi+1}(K(\mathbb{Z}_\pi, 1) \times K(\mathbb{Z}_\pi, 1))$ such that $\pi z = 0$ and z is not representable by a smooth oriented manifold.

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(4.6) let n and γ be positive integers; suppose γ is not divisible by any odd prime $\pi < (n+1)/2$. Then any $\mathbb{Z}_\gamma$-homology class of dimension n is representable by a smooth $\mathbb{Z}_\gamma$-manifold (note the special case γ prime $\geqslant (n+1)/2$).

(4.8) for $n \geqslant 0$ there exists a natural transformation of functors $\xi(\Omega)$: $\Omega_n(X) \to \sum_{i+j=n} H_i(X; \Omega_j)$ which is defined on the category of *all* topological spaces and is an $\alpha(n, \Omega)$-isomorphism.

$\xi(\Omega)$ is constructed geometrically: it provides an approximation of bordism by homology which depends on the dimension n and is more accurate than the one obtained by Conner and Floyd in [2] (14.2, p. 41).

(4.9) let $n \geqslant 0$ and let X be a topological space such that $H_*(X)$ has no π-torsion, for all odd primes $\pi < (n+3)/2$. Then there is an isomorphism $\Omega_n(X) \cong \sum_{i+j=n} H_i(X; \Omega_j)$.

Consider the following: given integers n and q and a map $f: M^n \to X$ (M an oriented smooth manifold, X *any* topological space), when does there exist an oriented homology of f to zero carrying singularities in codimension $\geqslant q$? In answer to this question we have the following result:

(4.10) suppose that, for $0 \leqslant 4k \leqslant q-2$, $H_{n-4k}(X)$ has no π-torsion, for any odd prime $\pi < (q+1)/2$. Then $f: M^n \to X$ is bordant to zero in X with singularities in codimension $\geqslant q$ if and only if certain a priori obstructions $\theta_I \in H_{n-4k}(X)$ and $\bar{\theta}_h \in H_*(X; \mathbb{Z}_2)$ vanish.

Analogous results hold for almost complex bordism $U_*(-)$. In this case many constructions are simplified, due to the fact that $U_*(\text{point})$ is free and its elements are charachterized by Chern numbers. Therefore we have preferred to deal with the almost complex resolution in detail (§§ 1, 2, 3) and indicate the modifications which are needed in the smooth oriented case (§ 4).

The methods which we use are geometric in nature, relying on the circumstance (observed by Sullivan in [7]) that the set of singularities of a cycle P gives rise to a natural homology class with coefficients in an appropriate bordism group of manifolds (see 1.6). This class is the only obstruction to reducing the dimension of the singularity of X and we use characteristic numbers as complete bordism invariants to study its torsion.

In this connection, we remark that, since this paper was written, the theorem that Chern numbers are complete bordism invariants in U_* has been given a geometric proof using no Steenrod algebra or formal groups (see [12] or [11], where the case of Stiefel Whitney numbers is considered).

0. – Notation and terminology.

Almost complex manifold = smooth oriented manifold with a complex structure on its stable tangent bundle.

$U_n(—)$ = almost complex bordism group of dimension n.

$\Omega_n(—)$ = smooth oriented bordism group of dimension n.

$T_n(—)$ = smooth unoriented bordism group of dimension n.

$H_n(—)$ = integral homology.

$U_n = U_n(\text{point})$; $\Omega_n = \Omega_n(\text{point})$; $T_n = T_n(\text{point})$.

An element in one of the groups $U_n(X)$, $\Omega_n(X)$, $T_n(X)$, $H_n(X)$ (or any other bordism group) will be denoted $[P]_X$ or $[P \xrightarrow{f} X]$.

$[a/b]$ = integral part of the rational number a/b.

cX = cone on X with vertex c.

$A - B = \{x \in A : x \notin B\}$.

If A is a simplex, then $\hat{A}$ = barycenter of A; $\dot{A}$ = boundary of A; $\mathring{A} = A - \dot{A}$.

For n a positive integer and X a topological space, nX = disjoint union of n copies of X; that is, $nX = \coprod_{i=1,\ldots,n} X \times \{i\}$.

Let $\xi \colon G \to H$ be a homomorphism of finitely generated abelian groups and α a positive integer; we say that ξ is an *α-monomorphism* (resp. *α-epimorphism*) if, for any $x \in \ker \xi$ (resp. $\operatorname{coker} \xi$), there exists a positive integer s such that $\alpha^s x = 0$. Of course these definitions depend only on the primes dividing α.

1. – Complex cycles.

1.1 DEFINITION. A *complex cycle* of dimension n (*n-cycle*) is a polyhedron P such that:

(1) P is a PL cycle of dimension n with singularity SP (see [6], p. 98);

(2) $P - SP$ is an almost complex manifold compatible with the given PL structure.

1.2. DEFINITION. A *complex n-cycle with boundary* (or *relative n-cycle*) is a PL n-cycle with boundary P, ∂P (ibid.) such that $P - SP$ is an almost complex manifold with boundary $\partial P - SP$.

We set $p = \dim SP \leqslant n - 2$.

1.3 REMARK. It is immediately seen that the bordism theory of complex cycles is integral homology theory.

Let K, L be a triangulation of P, SP with L full in K and oriented (*i.e.* each simplex is oriented); $K^{(1)}/L$ the first barycentric derived of K away from L; A a p-simplex of L; $D(A, K)$ the dual cone of A in K. Then $\mathrm{st}\,(A, K^{(1)}/L) = \dot{A} * D(A, K)$ and $\dot{D}(A, K) = \mathrm{lk}\,(A, K^{(1)}/L) \cong \mathrm{lk}\,(A, K)^{(1)}$. For the sake of simplicity, we shall often make no difference in the notation between the triangulation K and the underlying polyhedron P.

Because $P - SP$ is an almost complex manifold, the smoothing theorems of [3] imply that the PL manifold $\mathrm{st}\,(A, K^{(1)}/L) - A$ has a well-defined almost complex structure with boundary $\dot{A} * \dot{D}(A, K) - \dot{A} \cong \mathring{A} \times \dot{D}(A, K)$, which induces an almost complex structure on $\dot{D}(A, K)$. Therefore, associated to K, there is the « singularity » p-chain with coefficients in U_{n-p-1} defined by Sullivan in [7], that is $C(K) = \sum_A [\dot{D}(A, K)]A$, $\dim A = p$.

Remark on combinatorial chains.

It is well-known that, given an integral cycle as a formal sum of p-simplexes of an oriented simplicial complex, one can realise it geometrically by glueing the $(p-1)$-faces together in pairs according to a cancellation rule. It is clear that this construction can be extended to the more general case of a simplicial p-chain with coefficients in an abelian group G.

Precisely, we define a *combinatorial p-chain with coefficients in* G to be an oriented simplicial complex $\Gamma(p)$, purely p-dimensional, in which every p-simplex A is labelled by an element $g(A) \in G$.

Given an oriented simplicial p-chain $c = \sum_A A \otimes g(A)$ in K, its *realisation* is the combinatorial p-chain which consists of the subcomplex of K formed by the union of those closed p-simplexes A for which $g(A) \neq 0$, each simplex A being labelled by $g(A)$.

Let B be a $(p-1)$-simplex of $\Gamma(p)$ which is the face of exactly the simplexes $A_1, .., A_r$ and set $g(B) = \sum_{i=1,\dots,r} \varepsilon(B, A_i) g(A_i)$ ($\varepsilon(B, A_i)$ = incidence number). We define $\delta\Gamma(p)$, the *boundary* of $\Gamma(p)$, as the realisation of $\sum_B B \otimes g(B)$.

Two combinatorial p-chains $\Gamma(p)$, $\Gamma'(p)$ (with coefficients in G) are *isomorphic* if there exists an isomorphism of oriented simplicial complexes $\Gamma(p) \cong \Gamma'(p)$ which preserves the labels.

In the sequel, it will be convenient to regard a «singular p-chain with coefficients in G» in a topological space X as a map $g\colon \Gamma(p) \to X$ rather than as a linear combination of singular p-simplexes with coefficients in G (the two points of view are easily seen to be equivalent).

1.4 *Thickening of a combinatorial chain $\Gamma(p)$ with coefficients in U_q $(q > 0)$.*

If A is an oriented simplex and $V(A)$ is a closed almost complex manifold of positive dimension, then $\dot{A} * \hat{A}V(A)$ has a structure of relative complex cycle induced by the homeomorphism

$$\dot{A} * \hat{A}V(A) \cong A \times \hat{A}V(A)/(a, x) \sim (a, x') \qquad \text{for any } a \in \dot{A} \text{ and } x, x' \in \hat{A}V(A)$$

with boundary $\partial(\dot{A} * \hat{A}V(A)) = \dot{A} * V(A)$.

Given a combinatorial p-chain $\Gamma(p)$ with coefficients in U_q $(q > 0)$, the relative complex cycle $\tilde{\Gamma}(p) = \coprod_A \dot{A} * \hat{A}V(A)$ $(\dim A = p,\ [V(A)] = \text{label of } A)$ will be referred to as a *thickening* of $\Gamma(p)$. We shall often assume tacitly that $V(A)$ is equipped with a triangulation; in this way $\tilde{\Gamma}(p)$ is a simplicial complex via the join triangulations.

1.5 NOTE. If $B < A$ is a codimension one face of A, then

$$\operatorname{lk}\big(B, \dot{A} * V(A)\big) = \varepsilon(B, A)\, V(A)\,.$$

Let $[P \xrightarrow{f} X] \in H_n(X)$, with X a topological space, P a complex cycle and K, L a triangulation of P, SP as above.

1.6 THEOREM (Sullivan [7], p. 204). *The singularity p-chain $C(K)$ is a cycle and, if $[C(K)]_X = 0$, then $[P]_X = [P']_X$, with P' a complex cycle and* $\dim SP' < p = \dim SP$.

PROOF. In order to see that $C(K)$ is a cycle it is enough to look at the links of the $(p-1)$-simplexes of L (see [4], [7]).

After labelling each p-simplex A of L by $[\dot{D}(A, K)]$ we obtain a combinatorial chain $\Gamma(p)$ realising $C(K)$. Suppose $C(K) = 0$. Then, $\dot{D}(A, K)$ is an almost complex boundary, hence there exists an almost complex manifold $W(A)$ with $\partial W(A) = \dot{D}(A, K)$. We replace $\operatorname{st}(A, K) = \dot{A} * \hat{A}\dot{D}(A, K)$ by $\dot{A} * W(A)$ for each p-simplex $A \in L$; the resulting polyhedron is the required complex cycle P', which is a blow-up of P in the sense of [4], [7].

If it is only $[C(K)]_X = 0$, there exists a combinatorial chain $\Gamma(p+1)$ (with coefficients in U_{n-p-1}) with $\delta\Gamma(p+1) = \Gamma(p) = \Gamma(p+1) \cap K$ and a map $g\colon \Gamma(p+1) \to K$ extending $f|\Gamma(p)$.

In the simplicial product $K \otimes [0, 1]$ we identify $K \otimes 0$ with K and form the relative complex cycle

$$Q = K \otimes [0,1] \coprod_{\Gamma(p)\otimes 1} \tilde{\Gamma}(p+1)\,.$$

There is an obvious map $Q \xrightarrow{F} X$ extending $P \xrightarrow{f} X$, obtained via g and the projection $\tilde{\Gamma}(p+1) \to \Gamma(p+1)$. Thus $Q \xrightarrow{F} X$ provides a homology between

$$K = K \otimes 0 \xrightarrow{f} X \quad \text{and} \quad K'' = K \otimes 1 \coprod_{\Gamma(p)\otimes 1} \partial\tilde{\Gamma}(p+1) \to X\,.$$

At this point we observe that, by 1.5 and the definition of the boundary δ, we have $C(K'') = 0$, and so we are reduced to the previous case. □

1.7 REMARK. Theorem 1.6 is slightly more general than Sullivan's theorem ([7], p. 204). In fact we assume $[C(K)]_X = 0$, which is only implied by the hypothesis $[C(K)]_P = 0$ of [7] and we do not require the existence of a degree-one map $P' \to P$. On the other hand, the « only if » part of the statement will not be true in this more general case. We shall need our version of theorem 1.6 only in the proof of 3.5 (*b*).

2. – Almost complex resolution.

Let $\mathcal{T}(k)$ be the set of partitions of the positive integer k.

2.1 PROPOSITION. *There exists a basis* $\{[M_I], I \in \mathcal{T}(k)\}$ *of* U_{2k} *and polynomials in the (tangential) Chern classes* $\{s_I, I \in \mathcal{T}(k)\}$ *such that*:

$$(a)\quad s_I|M_J| = \begin{cases} 0 & \text{if } I \neq J \\ \varrho_I \neq 0 & \text{if } I = J \end{cases} \qquad (|\ | = \text{fundamental class})$$

(*b*) *each* ϱ_I *is a divisor of the L.C.M. of* $\{a_I,\ I \in \mathcal{T}(k)\}$, *where* $a_I = (i_1+1)(i_2+1)\dots(i_r+1)$, *I being the partition* $(i_1, \dots, i_r)$ *of* k.

PROOF. The existence of $\{s_I\}$ and $\{[M_I]\}$ follows easily from [5] (§ 16, theorem 16.7). □

Now, for each integer $n \geqslant 0$, define

$$\omega_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ \text{L.C.M. } \{a_I, I \in \mathfrak{I}(k)\} & \text{if } n = 2k \end{cases}$$

and $\alpha(n, U) = \prod_{i=1,\ldots,n} \omega_i$.

2.2 REMARK. Later we shall be interested in the primes dividing $\alpha(n, U)$; it is easily checked that $\omega_{2k} = \prod_p p^{[k/(p-1)]}$, where p is prime, $p \leqslant k+1$. It follows that, for each n, the primes dividing $\alpha(n, U)$ are precisely those primes p which are less than $(n+3)/2$.

2.3 MAIN LEMMA. *For any* $[P]_X \in H_n(X)$, *we have* $\omega_{n-p-1}[P]_X = [P']_X$, *where* P' *is a complex cycle with* $\dim SP' < p = \dim SP$. (*we call* $[P']_X$ *a resolution of* $[P]_X$).

PROOF (using the notation introduced previously). For $n-p-1$ odd, we have $U_{n-p-1} = 0$; therefore the obstruction chain $C(K)$ is zero and the result follows immediately from Sullivan's theorem 1.6. Suppose $n-p-1 = 2k$; let $K^{(1)}$ be the first barycentric derived of K; $N = N(K^{(1)})$ the simplicial neighbourhood of $L^{(1)}$ in $K^{(1)}$ and $\dot{\pi}: \dot{N} \to L^{(1)}$ the simplicial map induced by the pseudo-radial projection along the lines through the vertices of N (recall that N is a simplicial mapping cylinder of $\dot{\pi}$). If Q is the closed complement of N in $K^{(1)}$, then, by the smoothing theorems of [3], Q is a triangulated almost complex manifold with $\partial Q = \dot{N}$.

For each $I \in \mathfrak{I}(k)$, represent the Lefschetz duals of $s_I(Q)$, $s_I(\dot{N})$ by simplicial chains G_I, E_I respectively, with $\partial G_I = E_I$. Let E_I^+ be the chain of N given by the simplicial mapping cylinder of $\dot{\pi}|E_I$ and set

$$G = \sum_I (\omega_{n-p-1}/\varrho_I)(G_I + E_I^+) \otimes [M_I].$$

CLAIM. $\partial G = \omega_{n-p-1} C(K)^{(1)}$.

The claim follows from the

BOUNDARY RULE. Let A be a p-simplex of $L^{(1)}$. In the group U_{n-p-1}, $[\dot{\pi}^{-1}(\hat{A})]$ can be written uniquely as a linear combination of the generators $\{[M_I]\}$ with integral coefficients. If x_I is the coefficient of $[M_I]$, then A appears in the boundary $G_I + E_I^+$ with multiplicity exactly $x_I \varrho_I$.

PROOF OF THE BOUNDARY RULE. By transversality we have a commutative diagram

$$
(1) \qquad \begin{array}{ccc} \mathring{A}\times\dot{\pi}^{-1}\hat{A} & \longrightarrow & \dot{\pi}^{-1}\mathring{A} \\ & {\scriptstyle\text{projection}}\searrow \quad \swarrow{\scriptstyle\dot{\pi}} & \\ & \mathring{A} & \end{array} \qquad (A\subset B\in L)
$$

We U-orient $\dot{\pi}^{-1}\hat{A}$ so that the product of the orientations of $\mathring{A}$ and $\dot{\pi}^{-1}\hat{A}$ is compatible with the U-orientation of $\partial(N-SP)$. In this way, $\dot{\pi}^{-1}\hat{A}$ becomes an almost complex manifold isomorphic to $\dot{D}(B, K)$ (by the smoothing theorems of [3]). Now, by diagram (1), E_I intersects $\dot{\pi}^{-1}\hat{A}$ transversally in a finite set of oriented points whose algebraic sum is the coefficient ε_I of A in $\partial(G_I+E_I^+)$. On the other hand, as the intersection is transverse, the 0-cycle $E_I\cap\dot{\pi}^{-1}\hat{A}$ is Lefschetz dual to $s_I(\dot{N})|\dot{\pi}^{-1}\hat{A}$. Therefore, by diagram (1) and the naturality of the classes s_I, we have $\varepsilon_I=x_I\varrho_I$. □

At this point, the main lemma is a consequence of theorem 1.6 applied to the complex cycle $\omega_{n-p-1}P$ (rel SP), which is formed by taking the disjoint union of ω_{n-p-1} copies of P and identifying them along SP; that is,

$$\omega_{n-p-1}P\ (\mathrm{rel}\, SP)=\coprod_{i=1,\dots,\omega_{n-p-1}} P\times\{i\}/(x,i)\sim(x,j)$$

for any $x\in SP$, $1\leqslant i$, $j\leqslant\omega_{n-p-1}$.

Its homology class $[P']_X$ clearly equals

$$[P']_X=[\omega_{n-p-1}P\ (\mathrm{rel}\, SP)\xrightarrow{\amalg f}X]=\omega_{n-p-1}[P]_X \qquad \square$$

2.4 REMARK. We note that, if we are interested in a particular $[P]_X$, the complex cycle P' constructed in the above proof may not be the simplest way of finding a resolution of a multiple of P. For instance, suppose $C(K)$ is of the form $\left(\sum_A A\right)\otimes[M]$ and γ is a polynomial in the (tangential) Chern classes such that $\gamma|M|=m\neq 0$. Then, proceeding as in 2.3, we find a simplicial chain G on $K^{(1)}$ with $\partial G=mC(K)$; from G, a resolution P' of mP is constructed in the usual way.

An iterated application of the main lemma gives

2.5 THEOREM. *Any integral homology class of dimension n is $\alpha(n, U)$-representable by an almost complex manifold.*

In fact, it is $\alpha(n-3, U)$-representable.

SKETCH OF THE PROOF. In order to prove $\alpha(n-3, U)$-representability, it suffices to show that, if $\dim SP \leqslant 1$, then $[P]_X = [P']_X$ with $\dim SP' < \dim SP$. The case $\dim SP = 0$ is easily dealt with; therefore we assume $\dim SP = 1$ and, for the sake of simplicity, $C(K)$ of the form $\left(\sum_A A\right) \otimes [M_I]$, where M_I is as in 2.1 (the general case is analogous). Suppose $z = \sum_A A \neq 0$ in $H_1(P; \mathbb{Z})$; then there exists a prime q such that the reduction of z mod q (still denoted z) is not zero. Let $z^* \in H^1(P; \mathbb{Z}_q)$ be the dual of z, i.e. $\langle z, z^* \rangle = 1$. Now $H^1(P; \mathbb{Z}_q) = [P, L]$, where L is a high dimensional q-lens space. Note that the fundamental class $\alpha \in H^1(L; \mathbb{Z}_q)$ has a Poincaré dual cycle which is a «twisted $\mathbb{Z}_q$-manifold». Thus transversality gives that the homology class $z^* \cap |P| \in H_{n-1}(P; \mathbb{Z}_q)$ is represented by a polyhedron Q, which is a (weakly) almost complex manifold outside the disjoint union $* \coprod V$, where $*$ is a point and V is the twisted $\mathbb{Z}_q$-singularity set ($\dim V = n-2$); moreover $\mathrm{lk}(*, Q) = M_I$. Hence, after excising an open regular neighbourhood of $* \coprod V$, one obtains an almost complex bordism between M_I and a q-fold cover. This is a contradiction because of the choice of M_I. Thus $z = 0$ in $H_1(P; \mathbb{Z})$ as required. □

2.6 COROLLARY. *Let z be an n-dimensional homology class such that $\beta z = 0$; then z is ϱ-representable, with $\varrho = $ G.C.D.$(\alpha(n-3, U); \beta)$. In particular, if z belongs to the π-(primary) torsion with π prime and $\pi \geqslant n/2$, then z is representable by an almost complex manifold.* □

We conclude this section with some remarks on representability of homology classes with coefficients in a finite cyclic group $\mathbb{Z}_\gamma$. It is known that almost complex $\mathbb{Z}_\gamma$-manifolds (as defined by Sullivan) represent almost complex bordism with $\mathbb{Z}_\gamma$-coefficients and that, if $\gamma \neq 2$, not every $\mathbb{Z}_\gamma$-homology class is representable by an almost complex $\mathbb{Z}_\gamma$-manifold. We have the following:

2.7 THEOREM. *Let n and γ be positive integers and suppose γ is prime to $\alpha(n-2, U)$. Then any $\mathbb{Z}_\gamma$-homology class of dimension n is representable by an almost complex $\mathbb{Z}_\gamma$-manifold. (Note the special case γ prime, $\gamma \geqslant (n+1)/2$.*

PROOF. Let $U_n(X; \mathbb{Z}_\gamma)$ be the n-dimensional bordism group of the topological space X with coefficients in $\mathbb{Z}_\gamma$ and let $\lambda\colon U_n(X; \mathbb{Z}_\gamma) \to H_n(X; \mathbb{Z}_\gamma)$ be the natural map. We must prove that, under the assumption of the theorem, λ is onto. Let $M\mathbb{Z}_\gamma$ be a 2-dimensional Moore space for $\mathbb{Z}_\gamma$; then there are equivalences

$$U_n(X; \mathbb{Z}_\gamma) \cong U_{n+1}(X \wedge M\mathbb{Z}_\gamma) \quad \text{and} \quad H_n(X; \mathbb{Z}_\gamma) \cong H_{n+1}(X \wedge M\mathbb{Z}_\gamma)\,.$$

Thus we only have to prove that the Steenrod map $\mu\colon U_{n+1}(X\wedge M\mathbf{Z}_\gamma)\to$ $\to H_{n+1}(X\wedge M\mathbf{Z}_\gamma)$ is onto. This is a consequence of the hypothesis and corollary 2.6. $\square$

3. – Approximation of bordism by homology.

The aim of this section is to show that the n-dimensional part of $H_*(X)\otimes U_*$ approximates $U_n(X)$ up to an $\alpha(n, U)$-isomorphism. This result represents a « Conner-Floyd type » theorem at each dimension (compare [2], theorem 14.2, p. 41).

Let $U_n(X; q)$ be the bordism theory set up using complex cycles and bordisms P, SP such that $\dim P - \dim SP \geqslant q$ (i.e. singularities are in codimension at least q). There is, for each $n\geqslant 0$, a sequence of natural transformations

$$U_n(X)\cong U_n(X; n+2)\to\ldots\to U_n(X; q+1)\xrightarrow{\varphi_{q+1}} U_n(X; q)\xrightarrow{\varphi_q} U_n(X; q-1)\to$$
$$\to\ldots\to U_n(X; 2)\cong H_n(X)$$

defined by the identity on representatives.

The following lemma (and its proof) is essentially a relative version of 1.6 and it will insure that the main constructions of §§ 1-2 apply to this more precise context.

3.1 Lemma. *Let P_0 be a relative complex cycle of dimension n* $\operatorname{codim} SP_0 = q$, $\partial P_0 = P$, $\dim SP = n-q-1$; *let K_0, L_0 be a full triangulation of P_0, SP_0 and K, $L = K_0, L_0|P$. Then*

(1) *$C(K_0)$, $C(K)$ is a cycle in the pair P_0, P (with coefficients in U_{q-1});*

(2) *Let G_0, G be a simplicial homology of $C(K_0)$, $C(K)$ to zero in K_0, K. Then there exists an element $z(G_0)\in U_n(P_0, P; q+1)$ associated to G_0, G such that:*

  (*a*) $\varphi_{q+1}z(G_0) = [P_0\xrightarrow{\text{id}} P_0]\in U_n(P_0, P; q)$;

  (*b*) $\partial z(G_0) = z(G)\in U_{n-1}(P; q+1)$.

Proof. The proof is based on the same methods as 1.6.

In order to see that $C(K_0)$, $C(K)$ is a relative cycle, it is enough to look at the links of the $(n-q-1)$-simplexes of L_0.

Now we distinguish two cases: $G_0 = 0$ and $G_0\neq 0$. If $G_0 = 0$, $C(K_0) = 0$ and $C(K) = 0$. Then, for each $(n-q)$-simplex $A\in L_0$, $\dot D(A, K_0)$ is an almost complex boundary, hence there exists an almost complex manifold $W(A)$ such that $\partial W(A) = \dot{D}(A, K_0)$. We replace st $(A, K_0) = \dot{A} * \hat{A}\dot{D}(A, K_0)$ by $\dot{A} * W(A)$; at this point the $(n-q-1)$-simplexes $B \in L$ are dealt with similarly, using the pair st (B, K_0), st (B, K). Let P_0', P' be the resulting polyhedra; P_0' is a complex cycle of dimension n with boundary P' and there is a blow-up $f\colon P_0', P' \to P_0, P$ obtained by projecting $W(A)$ onto $D(A, K)$, using a collar of $\partial W(A)$. From the mapping cylinder of f one immediately obtains a bordism between P_0, P and P_0', P' in $U_n(P_0, P; q)$ so that the result follows in this particular case. We note that P_0', P' depends on the choice of the bordism $W(A)$; however it is immediately seen (by coning) that different choices do not alter its bordism class in $U_n(P_0, P; q+1)$.

If $G_0 \neq 0$, the same method as in 1.6 may be applied using a relative form of the thickening construction. We omit the details. □

3.2 Main lemma (relative version). *For any $[P_0, P]_{X,A} \in U_n(X, A; q)$, we have $\omega_{q-1}[P_0, P]_{X,A} = \varphi_{q+1}[P_0', P']_{X,A}$.*

Proof. The proof is essentially the same as that of 2.3, using 3.1 instead of 1.6. Details are omitted. □

For each integer $k \geqslant 0$, we fix a basis $\{[M_I]\colon I \in \mathfrak{I}(k)\}$ for U_{2k} and characteristic classes $\{s_I\}$ as in proposition 2.1.

3.3 Lemma. *On the category of all topological spaces, there exists a natural transformation $\sigma_q\colon U_n(X; q) \to U_n(X; q+1)$ which is an ω_{q-1}-splitting,* i.e. *$\varphi_{q+1}\sigma_q(z) = \omega_{q-1}z$, for each $z \in U_n(X; q)$* (ω_i has been defined previously).

Proof. As the maps from cycles to X do not play an essential role in the proof, we disregard them.

Write φ, σ, ω for φ_{q+1}, σ_q, ω_{q-1} respectively.

Let $[P]_X \in U_n(X; q)$. In the proof of the main lemma 2.3 we showed that $\omega[P]_X$ is represented in $U_n(X; q)$ by a complex cycle P' with codim $SP' > q$, so that P' is a cycle in the theory $U_n(X; q+1)$. We wish to define $\sigma[P]_X = [P']_X \in U_n(X; q+1)$. We need to make sure that σ is a well-defined map. Now the construction of P' in 2.3 depends on the following choices:

(*a*) the simplicial chains G_I, E_I in Q (of course, if G_I', E_I' is another choice, then G_I, E_I is homologous to G_I', E_I');

(*b*) the triangulation K, L of P, SP;

(*c*) the representative P of $[P]_X \in U_n(X; q)$.

(*d*) the bordisms $W(A)$ mentioned in the proof of 1.6 (compare also remark 3.4 below).

But using 3.2 it is easily seen that different choices a) b), c) do not alter the bordism class of P' in $U_n(X; q+1)$. In fact it is sufficient to apply 3.2 to a triangulated bordism (which clearly exists) joining the complex cycles on which we are going to make the two choices; in the cases (a) and (b), for instance, this bordism is just a cylinder with a suitable triangulation.

Moreover one checks immediately (by coning) that different choices (d) do not alter the bordism class.

Thus σ is a well-defined natural transformation, which is a homomorphism, because the assignments $K \to G_I$ are additive with respect to K. Finally, σ is an ω-splitting because φ is the identity on representatives and P' is bordant to ωP in $U_n(X; q)$ by construction. This concludes the proof of the lemma. □

We observe that the above lemma extends easily to pairs X, A.

3.4 Remark. It is convenient here to sum up the procedure for obtaining $\sigma[P]_X$, as it can be deduced from the constructions of 2.3 and 1.6, to which we refer for notation. Let $[P]_X \in U_n(X; q)$. Suppose q is even; then a representative P' of $\sigma[P]_X$ is obtained from K by replacing $\operatorname{st}(A, K) = \dot{A} * \hat{A}\dot{D}(A, K)$ with $\dot{A} * W(A)$, where A varies over the $(n-q)$-simplexes of $L \subset K$ and $W(A)$ is *any* bordism of $\dot{D}(A, K)$ to zero. Note that, in this case, $\omega_{q-1} = 1$ and σ is an isomorphism, inverse of φ.

Suppose now $q-1 = 2k$ and let $\{[M_I]\}$ and $\{s_I\}$ be as in proposition 2.1; then $\omega_{q-1}C(K) = \partial \sum_I \omega_{q-1}/\varrho_I(G_I + E_I^+) \otimes [M_I]$ (1). Let Γ be a combinatorial realisation of $\sum_I (G_I + E_I^+) \otimes \omega_{q-1}/\varrho_I[M_I]$ with $\delta\Gamma = \Gamma' =$ combinatorial realisation of $C(\omega_{q-1}K^{(1)}/L \operatorname{rel} L)$. Then $K'' = (\omega_{q-1}K^{(1)}/L \operatorname{rel} L)_- \coprod_{\Gamma'} \partial\tilde{\Gamma}$ (where $(\ )_-$ means change of U-orientation) is a complex cycle for which $C(K'') = 0$. At this point P' is constructed as in the previous case.

3.5 Theorem. *For $n \geqslant 0$, there exists a natural transformation of functors*

$$\xi(U)\colon U_n(X) \to \sum_{i+j=n} H_i(X) \otimes U_j$$

which is an $\alpha(n, U)$-isomorphism.

Proof. Again, for the sake of simplicity, we disregard the maps to X.

Step 1. There exists an ω-isomorphism

$$\psi\colon U_n(X; q+1) \to U_n(X; q) \oplus \ker\varphi \qquad (\varphi = \varphi_{q+1};\ \omega = \omega_{q-1};\ \sigma = \sigma_q).$$

(1) Here $C(K)$ is identified with $C(K)^{(1)}$ via some fixed homology.

This is a consequence of lemma 3.3, the isomorphism ψ being defined by

$$\psi(z) = \big(\varphi(z), \omega z - \sigma\varphi(z)\big)\,.$$

Step 2. There exists a natural ω-isomorphism $f\colon \ker\varphi \to H_{n-q+1}(X)\otimes U_{q-1}$. This statement is non-trivial only for q odd.

Let $\varphi[P]_X = 0$ in $U_n(X;q)$; then there exists a relative complex cycle $\bar{P}$, of dimension $n+1$, such that $\partial\bar{P} = P$ and $\dim S\bar{P} \leqslant n-q+1$. If $\bar{K}$ is a triangulation of $\bar{P}$, then $[C(\bar{K})]_X$ is uniquely written as

$$[C(\bar{K})]_X = \sum_{I\in\mathfrak{I}(k)} [\bar{C}_I]_X \otimes [M_I] \qquad (q = 2k+1)$$

and we define

$$f[P]_X = \sum_{I\in\mathfrak{I}(k)} \varrho_I[\bar{C}_I]_X \otimes [M_I] \quad \text{if } \dim S\bar{P} = n-q+1$$

$$f[P]_X = 0 \qquad \text{otherwise}\,.$$

Now we have several statements to verify:

(*a*) If $\tilde{K}$ is a triangulation of another bordism $\tilde{P}$ of P to zero in $\ker\varphi$, then $K' = \bar{K}\coprod_K \tilde{K}$ is a triangulation of $\bar{P}\coprod_P \tilde{P}_-$, where $\tilde{P}_-$ is obtained from $\tilde{P}$ by reversing the U-orientation on $\tilde{P} - S\tilde{P}$. Let

$$G' = \sum_{I\in\mathfrak{I}(k)} (G'_I + E'^{+}_I)\otimes [M_I]\,,$$

where G'_I and E'^{+}_I are the chains of K' constructed as G_I and E^+_I in 2.3. Then

$$\partial G' = \sum_{I\in\mathfrak{I}(k)} \varrho_I[\bar{C}_I]_X \otimes [M_I] - \sum_{I\in\mathfrak{I}(k)} \varrho_I[\tilde{C}_I]_X \otimes [M_I]\,.$$

This shows that f does not depend on the triangulation $\bar{K}$ or the bordism $\bar{P}$.

Thus, if $[P]_X = 0$ in $U_n(X; q+1)$, we have $f[P]_X = 0$, because we can choose the bordism $\bar{P}$ so that $\dim S\bar{P} < n-q+1$. This shows that f depends only on the bordism class of P in $U_n(X; q+1)$ and therefore is well-defined; f is trivially a homomorphism.

(*b*) f is an ω-monomorphism. Because, if $\sum_{I\in\mathfrak{I}(k)} \varrho_I[\bar{C}_I]_X \otimes [M_I] = 0$, then $\varrho_I[\bar{C}_I]_X = 0$, $\forall I \in \mathfrak{I}(k)$, which implies $\omega[\bar{C}_I]_X = 0$, $\forall I\in\mathfrak{I}(k)$. Thus the relative complex cycle $\omega\bar{K}$ has

$$[C(\omega\bar{K})]_X = \sum_I \omega[\bar{C}_I]_X \otimes [M_I] = 0$$

and therefore (by 1.6) we can reduce the dimension of the singularities of $\omega\bar{K}$ so as to obtain a bordism $\bar{K}_1$ of ωK to zero with singularities in codimension $\geqslant q+1$, that is $\omega[P]_X = 0$ in $U_n(X; q+1)$ as required.

(*c*) f is an ω-epimorphism.

Let $[C]_X \otimes [M_I] \in H_{n-q+1}(X) \otimes U_{q-1}$, where C is a (triangulation of a) complex cycle of dimension $n-q+1$.

Form the complex cycle $P = C \times cM_I/(a, x) \sim (a, x')$ if a belongs to the $(n-q-1)$-skeleton of C and $x, x' \in cM_I$.

It is immediately seen that $[\partial P]_X \in U_n(X; q+1)$ and $P \to X$ is a bordism of $\partial P \to X$ to zero in $U_n(X; q)$. Thus $[\partial P]_X \in \ker\varphi$ and, by the definition of f, we have $f(\omega/\varrho_I[\partial P]_X) = \omega[C]_X \otimes [M_I]$, which proves that f is an ω-epimorphism.

Therefore f is an ω-isomorphism, as required.

Step 3 (proof of the theorem). For each $q = 1, \ldots, n+2$ we have natural ω_{q-1}-isomorphisms

$$U_n(X; q+1) \cong U_n(X; q) \oplus \ker\varphi_{q+1} \cong U_n(X; q) \oplus H_{n-q+1}(X; U_{q-1})$$

which, by induction, prove the existence of a $\left(\prod_{i=1,\ldots,q-2} \omega_i\right)$-isomorphism

$$U_n(X; q) \cong H_n(X; U_0) \oplus \ldots \oplus H_{n-q+2}(X; U_{q-2}) .$$

For $q = n+2$, we have the required $\alpha(n, U)$-isomorphism

$$\xi(U) \colon U_n(X) \to \sum_{i+j=n} H_i(X) \otimes U_j . \qquad \square$$

3.6 THEOREM. *Let $n \geqslant 0$ and let X be a topological space such that $H_*(X)$ is finitely generated and has no π-torsion, for any prime $\pi < (n+3)/2$. Then*

$$U_n(X) \cong \sum_{i+j=n} H_i(X) \otimes U_j .$$

PROOF. First we prove that $U_n(X)$ has no π-torsion, for any prime $\pi < (n+3)/2$. We proceed by induction; suppose $U_n(X; q)$ has no π-torsion; in the proof of 3.5 we have defined an ω-isomorphism ($\omega = \omega_{q-1}$; $\varphi = \varphi_{q+1}$; $\sigma = \sigma_q$)

$$\psi \colon U_n(X; q+1) \to U_n(X; q) \oplus \ker\varphi$$

given by $\psi(z) = (\varphi(z), \omega z - \sigma\varphi(z))$.

Assume there exists $z \in U_n(X; q+1)$ with $\pi z = 0$. Then

$$\psi(\pi z) = 0 \Rightarrow \pi\varphi(z) = \varphi(\pi z) = 0 \Rightarrow \varphi(z) = 0\,,$$

because $U_n(X; q)$ has no π-torsion. Thus $z \in \ker \varphi$. Now consider the ω-isomorphism $f\colon \ker \varphi \to H_{n-q+1}(X; U_{q-1})$ constructed in the proof of 3.5 (we retain the notation introduced there). We claim that under our hypotheses f is actually a monomorphism. In fact, given $[P]_X \in \ker \varphi \subset \subset U_n(X; q+1)$, we have (see step 2 in the proof of 3.5)

$$f[P]_X = 0 \Rightarrow \omega[C(\bar{K})]_X = 0 \quad \text{in} \quad H_{n-q+1}(X; U_{q-1}) \Rightarrow [C(\bar{K})]_X = 0\,,$$

because of the lack of π-torsion in $H_*(X)$ for each π dividing ω.

But, if $[C(\bar{K})]_X = 0$, we can apply the usual techniques to find a bordism of P to zero in $U_n(X; q+1)$ as required.

Now let us return to our $z \in \ker \varphi$; $\pi z = 0 \Rightarrow \pi f(z) = 0 \Rightarrow f(z) = 0 \Rightarrow \Rightarrow z = 0$, because f is injective. This proves that $U_n(X; q+1)$ has no π-torsion, completing the induction step.

At this point, we have two finitely generated abelian groups G and H, which we want to prove to be isomorphic, and we know that:

(*a*) there exists an ω-isomorphism $\xi\colon G \to H$;

(*b*) G and H have no π-torsion, for each prime π dividing ω.

From (*a*) and (*b*) it follows easily that there is an isomorphism between G and H (although ξ is not one in general). The theorem is proved. □

Let $[P]_X$ be an element of $U_n(X; q)$; for $0 \leqslant 2k \leqslant q-2$ and I a partition of k, let $G_I + E_I^+ = \bar{G}_I$ be the chain of $K^{(1)}$ constructed from the characteristic class $s_I \in H^{2k}(Q)$ as in the proof of the main lemma 2.3 (to which we refer for notation). By dimensional arguments, $\bar{G}_I$ is an integral cycle and thus it defines a class $[\theta_I(n-2k)]_X \in H_{n-2k}(X)$ which does not depend on the representative G_I. In particular, if P is an almost complex manifold, then $\theta_I(n-2k)$ represents the Lefschetz dual of s_I on P.

In the proof of 3.5 we have constructed a $\Big(\prod_{i=1,\dots,q-2} \omega_i\Big)$-isomorphism

$$\xi_q\colon U_n(X; q) \to \sum_{2k=0,\dots,q-2} H_{n-2k}(X) \otimes U_{2k}\,.$$

3.7 THEOREM. *For any topological space X, ξ_q is given by the formula*

$$\xi_q[P]_X = \sum_{\substack{2k=0,\dots,q-2\\ I\in\mathcal{T}(k)}} \omega_{2k}[\theta_I(n-2k)]_X \otimes [M_I]$$

(note that the range of I depends on k).

PROOF. The isomorphism ξ_q was constructed inductively from the following compositions:

$$\underbrace{U_n(X;q) \xrightarrow{\psi} U_n(X;q-1) \oplus \ker\varphi \xrightarrow{\mathrm{id}\oplus f} U_n(X;q-1) \oplus H_{n-q+2}(X) \otimes U_{q-2}}_{\eta_q}$$

if q is even, and $\varphi\colon U_n(X;q) \rightarrow U_n(X;q-1)$ if q is odd.

Let $z = [P]_X$ and $2k = q-2$; then $\psi(z) = \big(\varphi(z), \omega_{2k} z - \sigma\varphi(z)\big)$ and, as $\varphi(z) = [P]_X$ and P has no singularities in codimension $q-1$, it follows from definitions:

$$\sigma\varphi(z) = \omega_{2k}[P]_X + \sum_{I\in\mathfrak{I}(k)} \omega_{2k}/\varrho_I[\partial\tilde{\Gamma}_I]_X\,,$$

where $\tilde{\Gamma}_I$ is a thickening of the combinatorial chain

$$\Gamma_I = [\theta_I(n-2k)]_X \otimes [M_I] \quad \text{(see remark 3.4)}\,.$$

Hence $\psi[P]_X = \Big([P]_X, \sum_{I\in\mathfrak{I}(k)} \omega_{2k}/\varrho_I[\partial\tilde{\Gamma}_I]_X\Big)$.

Now observe that $\tilde{\Gamma}_I$ is a bordism of $\partial\tilde{\Gamma}_I$ to zero in $U_n(X;q-1)$; thus, by the definition of f in the proof of 3.5, we have

$$f[\partial\tilde{\Gamma}_I]_X = \varrho_I[\theta_I(n-2k)]_X \otimes [M_I]$$

and

$$\eta_q[P]_X = \Big([P]_X, \sum_{I\in\mathfrak{I}(k)} \omega_{2k}[\theta_I(n-2k)]_X \otimes [M_I]\Big)\,.$$

By iteration on $[P]_X = \varphi(z) \in U_n(X;q-1)$ we obtain the result. □

3.8 COROLLARY. *Let X be a topological space such that, for $0 \leqslant 2k \leqslant q-2$, $H_{n-2k}(X)$ has no π-torsion, for any prime $\pi < (q+1)/2$. Then $[P]_X = 0$ in $U_n(X;q)$ if and only if all the homological obstructions $[\theta_I(n-2k)]_X \in H_{n-2k}(X)$ vanish.*

PROOF. From the above theorem we have that

$$\xi_q[P]_X = \sum_{\substack{2k=0,\dots,q-2\\ I\in\mathfrak{I}(k)}} \omega_{2k}[\theta_I(n-2k)]_X \otimes [M_I]\,.$$

The hypotheses on $H_{n-2k}(X)$ imply that ξ_q is a monomorphism (see proof of 3.6), which proves the result. □

3.9 REMARK. Note the special case $q = n+2$, i.e. $U_n(X;q) = U_n(X)$.

3.10 COROLLARY. Let $[P]_X \in U_n(X; q)$ and $r \leqslant q$; suppose that, for $0 \leqslant 2k \leqslant r-2$, $H_{n-2k}(X)$ has no π-torsion, for any prime $\pi < (r+1)/2$. Then $[P]_X = 0$ in $U_n(X; r)$ if and only if the homological obstructions $[\theta_I(n-2k)]_X \in H_{n-2k}(X)$ vanish, for $0 \leqslant 2k \leqslant r-2$.

PROOF. Obvious, by looking at the composition

$$U_n(X; q) \to U_n(X; r) \to \sum_{2k=0,\dots,r-2} H_{n-2k}(X) \otimes U_{2k}. \qquad \square$$

4. – Smooth resolution.

The analogy with the complex case is very close, except for the fact that the bordism group Ω_* has 2-torsion. Therefore we limit ourselves to sketching the modifications which are needed in this case.

All of § 1 translates immediately into the smooth category, replacing the words « complex » and « almost complex » by « smooth oriented ». By analogy with the complex case we have the following:

4.1 PROPOSITION. (1) *There exists a basis* $\{[M_I], I \in \mathcal{T}(k)\}$ *for the free part of* Ω_{4k} *and polynomials* $\{p_I, I \in \mathcal{T}(k)\}$ *in the (tangential) Pontryagin classes such that:*

(*a*) $$p_I|M_J| = \begin{cases} 0 & \text{if } I \neq J \\ \varrho_I' \neq 0 & \text{if } I = J. \end{cases}$$

(*b*) *Each* ϱ_I' *is a divisor of the* L.C.M. *of* $\{a_I', I \in \mathcal{T}(k)\}$, *where* $a_I' = (2i_1+1)\dots(2i_r+1)$, *I being the partition* $(i_1, \dots, i_r)$ *of* k.

(2) *For each* $n \geqslant 0$, *there exists a* $\mathbb{Z}_2$*-basis* $\{[L_j]\}$ *for the torsion part of* Ω_n *and Stiefel-Whitney polynomials* $\{w^{(i)}\}$ *such that* $w^{(i)}|L_j|$ *is the unit matrix.* $\square$

For each integer $n \geqslant 0$, we define

$$\omega_n' = \begin{cases} 1 & \text{if } n \not\equiv 0 \pmod 4 \\ \text{L.C.M.}\{a_I', I \in \mathcal{T}(k)\} & \text{if } n = 4k \end{cases}$$

and $\alpha(n, \Omega) = \prod_{i=1,\dots,n} \omega_I'$.

4.2 MAIN LEMMA. *For any* $[P]_X \in H_n(X)$, *we have* $\omega_{n-p-1}'[P]_X = [P']_X$, *where* P *and* P' *are smooth cycles and* $\dim SP' < p = \dim SP$.

PROOF. The proof is similar to the almost complex case, using the classes p_I and $w^{(i)}$ instead of the s_I, so that we now have $\omega_{n-p-1}' C(K)^{(1)} = \partial G$, where

$$G = \sum_I (\omega'_{n-p-1}/\varrho'_I)(G_I + E_I^+) \otimes [M_I] + \sum_i W_i \otimes [L_i],$$

with G_I, E_I Lefschetz dual of p_I and W_i Lefschetz dual of $w^{(i)}$. □

By iteration we have

4.3 THEOREM. *Every integral homology class of dimension n is $\alpha(n-3, \Omega)$-representable by a smooth oriented manifold.* □

4.4 REMARK. We have $\omega'_{4k} = \prod_p p^{[2k/(p-1)]}$ with p prime, $2 < p \leqslant 2k+1$. It follows that the primes dividing $\alpha(n, \Omega)$ are precisely those primes p such that $2 < p < (n+3)/2$.

Note that $\alpha(n, \Omega)$ is an *odd* integer. The fact that any integral homology class has an odd multiple which is representable by a smooth oriented manifold was proved by Wall (see [9] or [2], 15.3). Note also that $\alpha(n, U)$ is much greater than $\alpha(n, \Omega)$.

The following corollaries are proved as in the complex case.

4.5 COROLLARY. *Let z be an n-dimensional homology class such that $\beta z = 0$. Then z is ϱ-representable, with $\varrho = \text{G.C.D.}(\alpha(n-3, \Omega); \beta)$. In particular, if z belongs to the π-torsion, with π prime and $\pi = 2$ or $\pi \geqslant n/2$, then z is representable by a smooth oriented manifold.* □

4.6 COROLLARY. *Let n and γ be positive integers. Suppose γ is not divisible by any odd prime $\pi < (n+1)/2$. Then any $\mathbf{Z}_\gamma$-homology class of dimension n is representable by a smooth $\mathbf{Z}_\gamma$-manifold (note the special case: γ prime, $\gamma \geqslant (n+1)/2$).* □

4.7 REMARKS:

(*a*) It is not difficult to see that Thom's example ([8], p. 62) gives a class in $H_7(K(\mathbf{Z}_3, 1) \times K(\mathbf{Z}_3, 1))$ which is representable by a smooth cycle P with the « nicest possible » singularity structure, i.e. SP = orientable surface and a neighbourhood N of SP in P is of the form $SP \times \text{cone } \mathbf{CP}^2$ ($\mathbf{CP}^2$ = complex projective plane).

(*b*) The above example is immediately generalised to any odd prime π, providing a class of π-torsion in $H_{2\pi+1}(K(\mathbf{Z}_\pi, 1) \times K(\mathbf{Z}_\pi, 1))$ which is not representable by a smooth oriented manifold. This shows that the range $\pi = 2$, $\pi \geqslant n/2$ of corollary 4.5 is the best possible as regards representability of π-torsion homology classes.

The proofs of the following propositions are similar to those of their counterparts in the complex case.

4.8 THEOREM. For $n \geqslant 0$ *there exists a natural transformation of functors*

$$\xi(\Omega): \Omega_n(X) \to \sum_{i+j=n} H_i(X; \Omega_j)$$

which is defined on the category of all topological spaces and is an $\alpha(n, \Omega)$-*isomorphism.* □

4.9 COROLLARY. *Let* $n \geqslant 0$ *and let* X *be a topological space such that* $H_*(X)$ *has no* π-*torsion for all odd primes* $\pi < (n+3)/2$. *Then there is an isomorphism* $\Omega_n(X) \cong \sum_{i+j=n} H_i(X; \Omega_j)$. □

Let $[P]_X \in \Omega_n(X; q)$; as in the complex case, construct classes

$$[\theta_I(n-4k)]_X \in H_{n-4k}(X), \quad 0 \leqslant 4k \leqslant q-2, \; I \in \mathcal{T}(k),$$

corresponding to the integral characteristic classes $\{p_I\}$ of 4.1 and classes

$$[\bar{\theta}_h(n-m)]_X \in H_{n-m}(X, \mathbf{Z}_2), \quad 0 \leqslant m \leqslant q-2,$$

corresponding to the $\mathbf{Z}_2$-characteristic classes $\{w^{(h)}\}$ of 4.1; note that, for each m, h varies from 1 to dim $\mathcal{C}_m$, where $\mathcal{C}_m$ is the torsion part of Ω_m. The following theorems are complete analogues of 3.8, 3.10 and are proved in the same way:

4.10 THEOREM. *If, for* $0 \leqslant 4k \leqslant q-2$, $H_{n-4k}(X)$ *has no* π-*torsion, for all odd primes* $\pi < (q+1)/2$, *then* $[P]_X$ *is a boundary in* $\Omega_n(X; q)$ *if and only if all the homological obstructions* $[\theta_I(n-4k)]_X$ *and* $[\bar{\theta}_h(n-m)]_X$ *vanish (note the special case* $q = n+2$). □

4.11 THEOREM. *Let* $[P]_X \in \Omega_n(X; q)$ *and* $0 \leqslant r \leqslant q$. *Suppose that, for* $0 \leqslant 4k \leqslant r-2$, $H_{n-4k}(X)$ *has no* π-*torsion for any odd prime* $\pi < (r+1)/2$. *Then* $[P]_X$ *is zero in* $\Omega_n(X; r)$ *if and only if* $[\theta_I(n-4k)]_X = 0$ *for* $0 \leqslant 4k \leqslant r-2$ *and* $[\bar{\theta}_h(n-m)]_X = 0$ *for* $0 \leqslant m \leqslant r-2$. □

4.12 FINAL REMARKS. Let T be either smooth or PL unoriented bordism. The procedure for desingularising cycles described in this paper is considerably simplified when applied to unoriented smooth (or PL) cycles, using Stiefel-Whitney (resp. PL) characteristic numbers; note that $\alpha(n, T) = 1$, $\forall n$.

In this case the procedure gives a direct construction of an unoriented smooth (resp. PL) manifold representing a $\mathbf{Z}_2$-homology class (Thom [8]). Moreover, using the same kind of arguments as those of 4.8, one can construct an explicit equivalence of homology theories $T_*(X, A) \simeq$ $\simeq H_*(X, A; \mathbb{Z}_2) \otimes T_*$ on the category of all pairs of topological spaces.

For instance, this can be used to define a priori homological obstructions to the existence of a blow-up of a smooth (or *PL*) cycle which improves on a result of Kato (see [4], 4.5). As an example, one can prove

4.13 THEOREM (we refer to [4], 4.5 for notation and definitions). *Let (P, Q) be a PL (resp. smooth) m-variety. Suppose Q is compact and of dimension q. Then there is a PL (resp. smooth) blow-up $f\colon (P', Q') \to (P, Q)$ with* $\dim Q' \leqslant k$ *if and only if* $\nu_i(P, Q)_2 = 0$ *for* $i \geqslant k+1$. □

REFERENCES

[1] S. BUONCRISTIANO - C. P. ROURKE - B. J. SANDERSON, *A geometric approach to homology theory*, London Math. Soc., Lecture Note Series, **18** (1976).

[2] P. E. CONNER - E. E. FLOYD, *Differentiable periodic maps*, Ergebnisse der Math. und Grenz., Springer Verlag, Berlin (1964), Band 33.

[3] M. HIRSCH - B. MAZUR, *Smoothings of piecewise linear manifolds*, Annals of Mathematics Studies, **80** (1974).

[4] M. KATO, *Topological resolution of singularities*, Topology, **12** (1973), pp. 355-372.

[5] J. W. MILNOR - J. D. STASHEFF, *Characteristic classes*, Annals of Mathematics Studies, **76** (1974).

[6] C. P. ROURKE - B. J. SANDERSON, *Introduction to piecewise linear topology*, Ergebnisse der Math. und Grenz., Springer Verlag, Berlin (1972), Band 69.

[7] D. SULLIVAN, *Singularities in spaces*, Proc. of Liverpool Singularities Symposium II, Lecture Notes in Mathematics, **209** (1971), pp. 196-206.

[8] R. THOM, *Quelques propriétés globales des applications différentiables*, Comm. Math. Helv., **28** (1954), pp. 17-86.

[9] C. T. C. WALL, *Determination of the cobordism ring*, Ann. of Math., **72** (1960), pp. 292-311.

[10] E. Y. MILLER, *The construction of a manifold representing a homology class*, preprint MIT.

[11] S. BUONCRISTIANO - D. HACON, *An elementary geometric proof of two theorems of Thom*, to appear in Topology.

[12] S. BUONCRISTIANO - D. HACON, *The geometry of Chern numbers*, to appear.

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