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Local holomorphic extendability and non-extendability of *CR*-functions on smooth boundaries

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Local Holomorphic Extendability and Non-Extendability of CR-Functions on Smooth Boundaries.

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§ 1. – In this paper we deal with CR-functions on the smooth (C^∞) boundary of (a germ of) an open set $\Omega \subset C^n$.

By local extendability to C^n or to $C\bar{\Omega}$ at $p \in \partial\Omega$, we mean the existence of two neighbourhoods $A \subset B$ of p in C^n such that all CR-functions on $B \cap \partial\Omega$ are the trace of a holomorphic function in A or $A \cap C\bar{\Omega}$ respectively.

Our results are based on the notion of *supersector* of Ω at p. Let C be a regular analytic curve which has a contact of order precisely k at p with $\partial\Omega$. The k-jet of a defining function for $\partial\Omega$ in a complex coordinate on C or as we will say briefly, $\partial\Omega$, divides (the germ of) C in closed sectors, *interior sectors and exterior sectors of order k*. The first are in $\bar{\Omega}$ and the second in $C\Omega$ (more precisely, k-jets of these sets). If we remove all those exterior sectors which have width $\geqslant \pi/k$, the rest of C is formed by disjoint closed sectors which are called *supersectors of order k of Ω at p*. A supersector is *proper* if (as a germ) it contains points of (the k-jet of) Ω, otherwise the supersector is said to be *degenerate*.

DEFINITION. We shall say that Ω satisfies the *rays condition* at $p \in \partial\Omega$ if, for some $2 \leqslant k < +\infty$, there is an analytic curve, tangent of order precisely k to $\partial\Omega$ at p on which there are real lines $l_1, \ldots, l_\nu$, $\nu \geqslant 3$, issued from p (rays), contained in the (k-jet of) $\bar{\Omega}$, such that at least one of them is contained in the (k-jet of) Ω and their angles satisfy $\widehat{l_j\, l_{j+1}} < \pi/k$, for $j = 1, \ldots, \nu - 1$ and $\widehat{l_1\, l_\nu} > \pi/k$.

Supersectors and rays condition can be analytically presented as follows.

Let be $\Omega \equiv \{\phi(z_1, \ldots, z_n) < 0\}$, where ϕ is a smooth function and let $\mathbf{C} \ni \zeta \mapsto \varphi(\zeta) \in \mathbf{C}^n$ be a complex analytic curve having a contact of order k at $p = \varphi(0)$ with $\partial\Omega$.

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There is $\phi(\varphi(\zeta)) = P_k(\zeta) + O(|\zeta|^{k+1})$ where P_k is a real homogeneous polynomial of the variables $\operatorname{Re}\zeta$, $\operatorname{Im}\zeta$ of degree k. A *supersector* is the image by φ of any component of the complementary in $\mathbf{C}$ of the union of those open sectors of width $\geqslant \pi/k$ on which P_k is positive and the supersector is *proper* if P_k is negative at some point of the component.

The *rays condition* is satisfied whenever, for some φ, there exist numbers $0 \leqslant \theta_1 < \ldots < \theta_\nu < 2\pi$ with $\theta_{j+1} - \theta_j < \pi/k$ for $j = 1, \ldots, \nu - 1$ and $\theta_\nu - \theta_1 > \pi/k$ such that $P_k(\exp[i\theta_j]) \leqslant 0$ for $j = 1, \ldots, \nu$ and strict inequality holds for one value of j at least. *This condition is obviously equivalent to the property of Ω of having a proper supersector of width $> \pi/k$.* Somehow complementary to the rays condition is the following property which says that interior sectors of Ω are always gathered in groups of width $< \pi/k$ and those groups are supersectors, *i.e.* they are separated by sectors of width $\geqslant \pi/k$ and therefore by sectors of width $> \pi/k$.

DEFINITION. Ω has *grouped sectors* at $p \in \partial\Omega$ if, for some k, $2 \leqslant k < +\infty$, all supersectors of Ω at p have width $< \pi/k$. This happens for instance when Ω is pseudoconvex of type k at p. Our main result is the following.

THEOREM 1.

(i) *If Ω satisfies the rays condition at $p \in \partial\Omega$ then, near p, CR-functions on $\partial\Omega$ extend holomorphically to $C\overline{\Omega}$.*

(ii) *If $\Omega \subset C^2$ has grouped sectors at $p \in \partial\Omega$ then p is a local peak point or for Ω. In particular there are CR-functions on $\partial\Omega$ at p which do not extend holomorphically to $C\overline{\Omega}$.*

If $\Omega \subset \mathbf{C}^2$ then each of the hypothesis in th. 1, the *rays condition* and the *grouped sectors* condition, imply that k is the maximal order of contact with $\partial\Omega$ at p for a regular analytic curve; i.e. *p is a point of type k for Ω.*

Conversely, if we think in terms of supersectors, we see that only two kinds of points $p \in \partial\Omega$ of finite type (say type k) excape theorem 1 for $\Omega \subset C^2$. They are the following:

DEFINITION. – A point $p \in \partial\Omega$ of type k is said to be *exceptional of harmonic kind* if all supersectors of order k have width $< \pi/k$ but π/k is attained.

DEFINITION. A point $p \in \partial\Omega$ of type k is said to be *exceptional of rays kind* if all *proper* supersectors of order k have width $\leqslant \pi/k$ but there is a degenerate supersector of width $> \pi/k$. Indeed those germs are exceptional because, for fixed k, they form a closed subset with empty interior of the set of germs of type k.

In section 5 we shall prove by examples that for exceptional germs both extendability and non-extendability can hold. It is, as will also be noted, not clear to the authors whether exceptional germs of rays kind always have local peak functions. We shall use local co-ordinates $z = x + iy$, $w = u + iv$, $z' = (z_3, \ldots, z_n)$, with p as origin and choose $u = 0$ as the real tangent hyperplane to $\partial\Omega$ at p. So we have

$$\Omega \equiv \{u + P_k(z) + R(z, v, z') < 0\}, \quad \textit{with} \tag{1.1}$$
$$R(z, v, z') = O\Big(|z|^{k+1} + |z|\big(|v| + |z'|\big) + v^2 + |z'|^2\Big)$$

where $P_k(z) = P_k(x, y)$ is a real homogeneous polynomial of degree $k \geqslant 2$.

For $\Omega \subset C^2$ we have more simply

$$\Omega \equiv \{u + P_k(z) + R(z, v) < 0\}, \quad \textit{with} \tag{1.1$'$}$$
$$R(z, v) = O\big(|z|^{k+1} + |z||v| + v^2\big).$$

The first part of theorem 1 is derived in section 3 as a consequence of the following slightly more general result.

THEOREM 2. *If the polynomial P_k in* (1.1) *has no nontrivial* (*i.e.* $\not\equiv -\infty$) *subharmonic minorant in C, then CR-functions on $\partial\Omega$ extend* (*near* 0) *holomorphically to $C\bar{\Omega}$.*

The proof of this theorem makes use of the traditional technique of analytic discs as is based on a theorem of Stensønes-Henriksen on the envelopes of holomorphy of subgraphes ([13]).

The second part of theorem 1 is proved in section 4. For later convenience we want to avoid that the z axis contains real lines along which it is tangent to $\partial\Omega$ of higher order. So, by replacing w by $w + \lambda z^k$ with large $|\lambda|$, we can assume that

$$P_k(\exp[i\theta]) \quad \text{and} \quad D_\theta P_k(\exp[i\theta]) \quad \textit{never vanish together}. \tag{1.2}$$

First E. E. Levi noticed in [7] that, for strictly pseudoconvex $\Omega \subset C^2$, thanks to the existence of a regular analytic curve through p, contained in $C\Omega$, there is no extendability to $C\bar{\Omega}$. Then H. Lewy proved in [8] that, by the same reason, there is extendability to Ω. Later Kohn and Nirenberg showed in [6] by an example, that the curve may not exist in the weakly pseudoconvex case; however Bedford and Fornæss proved in [3] and [4] that both the conclusions of Levi and Lewy are still valid for weakly pseudoconvex Ω of finite type. Our construction of a peak function is a variation on the original Bedford and Fornæss method of [4]. Kohn-Nirenberg's example however showed that when the Levi form vanishes at p, extendability to Ω and non-extendability ot $C\bar{\Omega}$ have not the same origin (the analytic curve). This gave the hope to have extendability results to a full neighbourhood of p.

Recently it has been proved that this happens always for points of odd type and, for points of even type $\geqslant 4$, in an open set of cases, moreover at a point of any finite type there is extendability to at least one side. Therefore for points of finite type it is now convenient to forget CR-functions and to discuss the extendability to $C\bar{\Omega}$ of (germs of) functions in $\mathcal{O}(\Omega) \cap C^0(\bar{\Omega})$. Also only even types are interesting. We shall keep this point of view. Further a germ of type k has the extendability property to $C\bar{\Omega}$ if there is an interior sector of width $> \pi/k$. (The above remarks in this paragraph are contained in Baouendi-Treves [1], Rea [9], [10], [11] and Bogges-Pitts [14].)

The last hypothesis is evidently stronger than the rays condition. Indeed this is *strictly* stronger for an open subset of germs of even type $\geqslant 6$. This is shown by an example in the last section.

Similar results to the ones in this paper have been independently and simultaneously obtained by Bedford [2] using different techniques.

§ **2.** – DEFINITION 2.1. *Let $F(z)$ be a real upper semicontinuous function on the open set $E \subset C$. For $z \in E$, set $\phi(z) = \sup\{\psi(z),\ \psi$ is subharmonic in $E,\ \psi \leqslant F\}$. Then $\phi(z) (= \overline{\lim_{\zeta \to z}}\, \phi(\zeta))$ is said to be the largest subharmonic minorant of F in E.*

In this section we will only deal with the case $E = C$.

LEMMA 2.1. *Let $P(z)$ be an upper semicontinuous function, homogeneous of degree k. Assume that P has a nontrivial (i.e. $\not\equiv -\infty$) subharmonic minorant ϕ on C. Then $P(\exp[i\theta_1]) \leqslant 0$, $P(\exp[i\theta_2]) \leqslant 0$, $\theta_1 < \theta_2 < \theta_1 + \pi/k$ implies $\phi(\exp[i\theta]) \leqslant 0$ for $\theta_1 \leqslant \theta \leqslant \theta_2$, and $P(\exp[i\theta]) \geqslant 0$ for $\theta_1 + \pi/k \leqslant \theta \leqslant \theta_2 + \pi/k$. If in addition $P(\exp[i\theta_1]) < 0$ or $P(\exp[i\theta_2]) < 0$, then $\phi(\exp[i\theta]) < 0$ for $\theta_1 < \theta < \theta_2$.*

PROOF. Observe at first that the last statement of the lemma follows from the first part applied to $P(z) + \operatorname{Re}(\alpha z^k)$ where $\alpha \in C$ is such that $\operatorname{Re}(\alpha z^k) > 0$ for $\theta_1 < \operatorname{Arg} z < \theta_2$ and $\operatorname{Re}(\alpha z^k) = 0$ for $\operatorname{Arg} z = \theta_1$ or $\operatorname{Arg} z = \theta_2$. Then the first part implies that $\phi < \phi + \operatorname{Re}(\alpha z^k) \leqslant 0$ for $\theta_1 < \theta < \theta_2$.

To prove the first part, let $\operatorname{Re} \alpha z^k$ be a harmonic term which is strictly negative at $\theta = \theta_1, \theta_2$. Replacing P by $P + \varepsilon \operatorname{Re} \alpha z^k$, $0 < \varepsilon \ll 1$ we see that it clearly suffices to show that if $P(\exp[i\theta_1])$, $P(\exp[i\theta_2]) < 0$ then $\phi \leqslant 0$ for $\theta_1 < \theta < \theta_2$ and $\phi \geqslant 0$ for $\theta_1 + \pi/k \leqslant \theta \leqslant \theta_2 + \pi/k$.

By a theorem of Kiselman ([5], or [12] page 179) there exists a sequence of smooth (on $C - \{0\}$), k-homogeneous subharmonic functions on C, $\{\phi_n\}_{m=1}^{\infty}$, such that $\phi_n(0) = 0$ for all n and $\overline{\lim_{n\to\infty}} \phi_n(z) \leqslant \phi$, $\forall z$. Hence for all large n, $\phi_n(\exp[i\theta_j]) < 0$, $j = 1, 2$. Since the ϕ_n's are smooth it follows by a well known argument (see for instance [4], p. 556) that the sets $\{\phi_n < 0\}$ and $\{\phi_n \geqslant 0\}$ have components which are sectors of width respectively $\leqslant \pi/k$ and $\geqslant \pi/k$. Hence $\varphi_n < 0$ for $\theta_1 < \theta < \theta_2$ and $\phi_n \geqslant 0$ for $\theta_1 + \pi/k \leqslant \theta \leqslant \theta_2 + \pi/k$ and the same inequalities hold for ϕ. □

According to definitions of section 1 we define as a *supersector of a real homogeneous polynomial* Q_k of degree k, any component of $C - E$, where E is the union of those components of $\{Q_k > 0\}$ which have width $\geqslant \pi/k$. A supersector will be said to be *proper* if $Q_k < 0$ at some of its points, non-proper supersectors are called *degenerate*.

We want now to derive some analytic consequence of the grouped sectors condition, thus *we shall assume in the rest of the section that, for each* $a \in C$, *all supersectors of* $\operatorname{Re}(az)^k + P_k(z)$ have width $< \pi/k$. A trivial consequence of the continuity of P_k is the following.

(2.1) *Let* $\theta_1 \leqslant \arg z \leqslant \theta_2$ *be a proper supersector of* $\operatorname{Re}(a_0 z)^k + P_k(z)$. *There exist positive numbers* ε, δ *such that, for all* $a \in C$ with $|a - a_0| < \varepsilon$, *the sector* $\theta_1 - \delta < \arg z < \theta_2 + \delta$ *contains exactly one supersector of* $\operatorname{Re}(az)^k + P_k(z)$ *and this is proper. Furthermore the sector has no intersection with any other supersector of* $\operatorname{Re}(az)^k + P_k(z)$.

Also we shall need a corresponding statement for degenerate supersectors. Degenerate supersectors are finite collections of rays $\{\theta_1, \ldots, \theta_\nu\}$, $\theta_1 < \theta_2 < \ldots < \theta_\nu$.

LEMMA 2.2. *Let* $\{\theta_1, \ldots, \theta_\nu\}$ *be a degenerate supersector of* $\operatorname{Re}(a_0 z)^k + P_k(z)$. *There exist positive numbers* ε, δ *and an open subset* A *of* $N \equiv \{|a - a_0| < \varepsilon\}$ *with* $a_0 \in \partial A$, *such that the sector* $\theta_1 - \delta < \arg z < \theta_\nu + \delta$ *intersects a supersector of* $\operatorname{Re}(az)^k + P_k(z)$, *with* $a \in N$ *if and only if* $a \in \bar{A}$ *(closure in* N*) and in this case the sector contains a unique supersector of* $Re\,(az)^k + P_k(z)$ *and this is proper if and only if* $a \in A$. *Moreover the boundary of* A *can be done to be* C^1 *by a transformation of the type* $a \to (a - a_0)^\alpha$, *with* $\frac{1}{2} < \alpha \leqslant 1$.

PROOF. We shall prove that the lemma holds for $\nu = 1$ with $\alpha = 1$ and that in this case $A = A_1$ is bounded by a curve $t \to a_0 + \alpha_1(t)$, real analytic for $t \neq 0$ and such that

$$(*) \qquad \lim_{t\to 0} \alpha_1'(t)/|\alpha_1'(t)| = |a_0|^{k-1} a_0^{1-k} \exp[i(\pi/2 - k\theta_1)].$$

In the general case we will have $A = \bigcup_{j=1}^{\nu} A_j$ and by $(*)$ the tangent lines to

∂A_{j+h} and ∂A_j will form an angle $k(\theta_j - \theta_{j+h}) \in]-\pi, 0[$ and the lemma will be proved.

Solving a system we see that θ_1 is a critical zero of $\operatorname{Re}(a \exp[i\theta])^k + P_k(\exp[i\theta])$ only if $a^k = b(\theta_1)$ with

$$b(\theta) = \exp[-ik\theta]\left[-P_k(\exp[i\theta]) + \frac{i}{k} D_\theta P_k(\exp[i\theta])\right]. \tag{2.2}$$

Furthermore, direct derivation shows that

$$b'(\theta) = \frac{i}{k}\exp[-i\theta]\Delta P_k(\exp[i\theta]) \tag{2.3}$$

holds (Δ = Laplacian).

So by obvious transversality arguments the set A and N exist and ∂A is the curve $a_0 + \alpha_1(t)$ defined by $(a_0 + \alpha_1(t))^k = b(\theta_1 + t)$. Note that $a_0 \neq 0$ by (1.2), thus the definition is correct. Only (∗) is not yet proved. We have by (2.3), $k(a_0 + \alpha_1(t))^{k-1}\alpha_1'(t) = (i/k)\exp[-i(\theta_1 + t)]\Delta P_k(\exp[i(\theta_1 + t)])$. But since, for $|\theta - \theta_1| < \varepsilon$, with small $\varepsilon > 0$, we have $\operatorname{Re}(a_0 \exp[i\theta])^k + P_k(\exp[i\theta]) = c|\theta - \theta_1|^{2l} + (|\theta - \theta_1|^{2l+1})$, with $c > 0$, thus, by application of the identity $\Delta P_k(\exp[i\theta]) = D_{\theta\theta}P_k(\exp[i\theta]) + k^2 P_k(\exp[i\theta])$, we obtain $\Delta P_k(\exp[i\theta]) > 0$ for $0 < |\theta - \theta_1| < \varepsilon$, so (∗) holds even if ΔP_k vanishes at $\exp[i\theta_1]$. □

Finally we need a last lemma

LEMMA 2.3. *Fix a proper supersector* $\sigma_0 = \{\theta_1 \leqslant \theta \leqslant \theta_2\}$ *of the polynomial* $\operatorname{Re}(a_0 z)^k + P_k(z)$. *Then it is not possible to find for every continuous function* $a(t)\colon \{t \geqslant 0\} \to C$, $a(0) = a_0$ *two continuous real functions* $\theta_1(t)$, $\theta_2(t)$, $\theta_1(0) < \theta_1 < \theta_2(0)$ *such that* $\theta_1(t) < \theta_2(t) < \theta_1(t) + \pi/k$ *and such that* $\operatorname{Re} a^k(t) z_k + P_k(z)$ *is always negative for some* a, $\theta_1(t) < \arg z < \theta_2(t)$ *while this expression is strictly positive if* $\theta_1(t) - \pi/k \leqslant \arg z \leqslant \theta_1(t)$ *or* $\theta_2(t) \leqslant \arg z \leqslant \theta_2(t) + \pi/k$.

PROOF. If this is possible, it follows by a monodromy argument that there exists on $C(a)$ two continuous functions $\theta_1(a) < \theta_2(a) < \theta_1(a) + \pi/k$ with $\operatorname{Re} a^k z^k + P_k(z) > 0$ if $\theta_1(a) - \pi/k \leqslant \arg z \leqslant \theta_1(a)$ or $\theta_2(a) \leqslant \arg z \leqslant \theta_2(a) + \pi/k$ and strictly negative at some θ, $\theta_1(a) < \theta < \theta_2(a)$.

Now consider a path $a = R\exp[i\psi]$, $R \gg 1$, $\psi \in [0, 2\pi]$. Then

$$\operatorname{Re}\left(\exp[i(\psi + \theta_2(a))]\right)^k + \frac{1}{R^k} P_k(\exp[ik\theta_2(a)]) > 0 \qquad \forall \psi .$$

This forces $\psi + \theta_2(a(\psi))$ to oscillate less than $2\pi/k$. But since ψ increases by 2π this contradicts that θ_2 is single-valued. □

§ 3. – In this section we prove theorem 2 and theorem 1 part (i). We shall use the following

THEOREM 3.1 (Stensønes-Henriksen [13]). *Let E be a convex, open subset of $C_z^{m-1}\times R_v$ and $F(z, v)$ a real, lower semicontinuous function on E. Consider the set*

$$\tilde{\Omega} \equiv \{(z, u + iv) \in C^m,\ u < F(z, v),\ (z, v) \in E\}\ .$$

Then

(i) *the envelope of holomorphy $\tilde{\Omega}$ of Ω is a schlicht domain and there exists a lower semicontinuous function ϕ on E such that*

$$\tilde{\Omega} \equiv \{(z, u + iv) \in C^m,\ u < \phi(z, v),\ (z, v) \in E\}$$

(ii) *If F_m are also semicontinuous in $E_m, E_m \subset E_{m+1}$, $UE_m = E$, ϕ_m corresponds to F_m and $F_m \uparrow F$, then $\phi_m \uparrow \phi$.*

(iii) *If E has the form $A\times R$ with $A \subset C^{m-1}$ and F is independent of v, then ϕ is also independent of v, in fact ϕ is the smallest pluri-superharmonic majorant of F as a function in A.*

PROOF OF THEOREM 2. As we noticed in section 1, CR-functions are extendable at least on one side. So we must only prove that, in a suitable neighbourhood of p, p belongs to the envelope of holomorphy of Ω.

From theorem 3.1 (iii) it follows that the envelope of holomorphy $\tilde{U}$ of

$$U = \{(z, w, z') \in C^n;\ \operatorname{Re} w + P_k(z) < 0\}\ ,$$

is all of C^n. For $m\geqslant 1$, let

$$U_m = \Big\{(z, w, z') \in C^n;\ |z|\leqslant m,\ |v|\leqslant m,\ |z'|\leqslant m,\\ -C < \operatorname{Re} w < -P_k(z) - \frac{1}{m}\left(|z| + |v| + |z'|\right)\Big\},\qquad C\gg 1\ .$$

Then by theorem 3.1 (ii) there exists an m so that $0 \in \tilde{U}_m$.

For $t\gg 1$, let $\tilde{z}$, $\tilde{w}$, $\tilde{z}'$ be new co-ordinates, $z = t\tilde{z}$, $w = t^k\tilde{w}$, $z' = t^k\tilde{z}'$. In these co-ordinates U_m is given by

$$U_m = \Big\{(\tilde{z}, \tilde{w}, \tilde{z}') \in C^n;\ |\tilde{z}|\leqslant \frac{m}{t},\ |\tilde{v}|\leqslant \frac{m}{t^k},\ |\tilde{z}'|\leqslant \frac{m}{t^k},\\ -t^{-k}C < \operatorname{Re} \tilde{w} < -P_k(\tilde{z}) - \frac{1}{m}\left(\frac{t}{t^k}|\tilde{z}| + |\tilde{v}| + |\tilde{z}'|\right)\Big\}.$$

Obviously 0 is still in the envelope of holomorphy of U_m. Hence to show that 0 is in the envelope of holomorphy of Ω it suffices to show that for large enough t

$$P_k(z) + C\Big(|z|^{k+1} + |z|\big(|v| + |z'|\big) + v^2 + |z'|^2\Big) \leqslant P_k(z) + \frac{1}{m}\Big(\frac{t}{t^k}\,|z| + |v| + |z'|\Big)$$

for all (z, v, z') with $|z|\leqslant m/t$, $|v|\leqslant m/t^k$, $|z'|\leqslant m/t^k$, for C some fixed constant.

Clearly $C\big(|z|(|v| + |z'|) + v^2 + |z'|^2\big)\leqslant(1/m)\big(|v| + |z'|\big)$ if t is large enough so it suffices to show that

$$C|z|^{k+1}\leqslant\frac{1}{m}\Big(\frac{t}{t^k}\,|z|\Big) \qquad \text{for } |z|\leqslant\frac{m}{t}, t \qquad \text{large enough}$$

or equivalently that

$$Cm|z|^k\leqslant\frac{t}{t^k}.$$

This is true as soon as $t\geqslant Cm^{k+1}$.

This finishes the proof of theorem 2. □

The first part of theorem 1 is an easy consequence of theorem 2 as we now show.

Proof of theorem 1, (i). We noticed in section 1 that the hypothesis is equivalent to the fact that for some $\alpha\in C$, $Q_k(z) = \operatorname{Re}\alpha z^k + P_k(z)$ has a proper supersector of width $> \pi/k$. Hence there exist real numbers $a_1\leqslant b_1 < a_2\leqslant b_2 < \ldots < a_m\leqslant b_m$ with $Q_k\leqslant 0$ on each $\{a_j\leqslant\theta\leqslant b_j\}$, $a_{j+1} - b_j < \pi/k$, $b_m - a_1 > \pi/k$ and for some j there is $a_j < b_j$, $Q_k < 0$ on $\{a_j < \theta < b_j\}$.

We will show that the largest subharmonic minorant ϕ_C of Q_k (and hence of P_k) is $\equiv -\infty$. Assume $\phi_C\not\equiv -\infty$, so $\phi_C(0) = 0$. By Lemma 2.1 it follows that $\phi_C < 0$ on (b_j, a_{j+1}), $[a_{j+1}, b_{j+1})$, $[b_{j+1}, a_{j+2}) \ldots$ and (b_{j-1}, a_j), $(a_{j-1}, b_j] \ldots$ inductively, so in fact $\phi_C < 0$ on (a_1, b_m). This contradicts however lemma 2.1 applied to ϕ_C since $b_m - a_1 > \pi/k$. So $\phi_C\equiv -\infty$ as wanted. □

§ 4. – We shall show in this section the proof of theorem 1, (ii). This is based on the following

Proposition 4.1. *If for all $a\in C$ the supersectors of the polynomial* $\operatorname{Re}(az)^k + P_k(z)$ *have width $< \pi/k$, then there exists $\varepsilon > 0$ and a function h, holomorphic and nonvanishing in the cone*

$$\operatorname{Re} w^k + P_k(z) < \varepsilon|z|^k + \varepsilon|w|^k \tag{4.1}$$

which has limit 0 at the origin.

We assume for a moment the proposition.

PROOF OF THEOREM 1 (ii). Set $\tilde{g}(z, w) = \prod_{j=0}^{k-1} h(z, \exp[ij\pi/k]w)$. Since $\tilde{g}(z, w_1) = \tilde{g}(z, w_2)$ when $w_1^k = w_k^2$, a nonvanishing holomorphic function g is defined on

$$\Omega_\varepsilon \equiv \{u + P_k(z) < \varepsilon|z|^k + \varepsilon|w|\}$$

such that $g(z, w^k) = \tilde{g}(z, w)$. For small $\delta > 0$, $\Omega_\varepsilon \cap \{|z|, |w| < \delta\}$ is simply connected and therefore does not contain a closed curve whose g-image turns around the origin. Therefore a holomorphic function f is defined in this set such that $f^N = g$ for some integer N, and $|\mathrm{Arg}\, f| < \pi/4$. Moreover f has 0 limit at the origin. Hence e^f is a peak function for the set $\bar{\Omega}_{\varepsilon/2} \cap \{|z|, |w| \leqslant \delta\}$ and, by (1.1)′, this set contains $\bar{\Omega} \cap \{|z|, |w| \leqslant \delta\}$ for suitably small $\delta > 0$. □

PROOF OF PROPOSITION 4.1. Let us consider the cone

$$\bar{V} \equiv \{(z_0, w_0) | z_0 \neq 0 \textit{ is in a supersector of } \mathrm{Re}\,(w_0 z/z_0)^k + P_k(z)\} \cup \{(0, w_0), \textit{ with } \mathrm{Re}\, w_0^k \leqslant 0\}\,.$$

Set V for the part of $\bar{V}$ coming only from proper supersectors. Any conical neighbourhood of $\bar{V}$ contains the set (4.1) for suitably small $\varepsilon > 0$, so that $\bar{V}$ is actually the closure of V. For each complex line L through the origin of C^2 i.e. $L \in P_1(C)$, the components of $\bar{V} \cap L \setminus \{0\}$ are sectors of width $< \pi/k$ separated by sectors of width $> \pi/k$ unless L is $\{z = 0\}$, in this case we have equality. Consider the space $\bar{R}$ of those components with the quotient topology and set R for the part of $\bar{R}$ coming from V. Thus an element of R over $[a:1] \in P_1(C)$ is a supersector of the polynomial $\mathrm{Re}\,(az)^k + P_k(z)$, projection $\pi\colon R \to P_1(C)$. Now statement (2.1) says exactly that π gives R the structure of a Hausdorff Riemann surface over $P_1(C)$ and Lemma 2.3 says that R has no compact component. $\bar{R}$ is compact because $\bar{V}$ is closed in C^2.

Now we define $\partial R = \bar{R} \setminus R$ as boundary of R. Lemma 2.2 says that this is a correct boundary. With the notation of this lemma we can extend the co-ordinate patches at the boundary from A to N and obtain a bigger Riemann surface $\tilde{R} \subset \bar{R}$ which also has no compact component after suitable reduction. Let now $\mathfrak{L} \to P_1(C)$ be the holomorphic line bundle obtained by blowing up the origin of C^2. The pull-back $\pi^* \mathfrak{L} \to R$ is also a holomorphic line bundle: the fiber over a component of $V \cap L \setminus \{0\}$ is L. Note that, if we write $[\alpha:\beta]$ for an element of $P_1(C)$, then the complex structure of R is defined as the one for which α/β is a meromorphic function and $\pi^* \mathfrak{L}$ is exactly the line bundle associated to this function.

Now neither $P_k(z)$ (by (1.2)) nor $\operatorname{Re} w^k$ have degenerate supersectors. Therefore z and w are both nonzero on ∂R so that the transition function w/z of $\pi^* \mathfrak{L}$ extends to $\tilde{R}$ and we have a larger line bundle $(\pi^* \mathfrak{L})^\sim \to \tilde{R}$.

There is a natural embedding $\nu\colon V \to (\pi^* \mathfrak{L})$ which has (in the local product co-ordinates of $\pi^* \mathfrak{L}$) the form

$$(4.2) \qquad \begin{cases} (z, w) \to (w/z, z)\,, & \text{for } z \neq 0 \\ (z, w) \to (z/w, w)\,, & \text{for } w \neq 0. \end{cases}$$

So, after a further possible reduction of $\tilde{R}$, ν extends to a biholomorphic map $\tilde{\nu}$ to $(\pi^* \mathfrak{L})^\sim$ of a conical neighbourhood $\tilde{V}$ of $\bar{V}$.

$\tilde{\nu}$ is C-linear on the components of $V \cap L \setminus \{0\}$ for all $L \in P_1(C)$. Finally, since $\tilde{R}$ is open, the dual of $(\pi^* \mathfrak{L})^\sim$ is trivial bundle by Behnke-Stein theorem and has a never vanishing holomorphic section H. H is a function on $(\pi^* \mathfrak{L})^\sim$. The required function h is $H \circ \tilde{\nu}$. □

§ 5. – In this section we discuss the exceptional cases and emphasize the criticality of type 6: only at points of even type $\geqslant 6$ can weakly pseudoconvex domains have no outer tangent disc as in Kohn-Nirenberg's example and this rays condition is strictly more general than the sector condition.

EXAMPLE 1. Let

$$P_6(x, y) = y^2 \left(y - \tan\frac{\pi}{9}\, x\right)^2 \left(y + \tan\frac{\pi}{9}\, x\right)^2.$$

Then $P_6(x, y)$ is exceptional of rays kind. The supersector $[-\pi/9, \pi/9]$has width $2\pi/9 > \pi/6$. One sees easily that P_6 is not exceptional of harmonic kind, does not satisfy the rays condition and does not have grouped sectors.

The bounded domain $\{\operatorname{Re} w + |w|^2 + P_6(z) + \varepsilon|z|^8 < 0\}$, $1 \gg \varepsilon > 0$, $\varepsilon > 0$, has (0) as a peakpoint and e^w as a peak-function. So CR-functions do not extend.

The authors do not know whether 0 is a peak-point for $\{\operatorname{Re} w + |w|^2 + P_6(z) - |z|^8 + \varepsilon|z|^{10} < 0\}$, $\varepsilon > 0$ some constant.

EXAMPLE 2. Let

$$P_6(x, y) = y^2 \left(y - \tan\frac{\pi}{9}\, x\right)^2 \left(y + \tan\frac{\pi}{9}\, x\right)^2 + \varepsilon \left(y - \tan\frac{\pi}{9}\, x\right)^3 \left(y + \tan\frac{\pi}{9}\, x\right)^3$$

$$\text{for } \varepsilon > 0\,, \quad \varepsilon \ll 1\,.$$

Then P_6 satisfies the rays condition. Observe that if ε is small enough, it is not possible to add a term of the form $\operatorname{Re}\alpha z^6$ to make $P_6 + \operatorname{Re}\alpha z^6$ have a negative sector of width $\geqslant \pi/6$ because $P_6(\exp[i\theta] + P_6(\exp[i[\theta + \pi/6]]) \geqslant 0$. So for $k \geqslant 6$, the rays condition is strictly more general than the sector property.

EXAMPLE 3. Let

$$P_6(x, y) = y^2\left(y - \tan\frac{\pi}{12}\,x\right)^2\left(y + \tan\frac{\pi}{12}\,x\right)^2.$$

Then P_6 is exceptional of harmonic kind and (0) is a peak point for

$$\{\operatorname{Re} w + |w|^2 + P_6(x, y) + \varepsilon|z|^8 < 0\}$$

with peak function e^w.

EXAMPLE 4. Let

$$P_6(x, y) = y^2\left(y - \tan\frac{\pi}{12}\,x\right)^2\left(y + \tan\frac{\pi}{12}\,x\right)^2 + \varepsilon\left(y - \tan\frac{\pi}{12}\,x\right)^3\left(y + \tan\frac{\pi}{12}\,x\right)^3$$

Then (0) is in the envelope of holomorphy of $\{\operatorname{Re} w + |w|^2 + P_6(z) + \varepsilon|z|^8 < 0\}$ and so CR-functions extend to a full neighbourhood of (0). But P_6 is still exceptional of harmonic kind.

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