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An Estimate of the Gap of the First Two Eigenvalues in the Schrödinger Operator.

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1. – Introduction.

We shall consider the following Dirichlet eigenvalue problem on a smooth bounded domain $\Omega \subset \mathbb{R}^n$,

$$\begin{cases} -\Delta u + Vu = \lambda u \\ u \equiv 0 \qquad \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

where V is a nonnegative function defined on $\bar{\Omega}$. As is well-known, the eigenvalues of problem (1.1) can be interpreted as the energy levels of a particle travelling under an external force field of a potential q in $\mathbb{R}^n$, where

$$q(x) = \begin{cases} V(x) & x \in \bar{\Omega} \\ +\infty & x \notin \Omega, \end{cases}$$

and the corresponding eigenfunctions are wave functions of the Schrödinger equation $-\Delta u + qu = \lambda u$. Furthermore, the set of eigenvalues $\{\lambda_k\}$ of (1.1) are nonnegative and can be arranged in a nondecreasing order as follows,

$$0 < \lambda_1 < \lambda_2 \leqslant \lambda_3 \ldots \leqslant \lambda_m \leqslant \ldots\,.$$

It is a significant problem to find a lower bound for λ_1 in terms of the geometry of Ω. This subject has been studied extensively by many authors. A rather precise bound in the case $V \equiv 0$ was worked out not only for a bounded domain in $\mathbb{R}^n$ but actually valid for a general Riemannian manifold with certain curvature conditions; we refer to [4] for these recent developments. Nevertheless, very little is known about the obvious interesting question of how big the gap is between λ_2 and λ_1. There are both physical

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and mathematical interests in finding out a lower bound for $\lambda_2 - \lambda_1$ in terms of the geometrical invariants of Ω and the given potential function V. Our main result is the following.

THEOREM (1.1). *Let Ω be a smooth convex bounded domain in $\mathbb{R}^n$ and $V\colon \bar{\Omega} \to \mathbb{R}$ a nonnegative convex smooth potential function.*

Suppose λ_2 and λ_1 are the first and second nonzero eigenvalues of (1.1), *then the following pinching inequality holds*

$$\frac{\pi^2}{4d^2} \leqslant \lambda_2 - \lambda_1 \leqslant \left(\frac{4n\pi^2}{D^2} + \frac{4(M-m)}{n}\right), \tag{1.2}$$

where $d =$ diameter of Ω, $D =$ the diameter of the largest inscribed ball in Ω, $M = \sup_{\bar{\Omega}} V$, and $m = \inf_{\bar{\Omega}} V$.

In the last section, we demonstrate how to make use of the main theorem here to obtain a similar theorem when $\Omega = \mathbb{R}^n$.

In Appendix B), we give a short proof of a theorem of Brascamp and Lieb on the log concavity of the first eigenfunction. A similar method of gradient estimate was used by Li and the third author in [4].

2. – A gradient estimate.

Let f_1 and f_2 be the first and second eigenfunctions of (1.1). It is a known fact that f_1 must be a positive function (a theorem of Courant [3]), and thus $u = f_2/f_1$ is a well-defined smooth function on Ω. Using the Hopf lemma and the Malgrange preparation theorem, one can actually verify that u is smooth up to the boundary $\partial\Omega$ (for a short proof of the case we need, see § 6). In this section, the following gradient estimate will be established, which is the key step to derive the lower bound for $\lambda_2 - \lambda_1$.

THEOREM 2.1. *With the same conditions stated in Theorem* (1.1), *we have the following estimate for the gradient of u,*

$$|\nabla u|^2 + \lambda(\mu - u)^2 \leqslant \sup_{\Omega} \lambda(\mu - u)^2 ,$$

where $\lambda = \lambda_2 - \lambda_1$, μ is a constant not less than $\sup_{\Omega} u$.

We proceed to give the proof by dividing our argument into two propositions. In the sequel of this, we denote by $G = |\nabla u|^2 + \lambda(\mu - u)^2$, which is a smooth function on $\bar{\Omega}$ as u is.

PROPOSITION 2.2. *With the same conditions in Theorem* (1.2), *if* G *attains its maximum in an interior point of* Ω, *we have the following inequality*

$$G \leqslant \sup_{\Omega} \lambda(\mu - u)^2 .$$

PROOF. By direct computation, we have

$$G_i = \sum_{j=1}^{n} 2u_j u_{ji} - 2\lambda(\mu - u)u_i \tag{2.1}$$

$$\Delta G = \sum_{i=1}^{n} G_{ii} = 2\sum_{i,j=1}^{n} u_{ij}^2 + \sum_{i,j=1}^{n} 2u_j u_{jii} + 2\lambda\sum_{i=1}^{n} u_i^2 - 2\lambda(\mu - u)\Big(\sum_{i=1}^{n} u_{ii}\Big). \tag{2.2}$$

It is by straightforward computation that

$$\Delta u = -\lambda u - 2(\nabla u \cdot \nabla \log f_1) \tag{2.3}$$

($\log f_1$ is well-defined since $f_1 > 0$ on Ω).

We substitute (2.3) into (2.2) and obtain

$$\Delta G = \Big\{2\sum_{i,j=1}^{n} (u_{ij})^2 + 2\lambda^2 u(\mu - u)\Big\} + \{4\lambda(\mu - u)(\nabla u \cdot \nabla \log f_1)\} - \{4(\nabla u)\cdot[\nabla\cdot(\nabla u \cdot \nabla \log f_1)]\} . \tag{2.4}$$

Suppose G attains its maximum in an interior point $p \in \Omega$. If $(\nabla u)(p) \neq 0$, then we can choose a coordinate such that $u_1(p) \neq 0$, $u_i(p) = 0$ for $2 \leqslant i \leqslant n$. Furthermore, since $\nabla G(p) = 0$, one easily deduces from (2.1) that relative to the above coordinate the following is true

$$u_{11}(p) = \lambda(\mu - u)(p) \tag{2.5}$$

$$u_{1i}(p) = 0 , \quad 2 \leqslant i \leqslant n . \tag{2.6}$$

Putting (2.5) and (2.6) into (2.4), we find a simplification for $\Delta G(p)$ with respect to this particular coordinate system,

$$\Delta G(p) = \Big\{2\sum_{i,j=1}^{n} u_{ij}^2 + 2\lambda^2(\mu - u)u\Big\} - \{4u_1^2(\log f_1)_{11}\} \leqslant 0 . \tag{2.7}$$

Since both V and Ω are convex by assumption, according to a result of Brascamp and Lieb [1], $\log f_1$ is concave, in particular $(\log f_1)_{11}(p)\leqslant 0$. Consequently, the second term of (2.7), namely $-4u_1^2(\log f_1)_{11}$, is nonnegative. Therefore, we have

$$\left\{\sum_{i,j=1}^{n} u_{ij}^2 + \lambda^2(\mu-u)u\right\}(p)\leqslant 0\,. \tag{2.8}$$

Furthermore, $u_{ij}^2(p)\geqslant 0\ \forall i,j$ implies

$$\{u_{11}^2+\lambda^2(\mu-u)u\}(p)\leqslant 0\,.$$

Again from (2.5), this leads to

$$\mu(\mu-u(p))\leqslant 0\,. \tag{2.9}$$

We can assume that $\sup_\Omega u$ is positive. On the other hand, $\sup_\Omega u$ is greater than $u(p)$ as $u_1(p)\neq 0$. If $\mu\geqslant\sup_\Omega u>0$, it gives rise to a contradiction of (2.9).

Our argument above shows that $\nabla u(p)=0$ and establishes the inequality $G\leqslant\sup_\Omega \lambda(\mu-u)^2$ as desired.

PROPOSITION 2.3. *Let us assume equation* (1.1) *satisfying all the conditions in Theorem* (2.1). *If G attains its maximum on $\partial\Omega$, then we have the same estimate*

$$G\leqslant\sup_\Omega \lambda(\mu-u)^2\,.$$

REMARK. We recall a differential geometric description of convexity here which will be used later. Suppose $H=(h_{\alpha\beta})_{2\leqslant\alpha,\beta\leqslant n}$ is the second fundamental form of $\partial\Omega$ relative to a unit normal of $\partial\Omega$ pointing outward to Ω. It is known that $\partial\Omega$ is convex iff H is positive definite.

PROOF OF PROPOSITION 2.3. Suppose G attains its maximum on $\partial\Omega$ at a point p. We can choose an orthonormal frame $\{l_1, l_2, \dots, l_n\}$ around p such that l_1 is perpendicular to $\partial\Omega$ and pointing outward. We also use the notation $\partial/\partial x_1$ to denote the restriction of l_1 on $\partial\Omega$, that is the normal unit vector field along $\partial\Omega$.

A simple computation shows

$$\begin{aligned}\frac{\partial G}{\partial x_1}(p) &= 2\sum_{i=1}^{n} u_i u_{i1} - 2\lambda u_1(\mu-u)\\ &= 2u_1u_{11} + 2\sum_{i=2}^{n} u_i u_{i1} - 2\lambda u_1(\mu-u)\geqslant 0\,.\end{aligned} \tag{2.10}$$

Consider the equation $\Delta u = -\lambda u - 2(\nabla u \cdot \nabla \log f_1)$, where both Δu and u are smooth up to the boundary and thus attain finite values on $\partial\Omega$. Hence, $(\nabla u \cdot \nabla \log f_1) = (1/f_1)\left[u_1(f_1)_1 + \sum_{2\leqslant i\leqslant n} u_i(f_1)_i\right]$ achieves finite values on $\partial\Omega$ as well. Nevertheless, since $f_1 \equiv 0$ on $\partial\Omega$, we have $(f_1)_i = 0\ \forall 2\leqslant i\leqslant n$ (l_i, $2\leqslant i\leqslant n$, is in the tangential direction). This implies that $\{(1/f_1)u_1(f_1)_1\}$ must be finite. By the Hopf lemma, $(f_1)_1 = \partial f_1/\partial x_1 \neq 0$ on $\partial\Omega$, we get the important observation that

$$u_1 \equiv 0 \quad \text{on } \partial\Omega\,. \tag{2.11}$$

Using (2.11) one can rewrite (2.10) as follows

$$\frac{\partial G}{\partial x_1}(p) = 2\sum_{i=2}^{n} u_i u_{i1} \geqslant 0\,. \tag{2.12}$$

From the definition of second fundamental form of a hypersurface in $\mathbb{R}^n$, one can derive

$$u_{i1} = -\sum h_{ij}u_j + \sum b_{ij}u_i\,, \qquad 2\leqslant i,\ j\leqslant n \tag{2.13}$$

where (b_{ij}) is a skew symmetric matrix *i.e.* $b_{ij} = -b_{ij}$. Putting (2.13) into (2.12), we have

$$\frac{\partial G}{\partial x_1}(p) = -2\sum_{i,j=2}^{n} u_i h_{ij} u_j \geqslant 0\,. \tag{2.14}$$

This contradicts the convexity of $\partial\Omega$. Thus $u_i(p) = 0$ for all $2\leqslant i\leqslant n$, and yields our inequality $G \leqslant \sup_\Omega \lambda(\mu - u)^2$.

Theorem 2.1 follows from the above two propositions.

3. – Lower bound.

In this section, we shall derive our lower bound $\pi^2/4d^2 \leqslant \lambda_2 - \lambda_1$.

Recall our basic estimate (Theorem 2.1) which says that for $\mu \geqslant \sup u$:

$$|\nabla u|^2 \leqslant \lambda\left\{\sup_\Omega(\mu - u)^2 - (\mu - u)^2\right\}. \tag{3.1}$$

In particular, we have

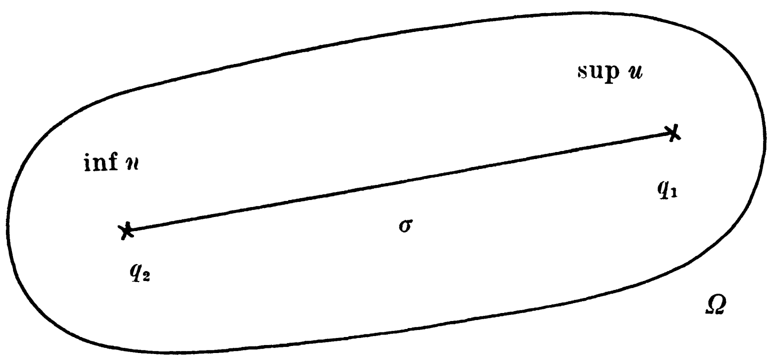
$$|\nabla u|^2 \leqslant \lambda\{(\sup u - \inf u)^2 - (\sup u - u)^2\}\,. \tag{3.2}$$

Furthermore,

$$\sqrt{\lambda} \geqslant \frac{|\nabla u|}{\sqrt{(\sup u - \inf u)^2 - (\sup u - u)^2}} . \tag{3.3}$$

Let $A = \sup u - \inf u$ and $W = \sup u - u$. One can rewrite (3.3) as

$$\sqrt{\lambda} \geqslant \frac{|\nabla W|}{\sqrt{A^2 - W^2}} . \tag{3.4}$$



Let q_1, q_2 be two points of $\bar{\Omega}$ such that $u(q_1) = \sup u$, $u(q_2) = \inf u$, and σ is the line segment joining them. σ lies in Ω since it is convex by assumption. We integrate both sides of (3.3) along σ from q_1 to q_2 and obtain

$$\int_{\sup u}^{\inf u} \frac{|\nabla u| ds}{\sqrt{(\sup u - \inf u)^2 - (\sup u - u)^2}} \leqslant \int_{q_1}^{q_2} \sqrt{\lambda}\, ds .$$

Changing variables, we have

$$\int_0^A \frac{|dW|}{\sqrt{A^2 - W^2}} \leqslant \int_{q_1}^{q_2} \sqrt{\lambda}\, ds .$$

By elementary calculus, one has

$$\frac{2}{\pi} \leqslant \sqrt{\lambda}\, \ell(\sigma) \leqslant \sqrt{\lambda}\, d ,$$

where $\ell(\sigma) =$ length of σ, $d =$ diameter of Ω. This proves $\lambda_2 - \lambda_1 \geqslant \pi^2/4d^2$ as has been claimed.

4. – Upper bound.

The major step to establish our upper bound $\lambda_2 - \lambda_1 \leqslant 4\pi^2/D^2 + \dfrac{4(M-m)}{n}$ is the following.

LEMMA 4.1. *Let Ω be a smooth bounded domain in $\mathbb{R}^n$ and V a bounded nonnegative potential defined on $\bar{\Omega}$. Suppose λ_1, λ_2 are the first and second nonzero eigenvalues of the Dirichlet boundary problem*

$$\begin{cases} \Delta f - Vf = -\lambda f \\ f \equiv 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

then

$$\lambda_2 - \lambda_1 \leqslant \frac{4}{n}(\lambda_1 - m).$$

where $m = \inf\limits_{\Omega} V$.

Some results of this sort in the case of $V \equiv 0$ were given by Payne, Pólya and Weinberger [6].

PROOF. Let f_1 be the first eigenfunction of (4.1). Take a trial function $f = x_i f_1 - a f_1$, where x_i is any fixed coordinate function for some $1 \leqslant i \leqslant n$ and a is a constant chosen to satisfy $\int\limits_{\Omega} f \cdot f_1 = 0$. The following computation shows that

$$\begin{aligned} -\Delta f + Vf &= -2\frac{\partial f_1}{\partial x_i} + (x_i - a)(-\Delta f_1 + Vf_1) \\ &= -2\frac{\partial f_1}{\partial x_i} + \lambda_1 (x_i - a) f_1 \\ &= -2\frac{\partial f_1}{\partial x_i} + \lambda_1 f. \end{aligned} \tag{4.1}$$

Multiplying both sides of (4.1) by f, integrating over Ω and then dividing by $\int\limits_{\Omega} f^2$, we have

$$\frac{\int\limits_{\Omega} f(-\Delta f + Vf)}{\int\limits_{\Omega} f^2} = \frac{-2\int\limits_{\Omega} (\partial f_1/\partial x_i)\cdot f}{\int\limits_{\Omega} f^2} + \lambda_1. \tag{4.2}$$

The following formula is well-known,

$$\lambda_2 = \inf_{\begin{cases} g \perp f_1 \\ g \equiv 0 \text{ on } \partial\Omega \end{cases}} \frac{\int\limits_{\Omega} (-g\Delta g + Vg^2)}{\int\limits_{\Omega} g^2} = \inf_{\begin{cases} g \perp f_1 \\ g \equiv 0 \text{ on } \partial\Omega \end{cases}} \frac{\int\limits_{\Omega} |\nabla g|^2 + \int\limits_{\Omega} Vg^2}{\int\limits_{\Omega} g^2}. \tag{4.3}$$

(4.3) together with (4.2) and the fact that $f \perp f_1$ imply

$$\lambda_2 \leqslant -2\left(\frac{\int_\Omega (\partial f_1/\partial x_i)\cdot f}{\int_\Omega f^2}\right) + \lambda_1\,, \tag{4.4}$$

$$\lambda_2 - \lambda_1 \leqslant -2\left(\frac{\int_\Omega (\partial f_1/\partial x_i)\cdot f}{\int_\Omega f^2}\right). \tag{4.5}$$

Substituting $f = x_i f_1 - a f_1$ and integrating by parts, gives

$$\begin{aligned}\int_\Omega f\cdot\frac{\partial f_1}{\partial x_i} &= \int_\Omega (x_i f_1 - a f_1)\frac{\partial f_1}{\partial x_i} = \frac{1}{2}\int_\Omega (x_i - a)\frac{\partial (f_1^2)}{\partial x_i}\\ &= \frac{1}{2}\int_\Omega x_i \frac{\partial (f_1^2)}{\partial x_i}\\ &= -\frac{1}{2}\int_\Omega f_1^2\,.\end{aligned} \tag{4.6}$$

We can always normalize f_1 such that $\int f_1^2 = 1$. Combining (4.5) and (4.6), we have

$$\lambda_2 - \lambda_1 \leqslant \frac{1}{\int_\Omega f^2}\,. \tag{4.7}$$

Again from (4.6) $\int f(\partial f_1)/\partial x_i) = -\frac{1}{2}$; moreover, the Schwarz lemma says that

$$\left(\int_\Omega \left(\frac{\partial f_1}{\partial x_i}\right)^2\right)\left(\int_\Omega f^2\right) \geqslant \frac{1}{4}\,. \tag{4.8}$$

This implies that

$$\left(\int_\Omega |\nabla f_1|^2\right)\cdot\left(\int_\Omega f^2\right) \geqslant \frac{n}{4} \tag{4.9}$$

since $|\nabla f_1|^2 = \sum_{j=2} (\partial f_1/\partial x_i)^2$. Bringing (4.7) and (4.9) together, we have

$$\lambda_2 - \lambda_1 \leqslant \frac{4}{n}\int_\Omega |\nabla f_1|^2\,. \tag{4.10}$$

Since $-\Delta f_1 + V f_1 = \lambda_1 f_1$ and $V \geqslant m$, it is easy to see that $\int_\Omega |\nabla f_1|^2 \leqslant \lambda_1 - m$.

Using this fact, one can conclude from (4.10) that

$$\lambda_2 - \lambda_1 \leqslant \frac{4}{n}(\lambda_1 - m).$$

This completes the proof.

REMARK. It is in general true that $\lambda_{i+1} - \lambda_i \leqslant \left(2\sum_{j=1}^{i}\lambda_j\right)\Big/ i$, where $1 \leqslant i \leqslant n-1$.

PROOF OF UPPER BOUND OF $\lambda_2 - \lambda_1$. Recall the identity (4.3)

$$\lambda_1 = \inf_{f\equiv 0 \text{ on } \partial\Omega} \frac{\int_\Omega |\nabla f|^2 + \int_\Omega V f^2}{\int_\Omega f^2}.$$

Let us choose g vanishing on $\partial\Omega$ s.t. $\int_\Omega |\nabla g|^2 / \int_\Omega g^2 = \mu_1$, where μ_1 is the first-nonzero eigenvalue of the Dirichlet problem (1.1) on Ω with $V\equiv 0$. Clearly we have

$$\lambda_1 \leqslant \frac{\int_\Omega |\nabla g|^2 + \int_\Omega V g^2}{\int_\Omega g^2} \leqslant \frac{\int_\Omega |\nabla g|^2}{\int_\Omega g^2} + M = \mu_1 + M.$$

Using a theorem of Cheng [2], we have

$$\mu_1 \leqslant \frac{n^2\pi^2}{D^2}, \qquad \text{when } n = \dim \Omega$$

and $D =$ the diameter of the largest inscribed ball in Ω. With Lemma 4.1, we can now establish our upper bound for $\lambda_2 - \lambda_1$ asserted in Theorem 1.1.

$$\lambda_2 - \lambda_1 \leqslant \frac{4}{n}(\lambda_1 - m) \leqslant \frac{4}{n}[\mu_1 + M - m] \leqslant \frac{4}{n}\left[\frac{n^2\pi^2}{D^2} + M - m\right] \leqslant \frac{4n\pi^2}{D^2} + \frac{4}{n}(M-m).$$

5. – Gap of eigenvalues over R^n.

In this section, we extend the estimate for eigenvalues of bounded domain to eigenvalues of R^n. We need the following well-known fact.

PROPOSITION 5.1. *Let $\lambda_2(R)$ be the second eigenvalue of $\Delta - V$ defined on the ball $B(R)$ with Dirichlet boundary condition. Then $\lambda_2(R)$ is a con-*

tinuous piecewise smooth function of R when $R > 0$. When it is smooth,

$$\frac{d}{dR}\lambda_2(R) = -\int\limits_{\partial B(R)} \left(\frac{\partial \varphi}{\partial r}\right)^2 \tag{5.1}$$

where φ is a normalized second eigenfunction of $\Delta - V$ defined on $B(R)$.

PROOF. Let $\varphi(x; r_2)$ be the normalized second eigenfunction of $\Delta - V$ defined on the ball $B(r_2)$ with Dirichlet boundary condition. In polar coordinates, φ is a function of the form $\varphi(\theta, r_1; r_2)$ where $\theta \in S^{n-1}$, the unit sphere, and $0 < r_1 \leqslant r_2 < \infty$.

It is well-known that we can assume φ to be piecewise smooth as a function of r_2. At the points where u is smooth, we can differentiate the equantion for φ and obtain

$$\int\limits_{B(r_2)} \varphi \Delta \left(\frac{\partial \varphi}{\partial r_2}\right) = \int\limits_{B(r_2)} \varphi (V - \lambda_2) \left(\frac{\partial \varphi}{\partial r_2}\right) - \int\limits_{B(r_2)} \frac{d\lambda_2}{dr_2}\varphi^2 . \tag{5.2}$$

Integrating by parts, we derive

$$\frac{d\lambda_2}{dr_2} = \int\limits_{\partial B(r_2)} \frac{\partial \varphi}{\partial r_1}\frac{\partial \varphi}{\partial r_2} . \tag{5.3}$$

Notice that $\varphi(\theta, r, r) = 0$ for all r. Hence

$$0 = \frac{d}{dr}\varphi(\theta; r, r) = \frac{\partial \varphi}{\partial r_1}(\theta; r, r) + \frac{\partial \varphi}{\partial r_2}(\theta, r, r) . \tag{5.4}$$

Putting this into (5.3) we have

$$\frac{d\lambda_2}{dr_2}(R) = -\int\limits_{\partial B(r_2)} \left(\frac{\partial \varphi_2}{\partial r_1}\right)^2 . \tag{5.5}$$

PROPOSITION 5.2. *Let φ be an eigenfunction of $\Delta - V$ defined on the ball $B(R) \subset \mathbb{R}^n$ with Dirichlet boundary condition and eigenvalue λ. Then*

(i) *If $2n - 2 \geqslant k > 2$,*

$$\int\limits_{\partial B(R} \left(\frac{\partial \varphi}{\partial r}\right)^2 \leqslant R^{-k+n-1}\left[-k\int\limits_{B(R)} r^{k-n}(V - \lambda)\varphi^2 - \int\limits_{B(R)} r^{k-n+1}\frac{\partial V}{\partial r}\varphi^2\right] . \tag{5.6}$$

(ii) *If* $k \geqslant 2n-2$ *and* $k > 2$,

$$\int_{\partial B(R)} \left(\frac{\partial \varphi}{\partial r}\right)^2 \leqslant \frac{k-2n+2}{2} R^{-k+n-1} \int_{B(R)} \varphi^2 \Delta r^{k-n} \tag{5'7}$$

$$+ (2n-2-2k) R^{-k+n-1} \int_{B(R)} r^{k-n}(V-\lambda)\varphi^2 - R^{-k+n-1} \int_{B(R)} r^{k-n+1} \frac{\partial V}{\partial r} \varphi^2 .$$

Proof. Let $d\theta$ be the volume element of the unit sphere S^{n-1} in $\mathbb{R}^n$ and Δ_θ be the spherical Laplacian. Then

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{n-1}{r}\frac{\partial \varphi}{\partial r} + \frac{\Delta_\theta \varphi}{r^2} = (V-\lambda)\varphi \tag{5.8}$$

and

$$\frac{d}{dr} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r}\right)^2 (r,\theta)\, d\theta = 2\int_{S^{n-1}} \frac{\partial \varphi}{\partial r}\frac{\partial^2 \varphi}{\partial r^2} d\theta \tag{5.9}$$

$$= -2\int_{S^{n-1}} \frac{n-1}{r}\left(\frac{\partial \varphi}{\partial r}\right)^2 \mathrm{d}\theta - 2\int_{S^{n-1}} r^{-2}\frac{\partial \varphi}{\partial r} \Delta_\theta \varphi\, d\theta + 2\int_{S^{n-1}} (V-\lambda)\varphi \frac{\partial \varphi}{\partial r} d\theta$$

$$= -2(n-1) r^{-1} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r}\right)^2 d\theta + r^{-2} \frac{\partial}{\partial r} \int_{S^{n-1}} |\nabla_\theta \varphi|^2\, d\theta + \int_{S^{n-1}} (V-\lambda) \frac{\partial \varphi^2}{\partial r} d\theta$$

$$= -2(n-1) r^{-1} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r}\right)^2 d\theta + r^{-2} \frac{\partial}{\partial r} \int_{S^{n-1}} |\nabla_\theta \varphi|^2 d\theta + \frac{d}{dr}\int_{S^{n-1}} (V-\lambda)\varphi^2\, d\theta - \int_{S^{n-1}} \frac{\partial V}{\partial r}\varphi^2\, d\theta .$$

Multiplying this equation by r^k (with $k > 2$) and integrating from 0 to R, we have

$$\int_0^R r^k \frac{d}{dr} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r}\right)^2 (r,\theta)\, d\theta\, dr = -2(n-1)\int_0^R r^{k-1} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r}\right)^2 d\theta\, dr \tag{5.10}$$

$$+ \int_0^R r^{k-2} \frac{d}{dr} \int_{S^{n-1}} |\nabla_\theta \varphi|^2\, d\theta\, dr + \int_0^R r^k \frac{d}{dr} \int_{S^{n-1}} (V-\lambda)\varphi^2\, d\theta\, dr - \int_0^R r^k \int_{S^{n-1}} \left(\frac{\partial V}{\partial r}\right) \varphi^2\, d\theta\, dr .$$

Integrating by parts, we have the following

$$R^k \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r}\right)^2 (R,\theta)\, d\theta = k\int_0^R r^{k-1} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r}\right)^2 (r,\theta)\, d\theta\, dr \tag{5.11}$$

$$+ \int_0^R r^k \left[\frac{d}{dr} \int_{S^{n-1}} \left(\frac{\partial \varphi}{\partial r}\right)^2 (r,\theta)\, d\theta\right] dr ,$$

$$0=\int_0^R (k-2)r^{k-3}\int_{S^{n-1}}|\nabla_\theta\varphi|^2\,d\theta\,dr+\int_0^R r^{k-2}\left[\frac{d}{dr}\int_{S^{n-1}}|\nabla_\theta\varphi|^2\,d\theta\right]dr \tag{5.12}$$

$$0=k\int_0^R r^{k-1}\int_{S^{n-1}}(V-\lambda)\varphi^2\,d\theta\,dr+\int_0^R r^k\left[\frac{d}{dr}\int_{S^{n-1}}(V-\lambda)\varphi^2\,d\theta\right]dr\,. \tag{5.13}$$

Putting (5.11), (5.12) and (5.13) into (5.10), we have

$$\begin{aligned} R^k\int_{S^{n-1}}\left(\frac{\partial\varphi}{\partial r}\right)^2(R,\theta)\,d\theta &= (k-2n+2)\int_0^R r^{k-1}\int_{S^{n-1}}\left(\frac{\partial\varphi}{\partial r}\right)^2 d\theta\,dr \\ &-(k-2)\int_0^R r^{k-3}\int_{S^{n-1}}|\nabla_\theta\varphi|^2\,d\theta\,dr-k\int_0^R r^{k-1}\int_{S^{n-1}}(V-\lambda)\varphi^2\,d\theta\,dr-\int_0^R\int_{S^{n-1}}r^k\frac{\partial V}{\partial r}\varphi^2\,d\theta\,dr\,. \end{aligned} \tag{5.14}$$

Hence,

$$\begin{aligned} R^{k-n+1}\int_{\partial B(R)}\left(\frac{\partial\varphi}{\partial r}\right)^2 &\leqslant (k-2n+2)\int_{B(R)}r^{k-n}\left(\frac{\partial\varphi}{\partial r}\right)^2-k\int_{B(R)}r^{k-n}(V-\lambda)\varphi^2 \\ &-\int_{B(R)}r^{k-n+1}\frac{\partial V}{\partial r}\varphi^2\,. \end{aligned} \tag{5.15}$$

By the divergence theorem,

$$0=\int_{\partial B(R)}r^{k-n}\varphi\frac{\partial\varphi}{\partial r}=\int_{B(R)}r^{k-n}|\nabla\varphi|^2+\int_{B(R)}\varphi\nabla r^{k-n}\cdot\nabla\varphi+\int_{B(R)}r^{k-n}\varphi\Delta\varphi\,. \tag{5.16}$$

Hence,

$$\begin{aligned} \int_{B(R)}r^{k-n}\left(\frac{\partial\varphi}{\partial r}\right)^2\leqslant\int_{B(R)}r^{k-n}|\nabla\varphi|^2 &= -\int_{B(R)}\varphi\nabla r^{k-n}\cdot\nabla\varphi-\int_{B(R)}r^{k-n}\varphi\Delta\varphi \\ &=\frac{1}{2}\int_{B(R)}\varphi^2\Delta r^{k-n}-\int_{B(R)}r^{k-n}(V-\lambda)\,\varphi^2\,. \end{aligned} \tag{5.17}$$

The proposition follows from (5.15) and (5.17).

It is straighforward to derive from Theorem 1.1 and the last two propositions the following theorem.

THEOREM 5.1. *Let V be a C^1-function defined on R^n with $n>4$. Let $\lambda_2(\varrho)$ be the second eigenvalue of the operator $-\Delta+V$ defined on the ball*

$B(\varrho)$ with Dirichlet boundary condition. Suppose that V is convex in the ball $B(R)$, then

$$\text{(i)}\quad \lambda_2-\lambda_1 \geqslant \frac{\pi^2}{4R^2}-\frac{1}{k-n}R^{-k+n}\cdot\sup\left\{-k|x|^{k-n}\big(\lambda_2(|x|)-V(x)\big)-|x|^{k-n+1}\frac{\partial V}{\partial r}\right\}_+ .$$

where $2n-2\geqslant k>n$ and $\{f\}_+$ stands for the positive part of f.

(ii) *When $k\geqslant 2n-2$, $k>n$ and $k>2$,*

$$\lambda_2-\lambda_1 \geqslant \frac{\pi^2}{4R^2}-\frac{1}{k-n}R^{-k+n}\cdot\sup_{|x|<R}\left\{\frac{k-2n+2}{2}\Delta r^{k-n}+(2n-2-2k)|x|^{k-n}\big(V(x)-\lambda_2(x)\big)-r^{k-n+1}\frac{\partial V}{\partial r}\right\}_+ .$$

REMARK. If $\lim_{|x|\to\infty} V(x)=\infty$ and $\partial V/\partial r\geqslant 0$, $k-n>2$ and R large, we can obtain a positive lower estimate for $\lambda_2-\lambda_1$. Note also that

$$\lambda_2(R)\leqslant\lambda_2(1)\leqslant\frac{n+4}{n}\lambda_1(1)-\frac{4}{n}\inf_{B(1)}V\leqslant n(n+4)\pi^2+\frac{n+4}{n}\sup_{B(1)}V-\frac{4}{n}\inf_{B(1)}V .$$

Hence $(\lambda_2(R)-V(x))_+$ can be estimated easily if $\lim_{|x|\to\infty}V(x)=\infty$.

6. – Appendix.

A) Here we shall give a quick argument to verify the « standard » fact that $u=f_2/f_1$ is smooth up to the boundary $\partial\Omega$. In the whole discussion, we assume Ω to be smooth convex. Our conditions in Theorem 2.1 allow us to apply the classical Hopf lemma to f_1.

Let us choose local coordinates $\{x_1, x_2, \ldots, x_n\}$ on a sufficiently small open set U such that $U\cap\partial\Omega=U\cap\{x_1=0\}$. Since f_1 is identically equal to zero on $\partial\Omega$ and $f>0$ in Ω, by the Hopf lemma we have $\partial f_1/\partial x_1<0$ on $\partial\Omega$. Furthermore, f_1 is smooth up to the boundary, thus one can consider f_1 as a smooth function which is defined on U restricted to $U\cap\bar{\Omega}$. Using the Malgrange preparation theorem [5], together with the fact that $\partial f_1/\partial x_1\neq 0$ on $\partial\Omega$, we have locally

$$f_1=g_1\cdot x_1 , \tag{6.1}$$

where g_1 is a unit which is smooth on $\bar{\Omega}\cap U$.

Moreover, f_2 is identically zero on $\partial\Omega$; applying the Malgrange's theorem again, one can write locally

$$f_2 = g_2 \cdot x_1 \cdot h_2 , \tag{6.2}$$

where g_2 is a unit which is smooth in $\bar{\Omega} \cap U$, and h_2 is also a smooth function in $\bar{\Omega} \cap U$. Now it is clear

$$u = \frac{f_2}{f_1} = \frac{g_2 \cdot h_2}{g_1}$$

must be smooth on $U \cap \bar{\Omega}$.

B) Here we give a proof of a theorem of Brascamp and Lieb.

Let f_1 be the first positive eigenfunction of the operator $\Delta - V$ on a convex domain Ω with Dirichlet condition. Then $u = \log f$ satisfies the equation

$$\Delta u = (V - \lambda) - |\nabla u|^2 .$$

By convexity of Ω, it is easy to see that u is concave in a neighborhood of $\partial\Omega$. If we consider the Hessian of u as a function of the frame bundle of Ω, it achieves a maximum in the interior of Ω. At such a point,

$$0 \geqslant \Delta u_{ii} = (V_{ii}) - 2 \sum_j u_{ij}^2 \tag{6.3}$$

and

$$u_{ij} = 0 \qquad \text{for } i \neq j , \tag{6.4}$$

Hence

$$u_{ii}^2 \geqslant \tfrac{1}{2} \min_{\Omega} V_{ii} . \tag{6.5}$$

By using (6.5), we can prove the concavity of u by the method of continuity. In fact, we can find family Ω_t and V_t so that $\Omega_1 = \Omega$ and $V_1 = V$. Furthermore, we may assume Ω_0 is a ball in Ω and V_0 is a quadratic function so that by computation, the theorem is valid in this case. In fact, we can let $V_t = tV + (1-t) V_0$ and $\Omega_t = \{(1-t) x_0 + tx_1 : x_1 \in \Omega_1 \text{ and } x_0 \in \Omega_0\}$. Then $\min_{\Omega} (V_t)_{ii} > 0$.

If for $t < 1$, u_t is not concave, $(u_t)_{ii}$ at the maximum point will be positive by (6.5). This is not possible if we have a sequence $t_\alpha \to t$ with $\max (u_{t_\alpha})_{ii} \leqslant 0$. Hence we have proven the log concavity of f_1.

The proof actually shows that $(\log f_1)_{ii} \leqslant -\sqrt{\tfrac{1}{2} \min_{\Omega} V_{ii}}$:

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