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Existence and convexity for hyperspheres of prescribed mean curvature

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Existence and Convexity for Hyperspheres of Prescribed Mean Curvature.

ANDREJS TREIBERGS

Let $Y: S^n \to \mathbb{R}^{n+1}$ be an embedding of the standard unit sphere into Euclidean space. We usually assume that Y is starlike with respect to the origin. Let $H(Y)$ denote the mean curvature of the hypersurface at Y with respect to the inner normal. We shall consider equations which if satisfied by one hypersphere Y, are also satisfied by the family of homothetic dilatations of Y. Our model equation for hyperspheres Y with this property is

$$H(Y) = \Phi(Y) \tag{1}$$

where $\Phi: \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}$ is a given homogeneous of degree minus one function. Given a solution Y and a positive constant θ, the dilated hypersurface θY also satisfies (1) by the homogeneity of mean curvature.

We derive properties of such equations from pointwise assumptions on Φ and its derivatives. The closer the data is to that of concentric spheres about the origin, the more the surfaces Y behave like spheres.

Aeppli [1] and Aleksandrov [2] have shown that uniqueness holds for starlike hypersurfaces that just satisfy a dilation invariant prescribed mean curvature equation.

THEOREM [Aeppli-Aleksandrov]. *Let Y_1, Y_2 be two orientable C^2 hypersurface of $\mathbb{R}^{n+1}$ such that Y_1 is a hypersphere which is strictly starlike with respect to the origin. Suppose they satisfy the homothetic relation*

$$H(Y_i) = \Phi(x, |Y_i|, N_i), \qquad i = 1, 2;$$

where $\Phi \in C^0(S^n \times \mathbb{R}^n \times S^n)$ is homogeneous of degree -1 in the second variable,

$$\Phi(x, \theta|Y|, N) = \theta^{-1}\Phi(x, |Y|, N), \qquad \theta > 0, \tag{2}$$

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$x = Y_i/|Y_i|$, and where N_i is the inner normal to the hypersurface Y_i. Then the surfaces are homothetic: for some $0 < \theta \in \mathbb{R}$, $Y_2 = \theta Y_1$.

Indeed, Aleksandrov and Aeppli have shown uniqueness theorems for the higher mean curvatures and various other curvature realization problems.

In Section 1 we formulate the problem as a quasilinear elliptic equation on S^n. In Section 2 we find an apriori gradient estimate for Y as a graph over the unit sphere, Theorem A, using ideas of [4] and [13]. The assumption of Theorem A, that a derivative bound hold on Φ, is similar to a sufficient condition of Serrin [12, p. 484] for the existence of a gradient bound and the solvability of the Dirichlet Problem for prescribed mean curvature in Euclidean domains.

In Section 3, the same estimate yields a stability result giving bounds on the closeness of Y to the standard sphere in terms of the closeness of Φ to constant on the sphere.

In Section 4, assuming more restrictions on the data, we give an existence theorem corresponding to the Aeppli-Aleksandrov uniqueness result, Theorem B, which may be regarded as an extension of the existence theorem of Bakelman-Kantor [4, 5]. We use the formulation and argument of Triebergs-Wei [13].

THEOREM (Bakelman-Kantor). *Let A_{r_1,r_2} be the annular region, $A_{r_1,r_2} = \{X \in \mathbb{R}^{n+1}: 0 < r_1 < |X| < r_2\}$ where $|X|$ is the Euclidean length. Let $\Phi \in C^{m,\alpha}(\bar{A})$, $m \geq 1$, satisfy*

$$\text{(i)}\quad \frac{\partial}{\partial \varrho}\, \varrho\Phi(\varrho X) \leq 0 \quad \forall \varrho X \in A$$

$$\text{(ii)}\quad \Phi(X) > \frac{1}{r_1} \qquad \text{if } |X| = r_1$$

$$\Phi(X) < \frac{1}{r_2} \qquad \text{if } |X| = r_2\,.$$

Then there exists a hypersphere given by radial projection

$$S^n \ni x \mapsto Y(x) = \varrho(x)x \subset A\,,$$

$$r_1 < \varrho(x) < r_2$$

where $\varrho(x) \in C^{m+2,\alpha}(S^n)$ such that the mean curvature

$$H(Y) = \Phi(Y)\,.$$

If there are two solutions $\varrho_1(x)$, $\varrho_2(x)$, they are homothetic to one another, $\varrho_1(x) = \theta\varrho_2(x)$ for some constant $\theta > 0$.

For homogeneous data, condition (ii) holds with equality:

$$\frac{\partial}{\partial \varrho} \varrho \Phi(\varrho, x) = 0 \quad \text{if } \varrho_1 < \varrho < \varrho_2 .$$

Condition (ii) is used to prove an apriori C^0 bound, which is in turn used in estimating the higher derivatives. We show that condition (ii) may be replaced in our homothetic problems by a restriction of Φ. However, since the solutions are undertermined up to dilation, the existence statement is reformulated as an eigenvalue problem. Oliker [10] has given an analog of the Bakelman-Kantor theorem for prescribed Gauss curvature.

We also show that the assumptions of Theorem A are necessary in the sense that if the data is just far enough from spherical to violate the conditions, then there are hypersurfaces that satisfy (1) but which are not hyperspheres about the origin with bounded gradient.

In Section 5 we give sufficient conditions, further restricting Φ, so that solutions of [1] are convex hypersurfaces. The curvatures of Y are shown to approach the spheres' as the data becomes spherical. Previously, Chen and Huang [6] and Korevaar [9] have shown convexity of solutions with infinite boundary gradient of certain prescribed mean curvature problems for graphs over convex Euclidean domains.

Theorem C resembles Pogoreiov's [11] sufficient conditions for the solvability of the Christoffel Problem, which asks for a convex hypersphere whose sum of principal radii of curvature is a given function of the normal. When parameterized by the Gauss map, the equation for the support function is readily solvable, but the solution must be shown to correspond to a convex hypersurface.

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1. – Formulation of the Problem.

A hypersphere radially graphed over S^n may be represented by

$$Y(x) = e^{u(x)}$$

where $u \in C^2(S^n)$ and x identifies a coordinate of S^n and vector in $\mathbb{R}^{n+1}$. It is shown in [13] that the mean curvature $H(Y)$ has the expression

$$\mathcal{M}u = -\frac{1}{n}\operatorname{div}_{S^n}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) + \frac{1}{\sqrt{1+|Du|^2}} = e^u H(Y)\,.$$

We seek solutions of the equation

$$H(Y) = \Phi(x, |Y|, N)$$

where Φ is homogeneous of degree -1 in the second variable. It is convenient to rewrite the right side as

$$e^u H = e^{u+c}\Phi(X, e^{u+c}, N) = \varphi(x, Du)\,,$$

and call $\varphi = rH$ the *reduced mean curvature* [1]. Denote the radius $r = e^u$, and the distance of the tangent plane $Y(x)$ from the origin by

$$p(x) = -\langle Y, N\rangle = e^u(1+|Du|^2)^{-\frac{1}{2}}\,.$$

Other examples of invariant problems we can consider are

$$H(Y) = \psi(x) r^{\theta-1} p^{-\theta}$$

for constant θ, for which the equation becomes

$$\mathcal{M}u = \varphi(x, Du) = \psi(x)(1+|Du|^2)^{\theta/2}\,.$$

2. – A gradient estimate.

THEOREM A. *Let $\varphi(x, q) \in C^1(TS^n)$ be given. Suppose there exist non-negative functions $k_i(x) \in C^0(S^n)$ and constants α, β so that for all $(x, q) \in TS^n$,*

$$\begin{aligned} &k_1(x)(1+|q|^2)^{\alpha/2} \leqslant \varphi(x, q) \leqslant k_2(x)(1+|q|^2)^{\beta/2} \\ &|D_x\varphi(x, q)| \leqslant k_3(x)(1+|q|^2)^{\beta/2}\,. \end{aligned} \tag{3}$$

Let

$$0 < \varepsilon_1 = \inf_{x\in S^n} k_1^2(x)\,,$$

$$\varepsilon_2 = \sup_{x\in S^n} k_2(x)\,,$$

$$\varepsilon_3 = \sup_{x \in S^n} k_3(x) ,$$

$$\varepsilon_4 = \inf_{x \in S^n} \left\{ \frac{n}{n-1} k_1^2(x) - k_3(x) + \frac{n-1}{n} \right\} .$$

Then a solution $u \in C^2(S^n)$ *of*

$$\mathcal{M}u = \varphi(x, Du) \tag{4}$$

satisfies an apriori gradient bound

$$|Du| < c_1(n, \alpha, \beta, \varepsilon_1, \varepsilon_2, \varepsilon_3) \tag{5}$$

in the following examplary cases:

(i) $\beta < \max(0, 2\alpha)$;

(ii) $\alpha < 0,\ \beta = 0,\ n\varepsilon_2 < n - 1$;

(iii) $0 < \beta = 2\alpha,\ n\varepsilon_1 \geqslant \varepsilon_3(n-1)$;

(iv) $0 = \alpha = \beta,\ \varepsilon_4 > 0$.

PROOF. Assuming $u \in C^3(S^n)$, let $v = |Du|^2$. Then

$$a^{ij} u_{ij} = n(1 + v) - n\varphi(x, Du)(1 + v)^{\frac{3}{2}} \tag{6}$$

where

$$a^{ij} = (1 + v)\delta_{ij} - u_i u_j \tag{7}$$

and the subscripts denote covariant derivatives in an orthogonal frame on S^n. We compute

$$\tfrac{1}{2} v_i = u_m u_{mi} \tag{8}$$

$$\tfrac{1}{2} v_{ij} = u_{mi} u_{mj} + u_m u_{mij} \tag{9}$$

$$a^{ij}_m = 2u_k u_{km} \delta_{ij} - u_i u_{jm} - u_j u_{im} . \tag{10}$$

At a point $x_0 \in S^n$ where v attains its maximum we have using (7), (8), (9),

$$0 = u_m u_{mi} , \tag{11}$$

$$0 \geqslant \tfrac{1}{2} a^{ij} v_{ij} = (1 + v) \sum u_{ij}^2 - u_i u_{im} u_{mj} u_j + u_m a^{ij} (u_{ijm} + u_p R_i{}^p{}_{jm}) . \tag{12}$$

Differentiating equation (6) we obtain with (11) and (12)

$$u_m a^{ij} u_{ijm} = -n\varphi_{x_m} u_m (1+v)^{\frac{3}{2}} . \tag{13}$$

Either $v(x_0) = 0$ so (5) holds trivially or we may rotate coordinates at x_0 so $u_1 = v^{\frac{1}{2}}$ and $u_{11} = 0$ by (11). If $n = 1$ we solve for v from the equation. Assuming $n \geq 2$ henceforth, (8) implies that (6) becomes

$$\Delta n = n - n\varphi(1+v)^{\frac{1}{2}} .$$

By the Schwarz inequality

$$(n-1)\sum u_{ij}^2 \geq \left(\sum_2^n u_{ii}\right)^2 = (\Delta u)^2 . \tag{14}$$

Finally, on the sphere

$$R_i{}^p{}_{jm} = \delta_{ij}\delta_{pm} - \delta_{im}\delta_{pj}$$

so that

$$u_m a^{ij} u_p R_i{}^p{}_{jm} = (n-1)(1+v)v . \tag{15}$$

Inserting (13), (14), (15) into (12) yields

$$0 \geq \frac{n}{n-1}(1-\varphi\sqrt{1+v})^2 - |D_x\varphi| v^{\frac{1}{2}}(1+v)^{\frac{1}{2}} + \frac{n-1}{n} v . \tag{16}$$

By collecting powers of v and estimating by (3),

$$\begin{aligned} 0 \geq \frac{n}{n-1} k_1(1+v)^{1+\alpha} - k_3(1+v)^{1+\beta/2} + \frac{n-1}{n}(1+v) \\ - 4k_2(1+v)^{(1+\beta)/2} + \frac{2}{n} . \end{aligned} \tag{17}$$

The gradient bounds follow as in the typical case (iv) when (17) becomes

$$0 \geq \varepsilon_4(1+v) - 4\varepsilon_2(1+\nu)^{\frac{1}{2}}$$

which implies

$$|Du| \leq c_1 = \frac{4\varepsilon_2}{\varepsilon_4} .$$

The gradient estimate for $u \in C^2$ follows by approximation as in [8, p. 302].

3. – Stability of the constant solution.

Restricting attention to solutions of $\mathcal{M}u = \varphi(x)$, we show that the (possibly infinite) optimal bound implied by (16),

$$|Du| \leqslant c_2(n, \varphi, |D_x\varphi|) \tag{18}$$

goes to zero as $|D\varphi|$ does. Thus we have the stability result that $|Du|$, a dilation invariant measure of the distance of a solution from a sphere is small when φ is C^1 close to sphere data, $\varphi \equiv 1$. We analyze two simple cases.

3.1. If $n|D\varphi| < n-1$ then by neglecting the first term of (16) we find

$$|Du| \leqslant c_2 \leqslant \frac{n|D\varphi|}{((n-1)^2 - n^2|D\varphi|^2)^{\frac{1}{2}}} .$$

We may use the first term to estimate $|\varphi - 1|$ in terms of $|D\varphi|$.

3.2. If $|D\varphi| < 2\varphi$ then we obtain an estimate in terms of $2\xi = \sup |D\varphi|\varphi^{-1}$ since (16) may be written

$$0 \geqslant \text{square} - \frac{|D\varphi|}{\varphi} v^{\frac{1}{2}} + \frac{n-1}{n}\left(1 - \frac{|D\varphi|^2}{4\varphi^2}\right) v .$$

It follows that

$$|Du| \leqslant c_2 \leqslant \frac{2n\xi}{(n-1)(1-\xi^2)} \leqslant c_3 \xi \text{ where } c_3 = c_3(n, \sup |D\varphi|/\varphi) .$$

4. – Existence theorem.

Since solutions are defined up to dilation constant giving the linearized operator a kernel, integrability conditions must be put on the data. Usually obtained by integrating the equation, such formulas give only aposteriori conditions here. For this reason we recast the problem as a nonlinear eigenvalue problem. For each given $\varphi(x, q)$ we will find a unique proportionality constant λ for which there exist hyperspheres whose reduced mean curvature is $\lambda\varphi$.

THEOREM B. *Let $\varphi(x, q) \in C^1(TS^n)$. Suppose there are constants α, β and functions $k_i \in C^0(S^n)$ such that for all (x, q),*

$$k_1(x)(1+|q|^2)^{\alpha/2} \leqslant \varphi(x, q) \leqslant k_2(x)(1+|q|^2)^{\beta/2}$$

$$|D_x\varphi(x, q)| \leqslant k_3(x)(1+|q|^2)^{\beta/2},$$

$$0 < \tilde{\varepsilon}_1 = \inf_{S^n} k_1(x)^2,$$

$$\tilde{\varepsilon}_2 = \sup_{S^n} k_2(x),$$

$$\tilde{\varepsilon}_3 = \sup_{S^n} k_3(x),$$

$$\tilde{\varepsilon}_4 = \inf_{S^n}\left\{\frac{n}{n-1}\frac{k_1^2}{\tilde{\varepsilon}_5} - \frac{k_3}{\tilde{\varepsilon}_6} + \frac{n-1}{n}\right\}, \tag{19}$$

$$\tilde{\varepsilon}_5 = \sup_{S^n} \varphi(x, 0),$$

$$\tilde{\varepsilon}_6 = \inf_{S^n} \varphi(x, 0).$$

Suppose one of the following conditions is fulfilled:

(20)

(i) $\beta < \max(0, 2\alpha)$;

(ii) $\alpha < 0, \quad \beta = 0, \quad n\tilde{\varepsilon}_2 < (n-1)\tilde{\varepsilon}_6$;

(iii) $0 < \beta = 2\alpha, \quad n\tilde{\varepsilon}_1\tilde{\varepsilon}_6 > (n-1)\tilde{\varepsilon}_3\tilde{\varepsilon}_5$;

(iv) $0 = \alpha = \beta, \quad \tilde{\varepsilon}_4 > 0$.

Then there exists a unique constant λ satisfying

$$\frac{1}{\tilde{\varepsilon}_5} \leqslant \lambda \leqslant \frac{1}{\tilde{\varepsilon}_6} \tag{21}$$

such that the nonlinear eigenvalue problem

$$\mathcal{M}u = \lambda\varphi(x, Du) \tag{22}$$

is solvable by a function $u \in C^{2,\gamma}(S^n)$, $\gamma \in (0, 1)$. If w is any other solution of (22) then it is homothetic to u by an e^c dilation

$$w = c + u.$$

PROOF. Suppose there were a $\lambda \in \mathbb{R}^+$, $u \in C^2(S^n)$ such that (22) holds. Let $x_1, x_2 \in S^n$ be points where u attains its minimum and maximum respectively. At these points

$$1 \geqslant \mathcal{M}u(x_1) = \lambda\varphi(x_1, 0)\,,$$

$$1 \leqslant \mathcal{M}u(x_2) = \lambda\varphi(x_2, 0)\,,$$

which imply (21) for any eigenvalues λ. Since the elliptic equation (22) does not depend on u, the uniqueness statement (the Aleksandrov-Aeppli Theorem) follows from the maximum principle. Let u_1 and u_2 be two solutions of (22) with corresponding eigenvalues λ_1, λ_2. Let c be a constant so that $\inf(u_1 + c - u_2) = 0$. At the point x_0 where $(u_1 + c)(x_0) = u_2(x_0)$ we have

$$\lambda_1\varphi(x, Du_1) = \mathcal{M}u_1 = \mathcal{M}(u_1 + c) \leqslant \mathcal{M}u_2 = \lambda_2\varphi(x, Du_2) = \lambda_2\varphi(x, Du_1)$$

so that $\lambda_1 \leqslant \lambda_2$: By reversing the roles u_1 and u_2 we find that the eigenvalue is unique. For a function $w \in C^{1,\gamma}(S^n)$ define $\Lambda_t(\omega)$ by the formula

$$(23)\qquad \int\limits_{S^n} \frac{d\,\mathrm{vol}}{\sqrt{1 + |Dw|^2}} = \Lambda_t(w) \int\limits_{S^n} \varphi_t(x, Dw(x))\, d\,\mathrm{vol}$$

where

$$(24)\qquad \varphi_t(x, Dw(x)) = (1 - t)\varphi(x_3, 0) + t\varphi(x, Dw(x))$$

and $x_3 \in S^n$ is a point where $k_1(x_3) = \sup k_1$. Define the Banach space $B = \{w \in C^{1,\gamma}(S^n) \colon \int\limits_{S^n} w = 0\}$. We shall apply the Leray-Schauder Fixed Point Theorem [8, p. 228] to solve the equation. By [3, p. 104] there exists a mapping $T_t \colon B \to B$ for each $0 \leqslant t \leqslant 1$ taking $w \in B$ to the solution $u \in C^{2,\gamma} \cap B$ of

$$(25)\qquad L_w u = \frac{1}{n} \operatorname*{div}_{S^n} \left\{ \frac{Du}{(1 + |Dw|^2)^{\frac{1}{2}}} \right\} = (1 + |Dw|^2)^{-\frac{1}{2}} - \Lambda_t(w)\varphi_t(x, Dw)\,.$$

This is solvable in B since (23) ensures that the right side is perpendicular to the kernel of the selfadjoint operator L_w. Thus T is compact and $T_0 w = 0$. The proof is completed if we show that any fixed point $T_t u = u$, $(u, t) \in B \times [0, 1]$, satisfies

$$\|u\|_{C^{1+\gamma}} < K$$

for some γ, where K is independent of t.

From the eigenvalue bound we see that Theorem A may be applied to functions that satisfy

$$\mathcal{M}u = \Lambda_t(u)\varphi_t(x, Du) .$$

The constants in Theorem A may be taken independent of t since (19) holds for all φ_t. In replacing φ by $\Lambda_t\varphi_t$ we see from the bounds

$$\begin{aligned} \varepsilon_1 &= \frac{\tilde{\varepsilon}_1}{\tilde{\varepsilon}_5^2} , \\ \varepsilon_2 &= \frac{\tilde{\varepsilon}_2}{\tilde{\varepsilon}_6} , \\ \varepsilon_3 &= \frac{\tilde{\varepsilon}_3}{\tilde{\varepsilon}_6} , \\ \varepsilon_4 &= \tilde{\varepsilon}_4 \leqslant \inf\left\{\frac{n}{n-1}\lambda^2 k_1 - \lambda k_2 + \frac{n-1}{n}\right\} . \end{aligned}$$

Thus if conditions (20) hold, Theorem A implies

$$|Du| \leqslant k(n, \alpha, \beta, \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_6) .$$

Since $u \in B$, by integrating this bound we obtain

$$|u| \leqslant k \operatorname{diam}(S^n) = k\pi .$$

Finally the $C^{1,\gamma}$ bound follows from, e.g., [8, p. 273].

4.1. COROLLARY. *Suppose $\varphi \in C^1(S^n)$ such that*

$$|D\varphi| < 2\varphi . \tag{26}$$

Then there exists a constant λ such that

$$\inf\varphi \leqslant \frac{1}{\lambda} \leqslant \sup\varphi \tag{27}$$

and a hypersphere Y about 0 whose mean curvature

$$H(Y) = \lambda\varphi(Y|Y|^{-1})|Y|^{-1}.$$

Any other hypersurface satisfying the equation is homothetic to Y. If either equality holds in (27) then Y is the standard sphere.

PROOF. (26) implies for any $\lambda \geq 0$

$$\lambda|D\varphi| < 2\lambda\varphi \leq \frac{n}{n-1}\lambda^2\varphi^2 + \frac{n-1}{n}.$$

Apply Theorem B in case (iv). By averaging equation (22) we see that if the left hand equality of (27) holds,

$$1 \geq \fint \frac{1}{\sqrt{1+|Du|^2}} = \lambda \fint \varphi = \frac{1}{\inf\varphi} \fint \varphi \geq 1.$$

Equality holds only if φ and u are constant. Similarly, if the right equality in (27) holds, we integrate (22) multiplied by exp $[nu]$,

$$1 \leq \frac{\int e^{nu}(1+|Du|^2)^{\frac{1}{2}}}{\int e^{nu}} = \frac{\int e^{nu}\varphi}{\sup\varphi \int e^{nu}} \leq 1.$$

4.2. EXAMPLES. In this section we indicate a sense in which condition $A(iv)$ of Theorem A is sharp. We consider surfaces which are rotationally symmetric about the x_{n+1} axis. Denote by θ the distance from the x_{n+1} axis on the unit sphere. The mean curvature of the family of cones

$$\tan^2\theta x_{n+1}^2 = \sum_{i=1}^{n} x_i^2$$

is given by

$$rH = \frac{n-1}{n}\cot\theta = \psi(x) \tag{28}$$

for which $A(iv)$ fails since

$$|D\psi| = \frac{n}{n-1}\psi^2 + \frac{n-1}{n}. \tag{29}$$

Any data that coincides with ψ on an interval cannot be expected to have a gradient estimate there.

Indeed we may realize hypersurfaces which « blow up » or « blow down » along any fixed cone $\theta = \theta_0$. Using the fact that on S^n for functions u of θ alone,

$$\Delta u = u_\theta(n-1)\cot\theta + u_{\theta\theta},$$

equation (4) becomes

$$-\frac{u_{\theta\theta}}{n(1+u_\theta^2)^{\frac{3}{2}}} + \frac{n-(n-1)u_\theta\cot\theta}{n(1+u_\theta^2)^{\frac{1}{2}}} = \psi(\theta).$$

Hence if in the vicinity of θ_0 a surface is given by

$$u(\theta) = M(\theta - \theta_0)^{-1}$$

then

$$\psi(\theta) = \frac{(n-1)\cot\theta}{n(1 + M^{-2}(\theta - \theta_0)^4)^{\frac{1}{2}}} + \frac{(\theta - \theta_0)^2}{(M^2 + (\theta - \theta_0)^4)^{\frac{1}{2}}} - \frac{2(\theta - \theta_0)^3 M}{n(M^2 + (\theta - \theta_0)^4)^{\frac{3}{2}}}$$

is close to (28) and barely violates A (iv) for large M.

5. – Sufficient conditions for convexity.

THEOREM C. *Let $Y = e^{u(x)}x\colon S^n \to \mathbb{R}^{n+1}$ be a C^4 starlike embedding of a hypersphere into Euclidean space. Suppose that Y satisfies the dilation invariant prescribed mean curvature equation*

$$\mathcal{M}u = \varphi(x)\,.$$

If in addition,

$$c_4(\varphi, |D\varphi|, \chi(D^2\varphi); n, \sup|D\varphi|) > 0 \quad \text{for all } x \in S^n\,, \tag{30}$$

where $\chi(D^2\varphi) = \sup\{\varphi_{ee}(x)|e \in T_x S_1^n, |e| = 1\}$, φ_{ee} is the second covariant derivative of φ in S^n and c_4 is a function to be described, then Y is a convex hypersurface. In this case, the second fundamental form h_{ij} of the surface satisfies

$$|\xi|^2 c_5(n, \varphi, D\varphi, D^2\varphi) \leqslant |Y| h_{ij}(x)\xi^i\xi^j \quad \text{for all } (x, \xi) \in TY\,,$$

where c_5, to be described later, tends to unity as φ tends to one in C^2.

PROOF. The hypersurface Y satisfies

$$H(Y) = \Phi(Y)$$

where $\Phi(Y) = \varphi(Y/r)/r$ and $r = |Y|$. We calculate intrinsically in Y. Let $\{f_1, \ldots, f_{n+1}\}$ be an orthonormal frame of $\mathbb{R}^{n+1}$ such that f_{n+1} is the interior normal to Y. Let $\{\theta^1, \ldots, \theta^{n+1}\}$ be the dual coframe. The connection forms are defined by $d\theta^A = \theta^B \wedge \theta^A_B$, $\theta^B_A + \theta^A_B = 0$. On Y,

$$0 = d\theta^{n+1} = \theta^A \wedge \theta^{n+1}_A$$

so that by Cartan's lemma, the second fundamental form of Y is given by

$h_{ij} = h_{ji}$,

$$\theta_i^{n+1} = h_{ij}\theta^j. \tag{31}$$

The covariant derivatives $h_{ijk}\theta^k = dh_{ij} - h_{ip}\theta_j^p - h_{jp}\theta_i^p$ satisfy the Codazzi equation $h_{ijk} = h_{ikj}$. By the flatness of $\mathbb{R}^{n+1}$, the curvature of Y is given by

$$-\tfrac{1}{2}S_{ikl}^j\theta^k\wedge\theta^l = d\theta_i^j - \theta_i^k\wedge\theta_k^j = \theta_{n+1}^k\wedge\theta_k^{n+1}.$$

By (31) we obtain the Gauss equation

$$S_{ikl}^j = h_{ik}h_{jl} - h_{il}h_{jk}. \tag{32}$$

To show $h_{ij}\theta^i\otimes\theta^j$ positive definite, we establish the positivity of

$$\zeta_0(u) = \inf\{h_{ff}(x)\colon (x, f)\in TS^n, |f| = 1\}.$$

If $(x_0, f_1(x_0))$, the point where $h_{11}(x_0) = \zeta_0$, is extended to $\{f_1, \dots, f_n\}$, an orthonormal frame in the neighborhood of x_0, then the function $\zeta(x) = h_{11}(x)$ has a minimum at $(x_0, f_1(x_0))$. Computing near x_0,

$$\begin{aligned} &0 = h_{1\alpha}, \quad \alpha > 1\\ &0 = \zeta_i = h_{11i} \\ &\zeta_{ij} = h_{11ij} = h_{i11j} = h_{i1j1} + h_{ip}S_{1j1}^p + h_{1p}S_{ij1}^p. \end{aligned} \tag{33}$$

Hence at x_0, by using (32), (33)

$$0 \leqslant \Delta\zeta = nH_{11} - \left(\sum h_{ij}^2\right)\zeta + nH\zeta^2. \tag{34}$$

A way to proceed is to use the Schwarz inequality

$$\sum h_{ij}^2 \geqslant n^{-1}\left(\sum h_{ii}\right)^2 = nH^2. \tag{35}$$

The second derivative $f_1f_1\Phi$ in the ambient $\mathbb{R}^{n+1}$ and the second covariant derivative Φ_{11} on Y are related by

$$f_1f_1\Phi = -(f_{n+1}\Phi)h_{11} + \Phi_{11}. \tag{36}$$

In terms of the orthonormal frame on $\mathbb{R}^{n+1}\setminus\{0\}$, $\{e_1, \dots, e_{n+1}\}$, where

$e_{n+1} = + X/|X|$, we may express f_1 and f_{n+1} by

$$\begin{aligned} f_{n+1} &= -\mu e_{n+1} + \nu e \\ f_1 &= \sigma e_{n+1} + \tau e_1 \end{aligned}$$

where $\mu^2 + \nu^2 = \sigma^2 + \tau^2 = 1$, $0 < \mu \leqslant \tau$, $0 < \nu$ are constants and e is a unit vector perpendicular to e_{n+1}. By Euler's relation on homogeneous functions

$$\begin{aligned} e_{n+1}\Phi &= \frac{\langle X, D\Phi\rangle}{r} = -\frac{\Phi}{r} \\ e_{n+1}e_{n+1}\Phi &= \frac{2\Phi}{r^2} \\ e_{n+1}(e\Phi) &= -\frac{2}{r}(e\Phi)\,. \end{aligned}$$

Hence

$$\begin{aligned} f_{n+1}\Phi &= \frac{\mu}{r}\Phi + \nu(e\Phi) \\ f_1 f_1 \Phi &= \frac{2\sigma^2}{r^2}\Phi - \frac{4\sigma\tau}{r}(e_1\Phi) + \tau^2(e_1e_1\Phi)\,. \end{aligned} \tag{37}$$

By using (35), (36), and (37) in (34) we obtain

$$0 \leqslant \left(\frac{2\sigma^2}{r^2}\Phi - \frac{4\sigma\tau}{r}(e_1\Phi) + \tau^2(e_1e_1\Phi)\right) + \left(\frac{\mu}{r}\Phi + \nu(e\Phi) - \Phi^2\right)\zeta + \Phi\zeta^2\,. \tag{38}$$

Similarly to (36), we express the ambient second derivatives in terms of the covariant derivatives on the unit sphere

$$e_1e_1\varphi = e_{n+1}\varphi + \varphi_{11}\,. \tag{29}$$

By using the homogeneity of Φ and its derivatives and (39) in (38) we obtain

$$0 \leqslant (r\zeta)^2 + (\mu + \nu\varphi_e/\varphi - \varphi)(r\zeta) + (2\sigma^2 - 4\sigma\tau\varphi_1/\varphi + \tau^2\varphi_{11}/\varphi - 1)\,. \tag{40}$$

Since $\nu = -(f_{n+1}, e_{n+1})$, it is bounded by the gradient of u on S^n, which by (18) is bounded by $c_2(n, \varphi, |\Omega\varphi|)$,

$$\begin{aligned} c_2(1 + c_2^2)^{-\frac{1}{2}} &\geqslant |Du|(1 + |Du|^2)^{-\frac{1}{2}} = \nu \geqslant |\sigma|\,, \\ (1 + c_2^2)^{-\frac{1}{2}} &\leqslant (1 + |Du|^2)^{-\frac{1}{2}} = \mu \leqslant \tau\,. \end{aligned} \tag{41}$$

In case $Z = \mu\tau^{-2}r\zeta > 0$, using (40) and (41) we obtain

$$(42)\quad 0 \leqslant (1 + c_2^2)Z^2 + \big(1 + c_2|D\varphi|/\varphi - \varphi\big)Z + \big(c_2^2 + 4c_2|D\varphi|/\varphi + \varphi_{11}/\varphi - 1\big)\,.$$

A condition that this form takes negative values for some positive Z is that the last coefficient be negative. If we set

$$c_4 = 1 - c_2^2 - 4c_2|D\varphi|/\varphi - \chi(D^2\varphi)/\varphi$$

then condition (30) implies that Y is convex. To see this, suppose that there is a continuous path of C^2 solutions u_t of $\mathcal{M}u_t = \psi_t$, $0 \leqslant t \leqslant 1$, connecting the unit sphere $u_0 \equiv \psi_0 - 1 \equiv 0$ to our given surface $u = u_1$, $\psi_1 = \varphi$, such that condition (30) holds for all ψ_t. Since $\zeta_0(u_0) = 1$ and $\zeta_0(t_t)$ is a continuous function of t, $\zeta_0(u_t)$ remains on the positive side of the zeroes of (42) so $\zeta_0(u_1) > 0$. A lower bound may be computed from the right hand zero, Z_2, of (42),

$$c_5 := e^{-\pi c_2}(1 + c_2^2)^{-\frac{1}{2}} Z_2 \leqslant \frac{\inf r}{r(x_0)}\big(r(x_0)\zeta(x_0)\big) \leqslant r h_{11}\,.$$

The properties of c_5 follow from those of c_2.

To see that there is a homotopy u_t satisfying (30) he let $\varphi_t = t\varphi + (1 - t)\sup_{S^n}\varphi$ and $\psi_t = \Lambda_t(u_t)\varphi_t$ where u_t is the fixed point of T_t, as in the proof of Theorem B. By the definition of φ_t and c_4, condition (30) holds for all t.

To see that u_t depends continuously on t, suppose there is a sequence $t_i \to \tau$ but $u_t \not\to u_\tau$. So for some $\varepsilon > 0$, a subsequence $\|u_{i'} - u_\tau\|_{C^2} \geqslant \varepsilon$. The apriori estimates on u_t imply uniform $C^{2,\gamma}$ bounds on $u_{i'}$, hence by compactness a subsequence $u_{i''}$ converges in C^2 to a fixed point of T_τ. We have reached a contradiction since by uniqueness in B, this fixed point is u_τ. Hence $\zeta_0(u_t)$ is continuous.

5.1. Remarks. Better conditions may result from a more careful exploitation of (42). Our condition, however, is homogeneous in φ, hence easily established for ψ_t. By using the estimate on c_2 in Section 3.2, we find a simpler condition

$$c_4 \geqslant c_6 - \frac{\chi(D^2\varphi)}{\varphi} = c_4 > 0$$

where

$$c_6 = c_6(n, \sup|D\varphi|/\varphi) = 1 - (c_3^2 + Bc_3)\xi^2\,.$$

Since the mean curvature is the sum of the principal curvatures, an upper bound tending toward unity as the data tends to spherical may be derived from the equation and lower bounds

$$rh_{11} = n\varphi - r\sum_{2}^{n} h_{ii} \leqslant n\tilde{\varepsilon}_5 - (n-1)c_5 .$$

In view of the apriori gradient estimates the result of this section may be regarded as one and two sided apriori bounds on the second derivatives of the solution.

It would be interesting to find conditions equivalent to convexity in this problem analogous to Firey's [7] necessary and sufficient condition in the Cristoffel problem.

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