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Some applications of Cauchy-Fantappie forms to (local) problems on $\bar{\partial}_b$

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Introduction.

Integrals kernels are well known as efficient tools to extend $\bar{\partial}_b$-closed forms, given on the boundary of some domain in $\mathbf{C}^n$, to $\bar{\partial}$-closed forms (or, which is closely related, to solve $\bar{\partial}_b$, or to solve $\bar{\partial}$ and $\bar{\partial}$ with given support). It is not so well known that they can be used for the local version of this problem, for wider purposes than done in [8], pp. 89-90 (and, it seems to me, in an easier way). It is the aim of this paper to illustrate the possibilities which arise from a systematic use of vanishing Cauchy-Fantappie kernels, a fact already much used in [8], from which this paper is much inspired.

In Part 0, we recall basic facts about Cauchy-Fantappie forms ([8], [7]), in particular to set the notations. But this provides us with an opportunity to make some rather trivial remarks, to be used later. These remarks will reveal the principles of the proofs.

Part I deals with the systematic writing of a $\bar{\partial}_b$-closed $(0, q)$ form defined on some real hypersurface of $\mathbf{C}^n$ as the « jump » between two $\bar{\partial}$-closed forms. This is first done locally (Proposition 1) and then when the localization is done just in the first variable (Proposition 2). These first results are obtained without convexity hypothesis, and for $0 \leqslant q \leqslant n$. Proposition 2′ specializes the conclusion of Proposition 2 under pseudo-convexity hypothesis, and for $q \leqslant n-3$ gives extensions with vanishing properties. Although the results of Propositions 1 and 2 are, to a large extent, included in the results of [2] (cf. I.0, below) we believe that Proposition 2′ is new.

The techniques of Part I allow one to solve some $\bar{\partial}_b$ problems. In Part II, I did not try to give the most general results that one can get this

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way. Rather, I preferred to discuss just two cases, giving an illustration of the method before getting submerged by the notations and technical hypotheses.

PROPOSITION 3. *Let S_n be the unit sphere in $\mathbf{C}^n$.*

(I) *Let Σ be the open set in S_n defined by the condition $|z_1| < \frac{1}{2}$.*

Let g be a smooth $\bar{\partial}_b$-closed form defined on Σ.

If $1 \leqslant q \leqslant n-3$, there exists u a smooth $(0, q-1)$ form defined on Σ verifying $\bar{\partial}_b u = g$.

(II) *Let Σ' be the open set in S_n defined by the condition $|z_1| > \frac{1}{2}$.*

Let g be a smooth $\bar{\partial}_b$-closed form defined on Σ'. If $1 \leqslant q \leqslant n-2$, there exists u a smooth $(0, q-1)$ form on Σ' verifying: $\bar{\partial}_b u = g$.

REMARK. Of course in (I) and (II) the case $q = n-1$ is quite hopeless since we meet the Lewy non solvability phenomenon.

And for (I), it is not difficult to construct a smooth $\bar{\partial}_b$-closed $(0, n-2)$ form g such that the equation $\bar{\partial}_b u = g$ cannot be solved. Indeed, the function $1/z_2$ defined on $\Sigma \cap \{z_3 = \ldots = z_n = 0\}$ cannot be extended as a $\bar{\partial}_b$-closed function on Σ, since it would then extend as a holomorphic function on the unit ball in the space $\{z_1 = 0\}$. This, as is well known (see [10] in the case of the $\bar{\partial}$ operator), gives rise to a non solvable $\bar{\partial}_b$ problem.

The second topic discussed in II is treated much in the perspective of yielding local results. Proposition 4 gives local solutions of the equation $\bar{\partial}_b u = g$ under rather weak pseudo-convexity hypotheses. The precise statement, being slightly technical, is only given later. Kernels allow one to deduce local results from the global results of J. Kohn [11].

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Part 0. – Notations, basic facts.

0.1. – *Notations.* If Ω is a domain in $\mathbf{C}^n$ its boundary will be denoted by $b\Omega$. If $z \in \mathbf{C}^n$, $z_1, \ldots, z_n$ will always denote the coordinates of z.

Let Σ be a piece of a real C^1 hypersurface in $\mathbf{C}^n$, given, as usual, by a defining function ϱ $(\Sigma = \{\varrho = 0\}$, and the normal is oriented towards the region where $\varrho > 0)$.

A section $S(\zeta, z)$ of the Leray bundle is, for z varying in some open set which does not meet Σ and ζ varying in Σ, a n-tuple of continuously differentiable functions $S(\zeta, z) = \big(s_1(\zeta, z), \ldots, s_n(\zeta, z)\big)$, on which the fundamental hypothesis is:

$$\operatorname{Re} \sum_{j=1}^{n} (\zeta_j - z_j)\, s_j(\zeta, z) > 0 \, .$$

Associated with S is the kernel:

$$K_S = \sum_{j=1}^{n} (-1)^{j+1} \frac{s_j(\partial_{\bar\zeta} + \partial_{\bar z})s_1 \wedge \ldots \widehat{(\partial_{\bar\zeta} + \partial_{\bar z})s_j} \ldots \wedge (\partial_{\bar\zeta} + \partial_{\bar z})s_n \wedge d\zeta}{\left(\sum_{k=1}^{n} (\zeta_k - z_k)s_k\right)^n}\,, \qquad d\zeta = d\zeta_1 \wedge \ldots \wedge d\zeta n.$$

The kernel K_S is often decomposed into components ω_q, of degree $(n-q-1)$ in $\bar\zeta$ and q in $\bar z$, $0 \leqslant q \leqslant n-1$. If $B(\zeta, z) = (\bar\zeta_1 - \bar z_1, \ldots, \bar s_n - \bar z_n)$ the kernel obtained is the Bochner-Martinelli kernel.

2.2. – *Jump Formula.* If g is a given $(0, q)$ form on Σ, say continuous and compactly supported, the Bochner-Martinelli transform is:

$$g^{\pm}(z) = (-1)^q \frac{(n-1)!}{(2i\pi)^n} \int_{\zeta \in \Sigma} g(\zeta) \wedge K_B(\zeta, z) \begin{cases} g^+ \text{ if } \varrho(z) < 0 \\ g^- \text{ if } \varrho(z) > 0 \end{cases}.$$

(In case Σ is the boundary of some bounded set Ω, g^+ is the Bochner-Martinelli transform on Ω and g^- is this transform outside $\bar\Omega$.) Of course, just the component of K of degree $(n-q-1)$ in $\bar\zeta$ and q in $\bar z$ is relevant.

Now g appears to be, mod $\bar\partial\varrho$, the jump between g^- and g^+, cf. [8]. To be more precise: for instance, if Σ and g are C^∞ it has been proved in [7] that g^+ (resp. g^-) extends smoothly on $\{\varrho \geqslant 0\}$ (resp. $\{\varrho \leqslant 0\}$) and that on Σ $\big(ie\{\varrho = 0\}\big)$ one has $g = g^+ - g^-$ mod $\bar\partial\varrho$, which means that there exists a C^∞ $(0, q-1)$ form h on Σ such that $g = g^+ - g^- + (\bar\partial\varrho \wedge h)$.

REMARK. It is perfectly clear that this jump phenomenon is a very local question about singular integrals.

I mean the following: fix some point $\zeta_0 \in \Sigma$ and W a neighborhood of ζ_0 in $\mathbf{C}^n$. Consider a section $S(\zeta, z)$ defined on $\big(\Sigma \times (W - \Sigma)\big)$. Assume that S and B coincide on $(\Sigma \cap W) \times (W - \Sigma)$ and that S extends as a smooth function of (ζ, z) on $(\Sigma \times W)$ outside the diagonal $\{\zeta = z\}$. Then define $g^{\pm}$ using K_S instead of K_B. Near ζ_0, the jump formula and smoothness results still hold, and *one can drop the hypothesis that g is compactly supported if the relevant component of K_S is compactly supported in the ζ variable.*

2.3. – *Homotopy Formula.* For convenience let us write this formula as a Lemma.

LEMMA 1. *Let $S_1(\zeta, z)$ and $S_2(\zeta, z)$ be sections of the Leray bundle defined on the neighborhood of some point z and for $\zeta \in \Sigma$. Let $q \in \{0, 1, \dots, n\}$ and g be a $(0, q)$ form, continuously differentiable on Σ.*
Assume:

(*a*) *$S_1 = S_2$ for ζ outside some compact set $\subset \Sigma$, and in case g is not compactly supported, that K_{S_1} and K_{S_2} are compactly supported in the ζ variable*;

or

(*b*) *S_1 and S_2 are «holomorphic» functions of ζ (in the sense C.R., i.e. $\bar\partial_b S_1 = \bar\partial_b S_2 = 0$) outside some compact set $\subset \Sigma$ and $0 \leqslant q \leqslant n - 3$*;

or

(*c*) *S_1 and S_2 are holomorphic in z for ζ outside some compact set $\subset \Sigma$ and $1 \leqslant q \leqslant n$.*

Then:

$$\int_{\zeta\in\Sigma} g(\zeta)\wedge K_{S_1}(\zeta, z) = \int_{\zeta\in\Sigma} g(\zeta)\wedge K_{S_2}(\zeta, z) + \bar\partial_z(Tg) + T(\bar\partial_b g) .$$

Wher› setting $\tilde S(\zeta, z, \lambda) = (1-\lambda)S_1 + \lambda S_2$ and $\delta = \partial_{\bar\zeta} + \partial_{\bar z} + d_\lambda$,

$$\tilde K(\zeta, z, \lambda) = \sum_{j=1}^{n} (-1)^{j+1} \frac{\tilde s_j\, \delta\tilde s_1 \wedge \dots \widehat{\delta\tilde s_j} \dots \wedge \delta\tilde s_n}{\left[\sum_k \tilde s_k \cdot (\zeta_k - z_k)\right]^n} \wedge d\zeta$$

and,

$$Tk(z) = \int_{\substack{\zeta\in\Sigma\\ \lambda\in[01]}} k(\zeta)\wedge\tilde K(\zeta, z, \lambda) \qquad (k \text{ a } (0, r) \text{ form}).$$

PROOF. First we check that the terms which appear in the formulas to be established, make sense. The hypotheses ensure that the relevant component of $\tilde K$ is compactly supported in ζ, and in case (*b*) and (*c*) that the relevant components of K_{S_1} and K_{S_2} are compactly supported. This is due to degree considerations.

Indeed, for ζ outside some compact set, in case (*a*), $\tilde S$ does not depend on λ, so no $d\lambda$ is available for the integration. In case (*b*), no $d\bar\zeta$ is available from K, and in case (*c*) no $d\bar z$ is available. The result, then, follows from the computation of $\int_{b(V\times[0,1])} g\wedge\tilde{K}$, using Stokes formula, as in [8], where V is a relatively compact open set in Σ, chosen large enough. One eliminates, again by degree consideration, the boundary term $\int_{(bV)\times[0,1]} g\wedge\tilde{K}$; and this is the reason for the restriction $q\leqslant n-3$, in case (*b*), bV being of real dimension $(2n-2)=n+(n-2)$.

LEMMA 2. *Consider $S(\zeta, z)$ a section of the Leray bundle defined for z in some open set ω and $\zeta\in\Sigma$. Let g be a continuously differentiable $(0, q)$ form defined and $\bar\partial_b$-closed on the neighborhood of some compact set H in Σ.*
If

(α) *S is « holomorphic » in ζ ($\bar\partial_b S=0$) for $\zeta\notin H$, and $0\leqslant q\leqslant n-3$;*

or

(β) *S is holomorphic in z when $\zeta\notin H$, and $0\leqslant q\leqslant n$, and, if $q=0$, g is defined on Σ and compactly supported,*

then, for $z\in\omega$, set:

$$\tilde{g}(z)=\int_{\zeta\in\Sigma} g(\zeta)K_S(\zeta, z)\,.$$

This definition makes sense and $\tilde{g}$ is $\bar\partial$ closed.

PROOF. Let us first assume, if we are in case (β), that $q\neq 0$.

The hypotheses ensure that the relevant component of K_S vanishes when $\zeta\notin H$, hence on the set where g fails to be continuous or even defined. And, for this reason, we can assume that g is defined and continuously differentiable on Σ. Fix $z^0\in\omega$. For z near z^0 define $S_1(\zeta, z)$ by « freezing the coefficients » (a well known trick): $S_1(\zeta, z)=S(\zeta, z^0)$. Define $\tilde{g}_1$ using K_{S_1} instead of K_S. When applying Lemma 1, (*b* or *c*) observe that the term $T(\bar\partial_b g)$ vanishes. Indeed, by considerations on the degree in $d\bar\zeta$ (case α) or in $d\bar z$ (case β), we get that the relevant component of $\tilde{K}$ vanishes at the points where $\bar\partial_b g\neq 0$. So the homotopy formula reduces to: $\tilde{g}_1=\tilde{g}+\bar\partial_z(...)$. Since S_1 is holomorphic in z, $\tilde{g}_1=0$ if $q\geqslant 1$, and, if $q=0$, $\tilde{g}_1$ is a holomorphic function. In both cases one sees that $\bar\partial\tilde{g}=0$.

Now, to treat the case $q=0$ in (β), we modify slightly the technique for freezing the coefficients. Let χ be a smooth function which is identically 1 on H, and is compactly supported in Σ, and such that $\chi(\zeta)=0$ if $\bar\partial_b g(\zeta)\neq 0$. Then, set: $S_1(\zeta, z)=\chi(\zeta)S(\zeta, z^0)+\big(1-\chi(\zeta)\big)S(\zeta, z)$.

Using now case (*a*) in Lemma 1, the proof ends as above. Because no $d\lambda$ will appear, the component of $\tilde{K}$ which is to be used in the integral $T(\bar\partial_b g)$, vanishes at the points where $\bar\partial_b g$ is not 0.

Part I.

I.0. – Let us recall a result of [2].

(AH): « Let U be a pseudo convex set in $\mathbb{C}^n$, and Σ be a smooth real hypersurface which divides U into two regions U^+ and U^-. Let g be a smooth $\bar\partial_b$-closed $(0, q)$ form defined on Σ $(0 \leqslant q \leqslant n)$, then there exist g^+ and g^- $(0, q)$ forms defined respectively on U^+ and U^-, smooth up to Σ, $\bar\partial$-closed, and such that on Σ $g = g^+ - g^-$ ».

For the convenience of the reader we sketch the very simple proof given in [2]. First, one can extend g to a smooth form $\tilde g$ defined on U and such that $\bar\partial\tilde g$ vanishes at infinite order on Σ ([2], Lemma 2.2). This is a question about the Taylor expansion of $\tilde g$ along Σ. Extend g. Formally, one can set: $\tilde g = g + \alpha_1\varrho + \ldots + \alpha_k\varrho^k + \ldots$ (ϱ a defining function for Σ); the coefficients ($(0, q)$-forms) α_k are inductively selected (see also the proof of Theorem 2.3.2′ in [10]). Whitney's extension theorem is then used to produce $\tilde g$, which is a smooth form having the above expansion near Σ. Set,

$$\theta = \left|\begin{array}{ll} \bar\partial\tilde g & \text{on } U_+ \\ -\bar\partial\tilde g & \text{on } U_- \\ 0 & \text{on } \Sigma\,. \end{array}\right.$$

Then θ is a smooth $\bar\partial$-closed $(0, q+1)$-form on U. So there exists γ a smooth $(0, q)$-form defined on U such that $\bar\partial\gamma = 0$. Set

$$g^+ = \frac{\tilde g - \gamma}{2} \quad \text{and} \quad g^- = -\frac{\tilde g + \gamma}{2}\,.$$

We remark that Propositions 1 and 2 (but not 2′) below are consequences of (AH), at least in the smooth case. But the smoothness assumption can also be relaxed in the proof above, and the real analyticity in the conclusion of Proposition 2 can also be obtained.

Our motivation for presenting these Propositions in the following. First, we want to emphasize the method (as it should be clear from the title of this paper). It is an extension of techniques used in [1] and [8], and Propositions 1 and 2 are easy applications. Also, it should be useful to have formulas involving explicit integral kernels, for example to establish estimates in various norms.

I.1. – Let us begin with a very general remark. A « global version » of the Proposition is due to Aizenberg and Dautov [1], Th. 2.11.

PROPOSITION 1. *Let Σ be a $\mathcal{C}^1$ real hypersurface in some neighborhood of* 0 *in* $\mathbf{C}^n$, $0 \in \Sigma$. *And let* ϱ *be a defining function for* Σ, Σ *divides* $\mathbf{C}^n$, *near* 0, *in two regions*: Ω^+ *defined by* $\varrho < 0$, *and* Ω^- *defined by* $\varrho > 0$.

For every U_1 *neighborhood of* 0, *there exists* U_2 *a smaller neighborhood of* 0 *such that*:

For every $q \in \{0, 1, \dots, n\}$ *and every continuously differentiable* $\bar\partial_b$*-closed form* g *defined on the set* $\Sigma \cap U_1$, *there exist smooth* $\bar\partial$*-closed forms* g^+ *and* g^- (*given, mod* $\bar\partial\varrho$, *by explicit integral formulas*) *defined respectively on* $U_2 \cap \Omega^+$ *and* $U_2 \cap \Omega^-$, *extending continuously to* $\Sigma \cap U_2$, *such that on* $\Sigma \cap U_2$, $g = g^+ - g^-$. *If the data* (Σ *and* g) *are smooth,* $g^\pm$ *are obtained smooth up to* Σ.

PROOF. We can assume that U_1 is open and bounded. Let H be a compact neighborhood of 0 in Σ, included in U_1. There exist $\alpha > 0$ and U_2 a neighborhood of 0 such that for every $z \in U_2$ and $\zeta \in (\Sigma \cap U_1) - H$

$$\operatorname{Re} \sum_{j=1}^{n} (\zeta_j - z_j)\bar\zeta_j > \alpha > 0 .$$

And we take $U_2 \subset U_1$.

Let χ be a smooth function defined on $\Sigma \cap U_1$ such that: $0 \leqslant \chi \leqslant 1$, $\chi(\zeta) = 0$ if $\zeta \notin H$ and $\chi(\zeta) = 1$ if $\operatorname*{Inf}_{z \in U_2} \left(\operatorname{Re} \sum_{j=1}^{n} (\zeta_j - z_j)\bar\zeta_j\right) < \alpha/2$ so, in particular, if $\zeta \in U_2$). One patches the sections $\bar\zeta$ and B by setting: $S(\zeta, z) = \chi(\zeta) B(\zeta, z) + \big(1 - \chi(\zeta)\big)\bar\zeta$, for $\zeta \in \Sigma \cap U_1$ and $z \in U_2$.

Then, if $z \notin \Sigma$, $\operatorname{Re} \sum_{j=1}^{n} (\zeta_j - z_j) s_j(\zeta, z) > 0$, since $\sum_{j=1}^{n} (\zeta_j - z_j)(\zeta_j - \bar z_j) > 0$ $0 \leqslant \chi \leqslant 1$ and $1 - \chi(\zeta) = 0$ if $\operatorname{Re} \sum_{j=1}^{n} (\zeta_j - z_j)\bar\zeta_j \leqslant 0$. So, this provides us with S a section of the Leray bundle, defined for $\zeta \in \Sigma \cap U_1$ and $z \in U_2 - \Sigma$, such that:

$$\text{if} \quad \zeta \in U_2 , \quad S(\zeta, z) = B(\zeta, z) ,$$

and

$$\text{if} \quad \zeta \notin H , \quad S(\zeta, z) = \bar\zeta .$$

In order to treat simultaneously the case $q = 0$, we can modify g outside some neighborhood of H in order to make it to be compactly supported in U_1 (but no longer $\bar\partial_b$-closed outside this neighborhood of H).

Define now

$$g_0^\pm(z) = (-1)^q \frac{(n-1)!}{(2i\pi)^n} \int_{\zeta \in \Sigma \cap U_1} g(\zeta) \wedge K_S(\zeta, z) .$$

First we treat the smooth case. It follows from Lemma 2, condition β, that $g_0^{\pm}$ are $\bar\partial$-closed, and from the remark 0.2 that there exists h a smooth $(0, q-1)$ form defined on $\Sigma \cap U_2$ such that $g = g_0^+ - g_0^- + \bar\partial\varrho \wedge h$. Let still denote by h a smooth extension of h to $U_2 \cap \Omega^+$. Set $g^- = g_0^-$ and $g^+ = g_0^+ + \bar\partial(\varrho h)$ (if $q = 0$, read $g^+ = g_0^+$); $g^{\pm}$ have the desired properties.

If the data are only $\mathcal{C}^1$ then $g_0^{\pm}$ are continuous up to Σ, and we find h continuous. Then we extend continuously h as a smooth function on $U_2 \cap \Omega^+$, in such a way that the extension has a gradient whose norm is $o(1/\varrho)$ when approaching Σ (use some harmonic extension). Again $g^+ = g_0^+ + \bar\partial(\varrho h)$ has the desired properties.

I.2. – Let us now do the localization in just one variable. This gives:

PROPOSITION 2. *Let $\mathcal{O}$ be a bounded open set in $\mathbf{C}^n$, whose boundary is continuously differentiable. Let ω_1 and $\tilde\omega_1$ be open sets in $\mathbf{C}$ such that the closure of ω_1 is included in $\tilde\omega_1$. Set*

$$\Sigma = \{z \in b\mathcal{O}, \quad z_1 \in \omega_1\} \qquad (z = (z_1, \dots, z_n))$$
$$\tilde\Sigma = \{z \in b\mathcal{O}, \quad z_1 \in \tilde\omega_1\}$$
$$\Omega^+ = \{z \in \mathcal{O}, \quad z_1 \in \omega_1\}$$
$$\Omega^- = \{z \in \mathbf{C}^n - \overline{\mathcal{O}}, \ z_1 \in \omega_1\}.$$

Let g be a $(0, q)$ form defined on $\tilde\Sigma$, continuously differentiable and $\bar\partial_b$-closed.

If $0 \leqslant q \leqslant n$, there exist $\bar\partial$-closed forms g^+ and g^- defined respectively on Ω^+ and Ω^-, real analytic on $\Omega^{\pm}$ and having continuous extension to Σ satisfying $g = g^+ - g^-$ on Σ.

If $b\mathcal{O}$ and g are smooth ($\mathcal{C}^\infty$), then $g^{\pm}$ can be chosen smooth up to Σ.

The next Proposition gives useful additional information in the case $q \leqslant n-3$

PROPOSITION 2′ (Same notations as in Proposition 2). *Let us assume that $\mathcal{O}$ is strongly pseudo convex or smooth and weakly pseudo convex. Then, if $0 \leqslant q \leqslant n-3$, one can obtain in addition that g^- extends as a $\bar\partial$-closed form on $\mathbf{C}^n - \overline{\mathcal{O}}$, which vanishes for z_1 outside some arbitrarily chosen neighborhood of $\bar\omega_1$.*

(Of course g^- is no longer real analytic outside Ω^-, and there is no « jump relation » with g on $b\mathcal{O} - \Sigma$.)

REMARK. The real analyticity in Proposition 2 will be used later only for $(n-2)$ forms, to which Proposition 2′ does not apply.

PROOF OF PROPOSITION 2. Let χ be a C^∞ function defined on $\mathbb{C}$, $0 \leqslant \chi \leqslant 1$, equal to 1 on some neighborhood of $\bar{\omega}_1$ and 0 outside some larger neighborhood of $\bar{\omega}_1$, whose closure is assumed to be contained in $\tilde{\omega}_1$.

For $z \in \Omega^+ \cup \Omega^-$ and $\zeta \in \tilde{\Sigma}$ set $S(\zeta, z) = \chi(\zeta_1) B(\zeta, z) + (1 - \chi(\zeta_1))(1/(\zeta_1 - z_1), 0, 0, \dots, 0)$ where $B(\zeta, z) = (\bar{\zeta}_1 - \bar{z}_1, \dots, \bar{\zeta}_n - \bar{z}_n)$ is the Bochner-Martinelli section. If $q \geqslant 1$, set

$$g_0^\pm(z) = \frac{(-1)^q(n-1)!}{(2i\pi)^n} \int_{\tilde{\Sigma}} g(\zeta) \wedge K_S(\zeta, z) \begin{cases} g_0^+ & \text{if } z \in \Omega_+ \\ g_0^- & \text{if } z \in \Omega_- . \end{cases}$$

Observe that the section S is holomorphic in z for ζ outside some compact set in $\tilde{\Sigma}$, so the remarks in Part 0 apply and one gets that: $g_0^\pm$ are well defined and $\bar{\partial}$-closed (lemma 2) and have continuous, resp. smooth if the data are smooth, extensions to Σ such that $g_0^+ - g_0^- = g \pmod{\bar{\partial}\varrho}$ (0. 2 Remark, ϱ being a defining function for Σ, smooth if Σ is smooth). Also it is clear that $g_0^\pm$ are real analytic. Let h be a continuous (resp. smooth) $(0, q-1)$ form defined on Σ such that $g_0^+ - g_0^- = g + \bar{\partial}\varrho \wedge h$ on Σ.

Consider a real analytic continuous extension of h to Ω^+ which in case the data are smooth is smooth on $\bar{\Omega}^+ \cup \Sigma$, and in case the data are only C^1 has a gradient whose norm is $o(1/\varrho)$ when approaching Σ (again using harmonic extension).

Set $g^- = g_0^-$ and $g^+ = g_0^+ - \bar{\partial}(\varrho h)$. Then $g^\pm$ have the desired properties.

Again, if $q = 0$, for the definition of $g_0^\pm$ we just have to replace g by some function compactly supported and agreeing with g on a neighborhood of the closure of the set $\{\zeta, \chi(\zeta_1) \neq 0\}$; and $g_0^\pm = g^\pm$.

PROOF OF PROPOSITION 2′. Since we could as well start from an enlarged open set ω_1, it is enough to prove Proposition 2′ with the conclusion that the jump formula $g(z) = g^+(z) - g^-(z)$, and the real analyticity of $g^\pm$, hold just for z_1 in any fixed compact set in ω_1.

Then, start with the same $g^\pm$ as previously; we will modify somewhat and extend g^-. But we will now benefit from the holomorphy of the section $(1/(\zeta_1 - z_1), 0, \dots, 0)$ in the ζ variable.

(i) Preleminary step: There exists a section of the Leray bundle $L(\zeta, z)$ defined for $z \notin \bar{\mathcal{O}}$ and $\zeta \in b\mathcal{O}$, smooth in both variables and holomorphic in ζ for all fixed z. Indeed, for each $z^0 = (z_1^0, \dots, z_n^0) \notin \bar{\mathcal{O}}$ it is possible to find holomorphic functions $\varphi_1, \dots, \varphi_n$ on $\mathcal{O}$, smooth on $\bar{\mathcal{O}}$ such that $\sum_1^n (\zeta_j - z_j^0)\varphi_j(\zeta) = 1$ on $\mathcal{O}$ ([9]). For z near z^0 one still has

$$\operatorname{Re}\left(\sum_1^n (\zeta_j - z_j)\varphi_j(\zeta)\right) > 0 \quad \text{on } \bar{\mathcal{O}} .$$

Using a partition of unity in the z variable on constructs $(L_1, \ldots, L_n)$ by patching together the local sections $\varphi_1, \ldots, \varphi_n$.

(ii) Let H be any compact set included in ω_1. Let ψ be a smooth function on $\mathbf{C}$ which is identically 1 on H and is compactly supported in ω_1:

Set $S^{\#}(\zeta, z) = \psi(z_1) S(\zeta, z) + \big(1 - \psi(z_1)\big) L(\zeta, z)$, and set

$$g^-_{\#}(z) = \frac{(-1)^q (n-1)!}{(2i\pi)^n} \int_{\tilde{\Sigma}} g(\zeta) \wedge K_{S^{\#}}(\zeta, z) \qquad \text{for } z \in \Omega^-.$$

If $z_1 \in H$, $g^-_{\#}(z) = g^-(z)$. For $z_1 \in \omega_1 - H$, $g^-_{\#}$ is well defined and $\bar\partial$-closed, now due to condition α in Lemma 2. For z_1 outside some compact set in ω_1, $\psi(z_1) = 0$. So $S^{\#}$ is holomorphic in ζ and the component of $K_{S^{\#}}$ of degree $n-q-1$ in $\bar\zeta$ is 0. Hence $g^-_{\#}(z) = 0$. Extend now $g^-_{\#}$ on $\mathbf{C}^n - \mathcal{O}$ outside Ω^- by setting it to be 0. The form $g^-_{\#}$ is the desired modification of g^-.

REMARK. The condition $q \neq n-2$ is used for the homotopy formula (Lemmas 1 and 2); that is, it is used to get that $g^-_{\#}$ is $\bar\partial$-closed, and not for the support condition.

If $q = 0$, the conclusion is, of course, that $g^- = 0$.

Part II

In this part we use the jump formulas studied in Part I, to solve the equation $\bar\partial_b u = g$. Let us however begin with a standard reduction of the problem.

II.1. – Let Σ be a smooth real hypersurface in $\mathbf{C}^n$ and assume that $\Sigma = \bigcup_{j \in N} \Sigma_j$ where each Σ_j is an open subset of Σ and $\bar\Sigma_j \subset \Sigma_{j+1}$. For $q \in \{1, \ldots, n\}$, consider the two following conditions:

(A_q) For every smooth and $\bar\partial_b$-closed $(0, q)$ form g on Σ, there exists u a smooth $(0, q-1)$ form verifying $\bar\partial_b u = g$ on Σ.

(B_q) For every $j \in N$ there exists $k_j \geqslant j$ such that for every smooth and $\bar\partial_b$-closed $(0, q)$ form g on Σ_{k_j}, there exists u a smooth $(0, q-1)$ form verifying $\bar\partial_b u = g$ on Σ_j.

LEMMA 3. *If* $q > 1$, *then* $\big((B_q)$ *and* $(B_{q-1})\big)$ *implies* (A_q).

For $q = 1$: *If every smooth* $\bar\partial_b$*-closed* (*i.e. CR*) *function defined on a neighborhood of* $\bar\Sigma_j$ *in* Σ *is the limit on* $\bar\Sigma_j$ (*in the* C^∞ *topology*) *of a sequence of functions which are smooth and* $\bar\partial_b$*-closed on* Σ *then* (B_1) *implies* (A_1).

PROOF. Assume that (B_q) and (B_{q-1}) are true. Without loss of generality one can assume that $k_{j+1} > k_j$ for every j. Let g be a $(0, q)$ form, smooth and $\bar\partial_b$-closed, defined on Σ. For each j there exists u_j such that $\bar\partial_b u_j = g$ on $\Sigma_{k_{j+1}}$. We have $\bar\partial_b(u_{j+1} - u_j) = 0$ on $\Sigma_{k_{j+1}}$. So, there exists γ_j such that $\bar\partial_b \gamma_j = u_{j+1} - u_j$ on Σ_{j+1}; restrict γ_j to Σ_j and then extend it to a smooth $(0, q-1)$ form $\tilde\gamma_j$. Set $u^{\#}_{j+1} = u_{j+1} - \bar\partial_b \tilde\gamma_j$. Then $\bar\partial_b u_{j+1} = g$ on $\Sigma_{k_{j+2}}$ and $u^{\#}_{j+1} = u_j$ on Σ_j. And this is the way to transform the sequence u_j into a convergent (stationary on each compact) sequence.

If $q = 1$, set $u^{\#}_{j+1} = u_{j+1} + P_j$, where P_j is a $\bar\partial_b$-closed function on Σ which approximates $u_j - u_{j+1}$ on $\bar\Sigma_j$. This leads to a convergent sequence.

II.2. – PROOF OF PROPOSITION 3 (stated in the introduction). In view of II.1 in order to establish (I) it is enough to prove (I′).

(I′): *Let* $\varepsilon > 0$, *and* g *be a smooth* $\bar\partial_b$*-closed* $(0, q)$ *form defined on the open set in* S_n (*the unit sphere in* $\mathbf{C}^n$) *given by* $|z_1| < \frac{1}{2} - \varepsilon/4$. *Set* $\Sigma_1 = \{z \in S_n, |z_1| < \frac{1}{2} - \varepsilon\}$. *Then, if* $1 \leqslant q \leqslant n-3$, *there exists* u *a smooth form defined on* Σ_1 *such that* $\bar\partial_b u = g$.

PROOF. Proposition 2′ allows us to write $g = g^+ - g^-$ on Σ_1, where g^+ is a $\bar\partial$-closed form defined on the set $\{|z| < 1$ and $|z_1| < 1 - \varepsilon/2\}$ and smooth on the closure of this set, and g^- is a smooth $\bar\partial$-closed form defined on the set $\{|z| > 1\}$, having a smooth extension to the set $\{z \in S_n, |z_1| < 1 - \varepsilon/2\}$.

Consider $\mathcal{O}^+$ a smooth strictly convex domain contained in the unit ball of $\mathbf{C}^n$, included in the region $\{|z_1| < 1 - \varepsilon/2\}$ and such that $\Sigma_1 \subset b\mathcal{O}^+$, and $\mathcal{O}^-$ a smooth strictly convex domain containing the unit ball of $\mathbf{C}^n$ and such that $b\mathcal{O}^- \cap S_n = \bar\Sigma_1$. Solving (cf. [12]) the equations $\bar\partial_b u^+ = g^+$ and $\bar\partial_b u^- = g^-$ respectively on $b\mathcal{O}^+$ and $b\mathcal{O}^-$ one gets on Σ_1: $\bar\partial_b(u^+ - u^-) = g$. So, $u = u^+ - u^-$ is a solution.

In order to prove (II) it is enough to prove:

(II′): *Let* $\varepsilon > 0$, *and* g *be a smooth* $\bar\partial_b$*-closed* $(0, q)$ *form defined on the open set in* S_n *defined by* $|z_1| > \frac{1}{2} + \varepsilon/8$. *Set* $\Sigma_1' = \{z \in S_n, |z_1| > \frac{1}{2} + \varepsilon\}$. *Then, if* $1 \leqslant q \leqslant n-2$, *there exists* u *a smooth form defined on* Σ_1' *such that* $\bar\partial_b u = g$.

PROOF. If $q \leqslant n-3$ the proof is not different from the proof given above. So let us concentrate on the case $q = n-2$.

(*a*) According to Proposition 2 we can write on $S_n \cap \{|z_1| > \frac{1}{2} + \varepsilon/4\}$, $g = g^+ - g^-$, where g^+ is a smooth $\bar\partial_b$-closed form on $\{|z| < 1, |z_1| > \frac{1}{2} + \varepsilon/4\}$

extending smoothly to the closure of this set, and where g^- is a real analytic $\bar\partial_b$-closed form on $\Omega^- = \{z \in \mathbf{C}^n, |z| > 1 \text{ and } |z_1| > \frac{1}{2} + \varepsilon/4\}$ extending smoothly to the set $\{z \in S_n, |z_1| > \frac{1}{2} + \varepsilon/4\}$. The solution of $\bar\partial_b u^+ = g^+$ is similar to the one in the previous case (considering a strictly pseudo convex domain $\mathcal{O}^+$ contained in the set $\{|z_1| > \frac{1}{2} + \varepsilon/2\} \cap \{|z| < 1\}$ and such that $\Sigma_1' \subset b\mathcal{O}^+$).

The change is about g^-. Instead of considering the problem of solving $\bar\partial_b v = g^-$ on Σ_1', let us first solve the equation $\bar\partial_b v = g^-$ on the hypersurface $\Lambda = \{z \in \mathbf{C}^n, |z| > 1, |z_1| = \frac{1}{2} + \varepsilon/2\}$!

(*b*) Solution of $\bar\partial_b v = g^-$ on Λ:

Of course, the problem is completely different, since Λ is foliated by $(n-1)$ dimensional complex manifolds (z_1 fixed). Therefore it reduces to solving for each z_1 fixed $\left(|z_1| = \frac{1}{2} + \varepsilon/2\right)$, and with smoothness in the parameter z_1, the following problem:

« Let τ be inclusion of $\mathcal{O}' = \{z' \in \mathbf{C}^{n-1}, |z'| > r = \sqrt{1 - (\frac{1}{2} + \varepsilon/2)^2}\}$ in $\mathbf{C}^n$ given by $\tau(z') = (z_1, z')$. One tries to solve the equation $\bar\partial\alpha = \beta$ on $\mathcal{O}'$ where β is the pull back of g via τ (and $\bar\partial$ is in the sense of $\mathbf{C}^{n-1}$) ».

Observe that we are precisely in the wrong degree: the degree $(n-2)$ in $\mathbf{C}^{n-1}$ with a ball deleted. Consider the restriction of β to S' the sphere of radius r in $\mathbf{C}^{n-1}$. The claim is that one can solve the equation $\bar\partial a = \beta$ on S'. Since β is a $(0, n-2)$ form (again the wrong, critical, degree) the necessary and sufficient condition to solve this equation is condition (cf. [8]):

$$(C) \quad \int_{z' \in S'} \beta(z') \wedge H(z')\, dz_2' \wedge \ldots \wedge dz_n' = 0, \text{ for all polynomial } H.$$

In order to check that this condition is satisfied, for H fixed, define, for $\xi \in \mathbf{C}$ and $|\xi| \geq \frac{1}{2} + \varepsilon/2$:

$$\Phi(\xi) = \int_{z' \in S'} g^-(\xi, z') \wedge H(z')\, dz_2' \wedge \ldots \wedge dz_n'.$$

This is a smooth function of ξ, real analytic for $|\xi| > \frac{1}{2} + \varepsilon/2$, (this is the place where we use the real analyticity of g^-), and condition (C) to be proved is that $\Phi(z_1) = 0$.

If $|\xi| > 1$, $g^-(\xi, z')$ defines a $\bar\partial$-closed form on the whole space $\{\xi\} \times \mathbf{C}^{n-1}$, so, applying Stokes formula, one gets that $\Phi(\xi) = 0$. Using the real analyticity of Φ one concludes that $\Phi(z_1) = 0$ as desired.

So, it is possible to extend β as a continuous $\bar\partial$-closed form on $\mathbf{C}^{n-1}$, smooth except possibly on S' (extending a to the ball of radius r). This, of course allows one to solve the equation $\bar\partial\alpha = \beta$ (β extended) on $\mathbf{C}^{n-1}$. Smooth dependence on z_1 can be obtained either by using the solutions of $\bar\partial$ and $\bar\partial_b$ given by the Neumann operator, or the explicit solutions of Henkin.

REMARK. Having in mind generalizations it is worth pointing out that, according to Stokes formula, in condition (C) we can replace the integration on S' by the integration on, say, any sphere in $\mathbf{C}^{n-1}$, of radius large enough (larger than r). This makes the argument which followed more adaptable.

(c) It follows from (b) that there exists v a smooth $(0, n-3)$ form defined on a neighborhood of A such that: $\bar\partial v = g^-$ on A. We can then modify v to make it defined on the set $\{z \in \mathbf{C}^n, |z| > 1\}$ and identically 0 for $|z_1| \geqslant \frac{1}{2} + \varepsilon$.

Consider now $g^= = g^- - \bar\partial v$. We have the following properties:

$g^=(z) = g^-(z)$ if $|z_1| > \frac{1}{2} + \varepsilon$;

$g^=$ is a smooth $\bar\partial$-closed form on the set $\{|z| > 1, |z_1| > \frac{1}{2} + \varepsilon/4\}$;

$g^=(z) = 0$ if $|z_1| = \frac{1}{2} + \varepsilon/2$.

The last property is the new feature. Consider now a smooth pseudoconvex set $\mathcal{U}$ whose boundary consists of Σ_1', the set $\{|z_1| = \frac{1}{2} + \varepsilon/2, |z| < 1 + \delta\}$ ($\delta > 0$ and small), and a surface patching these two pieces, lying in the set $\{|z| > 1, \frac{1}{2} + \varepsilon/2 < |z_1| < \frac{1}{2} + \varepsilon\}$. The interesting fact is that $g^=$ extends trivially (by 0) as a smooth $\bar\partial_b$-closed form defined on the whole boundary of $\mathcal{U}$.

So (cf. [12], [13]), one can solve on $b\mathcal{U}$ the equation $\bar\partial_b u^- = g^=$ with smooth u. On Σ_1' one has $\bar\partial_b(u^+ - u^-) = g$

REMARK. Instead of considering the open sets defined respectively by the conditions $|z_1| < \frac{1}{2}$ or $|z_1| > \frac{1}{2}$, one could consider the open set in S_n defined by: $z_1 \in \omega_1$, where ω_1 is some open set in $\mathbf{C}$. In general, one gets solvability results for $(0, q)$ forms only for $1 \leqslant q \leqslant n-3$. If ω_1 is connected and not included in the unit disk, then one gets solvability results for $(0, q)$ forms for $1 \leqslant q \leqslant n-2$.

II.3.

II.3.1. – We consider Ω a bounded set in $\mathbf{C}^n$ whose boundary consists of the real hyperplane $\Pi = \{\mathrm{Re}\, z_1 = 0\}$ and Σ a C^∞ real hypersurface located in the half space $\mathrm{Re}\, z_1 > 0$.

Still unpleasant hypotheses remain to be formulated. The first one, which is really restrictive, is to guarantee the possibility of « bumping » Σ near Π (and just there). More precisely we assume the following *bumping property*:

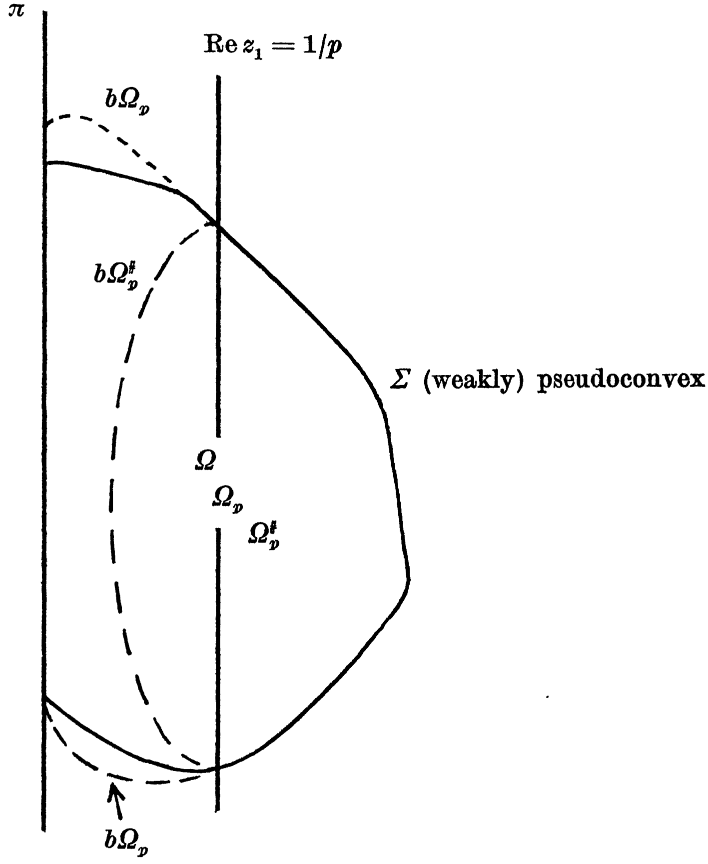
$\Omega = \bigcap_{p=1}^{\infty} \Omega_p$ where:

1) Ω_p is a C^∞, bounded, pseudo convex domain;

2) $\Omega_p \cap \{\operatorname{Re} z_1 > 1/p\} = \Omega \cap \{\operatorname{Re} z_1 > 1/p\}$;

$(*)_p$ 3) for any $\varepsilon > 0$, ε small enough (depending on p), $\Omega \cap \{\operatorname{Re} z_1 = \varepsilon\}$ is relatively compact in $\Omega_p \cap \{\operatorname{Re} z_1 = \varepsilon\}$.

The other hypothesis, that we will do, seems rather technical (and it is not clear that it is a restrictive one). We assume that, for every $p \in \mathbb{N}$, there exists a smooth pseudoconvex domain $\Omega_p^{\#}$ included in Ω such that $\Omega_p^{\#} \cap \{\operatorname{Re} z_1 > 1/p\} = \Omega \cap \{\operatorname{Re} z_1 > 1/p\}$, and the closure of $\Omega_p^{\#}$ is included in the set $\{\operatorname{Re} z_1 > 0\}$, see note below.



Observe that strict pseudo convexity of Σ near Π or geometric strict convexity is enough to guarantee the existence of such Ω_p and $\Omega_p^{\#}$. (For the construction of $\Omega_p^{\#}$ one can start with a domain bounded by Σ and some sphere and then apply the technique of [15] to smooth this domain.)

NOTE. J. E. Fornaess has shown me that this last hypothesis is always satisfied.

PROPOSITION 4. *Under the above hypotheses: Let g be any smooth $(0, q)$ form defined on Σ. If $\bar{\partial}_b g = 0$, and if $1 \leqslant q \leqslant n-3$, then there exists u a smooth $(0, q-1)$ form on Σ, such that on Σ, $\bar{\partial}_b u = g$.*

If the following hypothesis $(\mathcal{H})$ holds,

$(\mathcal{H})$ «*For $\varepsilon > 0$, ε small enough, the hypersurface $b\Omega \cap \{0 < \operatorname{Re} z_1 < \varepsilon\}$ is strictly pseudo convex, and the set $\bar{\Omega} \cap \{\operatorname{Re} z_1 = \varepsilon\}$ is polynomially convex*»,

then the above conclusion also holds for $q = n-2$ $(n \geqslant 3)$.

To simplify the next statement we will assume that the hypothesis $(\mathcal{H})$ holds.

COROLLARY (Assuming hypothesis $(\mathcal{H})$). (i) *If $1 \leqslant q \leqslant n-2$, every smooth and $\bar{\partial}_b$-closed $(0, q)$ form on Σ has a smooth $\bar{\partial}$-closed extension to the half space $\operatorname{Re} z_1 > 0$.*

(ii) *Let $1 \leqslant r \leqslant n-1$. If f is a smooth $(0, r)$ form defined on $\bar{\Omega} \cap \{\operatorname{Re} z_1 > 0\}$ vanishing on Σ, then there exists v a smooth $(0, r-1)$ form defined on $\bar{\Omega} \cap \{\operatorname{Re} z_1 > 0\}$, vanishing on Σ, such that $\bar{\partial} v = f$.*

PROOF OF THE COROLLARY. (i) Is immediate. Just extend u obtained in the proposition, and call $\tilde{u}$ the extension; $\bar{\partial}_b u = g$ means that for some k, $\bar{\partial}\tilde{u} = g + \bar{\partial}\varrho \wedge k$ on Σ. Then $\tilde{u} - \varrho k$ is the desired extension.

(ii) First solve $\bar{\partial}\alpha = f$ with α smooth on $\bar{\Omega} \cap \{\operatorname{Re} z_1 > 0\}$ ([11] and II.1, II.3.2). Then $\bar{\partial}\alpha = 0$ on Σ. Observe that α is of degree $(0, r-1)$.

According to (i), if $r > 1$ there exist $\alpha^{\#}$, a $\bar{\partial}$-closed extension of α/Σ to the set $\{\operatorname{Re} z_1 > 0\}$. Set $v = \alpha - \alpha^{\#}$.

If $r = 1$, $\alpha|\Sigma$ extends as a holomorphic function $\alpha^{\#}$ on Ω. Again set $v = \alpha - \alpha^{\#}$.

I have stated the corollary so that it may be compared with results by M. DERRIDJ [4].

REMARK. In $(\mathcal{H})$ the strict pseudo convexity hypothesis is done just to make sure that, for $\varepsilon > 0$, ε small enough, one can construct a smooth pseudo convex domain $\mathcal{U}$ (taking the place of the domain $\mathcal{U}$ in the proof of Proposition 3) such that $b\mathcal{U}$ consists of $b\Omega \cap \{\operatorname{Re} z_1 > 2\varepsilon\}$, a neighborhood of $\bar{\Omega} \cap \{\operatorname{Re} z_1 = \varepsilon\}$ in the set $\{\operatorname{Re} z_1 = \varepsilon\}$, and a surface (patching these two pieces) lying in $\bar{\Omega}^c$. This construction can be done, « bending » the surface $\{\operatorname{Re} z_1 = \varepsilon\}$ outside $\bar{\Omega}$ (using the polynomial convexity), « bumping » $b\Omega$, and smoothing the domain obtained this way (cf. [15]).

II.3.2. – Proof of Proposition 4, for $1 \leqslant q \leqslant n-3$.

Choose $\eta_p > 0$ such that condition $(*)_p$ holds for $\varepsilon \in (0, \eta_p)$, and such that $\Omega^{\#} \subset \{\operatorname{Re} z_1 > \eta_p\}$.

Claim. Fix $p \in \mathbb{N}$. Let $q \in \{1, \ldots, n-3\}$. If g is a smooth $(0, q)$ form defined on $\Sigma \cap \{\operatorname{Re} z_1 > \eta_p/8\}$ and $\bar{\partial}_b$-closed, then there exists u_p a smooth $(0, q-1)$ form on $\Sigma \cap \{\operatorname{Re} z_1 > 1/p\}$, such that on this set: $\bar{\partial}_b u_p = g$.

Proof of the claim. Proposition 2′ allows us to write $g = g^+ - g^-$ on $\Sigma \cap \{\operatorname{Re} z_1 > \eta_p/2\}$ where:

g^+ is a $\bar{\partial}$-closed form defined on $\Omega \cap \{\operatorname{Re} z_1 > \eta_p/2\}$, smooth up to $\Sigma \cap \{\operatorname{Re} z_1 > \eta_p/2\}$.

g^- is a smooth $\bar{\partial}$-closed form defined on $\bar{\Omega}^c$ (the complementary set of the closure of Ω), identically 0 on $\{\operatorname{Re} z_1 < \eta_p/4\}$, and having a smooth extension to $\Sigma \cap \{\operatorname{Re} z_1 > \eta_p/2\}$.

Since Proposition 2′ was stated for smooth domains Ω, to get g^+ and g^- one can, at this step, use instead of Ω the domain Ω_m for large m $(1/m < \eta_p/8)$.

Now, observe that g^- defines a smooth $\bar{\partial}$-closed form on $b\Omega_p$ (set it to be 0 if $\operatorname{Re} z_1 \leqslant 0$): the only points where this is not immediate are the points in $b\Omega_p \cap \Sigma$, but these are located in $\Sigma \cap \{\operatorname{Re} z_1 \geqslant \eta_p\}$ (and this is the point where I could not avoid the use of the bumping property). Therefore there exists u^-, a smooth $(0, q-1)$ form defined on $b\Omega_p$, such that $\partial_b u^- = g^-$ ([12], [13]). According to [11] one can solve the equation $\bar{\partial} u^+ = g^+$ on $\bar{\Omega}_p^{\#}$ with smooth u^+. So, ending the proof of the claim, we obtain $\bar{\partial}_b(u^+ - u^-) = g$ on $\Sigma \cap \{\operatorname{Re} z_1 > 1/p\}$.

The Proposition follows immediately from the claim, according to II.1, for $2 \leqslant q \leqslant n-3$. To get the result for $q = 1$ $(n \geqslant 4)$, it is enough to prove that: « Let $\varepsilon > 0$, and Φ a function holomorphic on $\Omega \cap \{\operatorname{Re} z_1 > \varepsilon\}$ which extends smoothly on $\bar{\Omega} \cap \{\operatorname{Re} z_1 > \varepsilon\}$. Then there exists a sequence of functions Φ_j holomorphic on Ω, smooth on $\bar{\Omega}$, such that Φ_j tends to Φ in the C^∞ topology, on $\overline{\Omega \cap \{\operatorname{Re} z_1 \geqslant 2\varepsilon\}}$ ». Set $Q(z) = Ae^{Bz}$; the constants A and B being chosen so that: $|Q| < \frac{1}{4}$ on $\{\operatorname{Re} z_1 \leqslant \varepsilon\}$ and $|Q| > 1$ on $\{\operatorname{Re} z_1 \geqslant 2\varepsilon\}$. Let χ be a smooth function on $\bar{\Omega}$ such that $\chi(z) = 1$ if $|Q(z)| > \frac{3}{4}$, and $\chi(z) = 0$ if $|Q(z)| \leqslant \frac{1}{2}$.

The approximations Φ_j are determined by setting $\Phi_j = \big((Q^j \Phi\chi) - \Psi_j\big)/Q^j$; Ψ_j has to satisfy $\bar\partial\Psi_j = Q^j\Phi\,\bar\partial\chi$ (and in order to solve this last equation one may use the solution of $\bar\partial$ in Ω_m for $1/m < \varepsilon$).

This ends the proof of Proposition 4 for $1 \leqslant q \leqslant n-3$. And one can remark that it is mainly a repetition of the proof of Proposition 2, with the trouble of heavier notations. It was however written to make sure that, using the results of [11] one gets results about weakly pseudo convex domains. I think (hope) therefore that it is not necessary to give full details of the proof for $q = n-2$.

II.3.3. – *Indications on the proof of Proposition* 4 *for* $q = n-2$ $(n \geqslant 3)$.

In view of II.3.2 it is quite clear how to transpose the proof of II.2. But one point at least deserves some comment.

Given $\varepsilon > 0$ such that $\bar\Omega \cap \{\operatorname{Re} z_1 = \varepsilon\}$ is polynomially convex, and g^- a smooth $\bar\partial$-closed $(0, n-2)$ form defined on the intersection of $\bar\Omega^c$ and of the set $\{\operatorname{Re} z_1 > \varepsilon/2\}$, one wants to solve $\bar\partial_b v = g^-$ on $\Lambda = \{z \in \mathbf{C}^n,\ \operatorname{Re} z_1 = \varepsilon,\ z \notin \tilde\Omega\}$, where $\tilde\Omega$ is a «small» open neighborhood of $\bar\Omega \cap \{\operatorname{Re} z_1 = \varepsilon\}$.

For each $\xi = \varepsilon + it$ $(t \in \mathbf{R})$ we can select Ω'_ξ a (possibly void) smooth strictly pseudo convex domain in $\mathbf{C}^{n-1}$, whose closure is polynomially convex, and such that $\bar\Omega \cap \{z_1 = \xi\} \subset \{\xi\} \times \Omega'_\xi \subset\subset \tilde\Omega$.

(i) For each ξ we can solve, as in II.2, an equation $\bar\partial\alpha = \beta$ in $(n-1)$ variables, namely on $\mathbf{C}^{n-1} - \bar\Omega'_\xi$, where β is the $\bar\partial$-closed $(n-2)$ form obtained by pulling back g^- on $\{z_1 = \xi\}$. Again, if Ω'_ξ is not void, and using Oka-Weil approximation theorem, the condition to be checked is that for every polynomial H, in $(n-1)$ variables:

$$\int_{b\Omega'_\xi} \beta(z') \wedge H(z')\, dz'_2 \wedge \ldots \wedge dz'_n = 0\,.$$

As already mentioned, according to Stokes formula, this condition is equivalent to the following:

$$\int_{|z'|=R} \beta(z') \wedge H(z')\, dz'_2 \wedge \ldots \wedge dz'_n = 0\,,$$

R being large enough so that Ω is included in the ball of radius R in $\mathbf{C}^n$.

Setting now, for $\eta = \varepsilon + it$ $(t \in \mathbf{R})$:

$$\Phi(\eta) = \int_{\substack{|z'|=R \\ z' \in \mathbf{C}^{n-1}}} g^-(\eta, z') \wedge H(z')\, dz'_2 \wedge \ldots \wedge dz'_n\,,$$

one ends the proof exactly as in II.2, observing that $\Phi(\eta) = 0$ for large t (and so for every t by analyticity), and using the results of [8] or [13] to solve $\bar{\partial}_b$.

(ii) One must now proceed in order to obtain solutions (as in (i)) depending smoothly on ξ. This is first done locally observing that for $\xi^{\#}$ near ξ one still has $\bar{\Omega} \cap \{z_1 = \xi^{\#}\} \subset \{\xi^{\#}\} \times \Omega'_{\xi} \subset \tilde{\Omega}$ so, one can work locally with a fixed domain Ω'_{ξ} and this makes easier to get smoothness (from [8] or [13]).

A global smooth solution is finally obtained, using a partition of unity in the ξ variable.

REMARKS. In view of [14], smoothness assumptions, in Proposition 4, seem hard to be discussed.

The case $1 \leqslant q \leqslant n - 3$, in Propositions 3 and 4, could be treated in the same way as the case $q = n - 2$. But the proof which was given seemed much simpler.

I wish to mention the paper [16] where a different approach is given for the local use of kernels, with different applications.

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