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Oblique derivative problems and invariant measures

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Oblique Derivative Problems and Invariant Measures (*).

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0. – Introduction.

The present paper is devoted to the study of some second-order oblique derivative problems for an operator A not in the divergence form and for a boundary operator B with Hölder continuous coefficients.

One major purpose is to describe the limiting behaviour of the solutions u_λ of the unilateral problem:

$$\begin{cases} \operatorname{Max}\,[u_\lambda - \psi;\ Au_\lambda + \lambda u_\lambda - f] = 0 & \text{in } \Omega \\ Bu_\lambda = 0 \quad \text{on } \Gamma\,, \end{cases}$$

as the positive parameter λ tends to zero. This is motivated by the control theory of stochastic processes. The above boundary value problem actually comprises the Bellman conditions for the optimal stopping of a diffusion process in Ω reflected at the boundary (see [3]). The case $\lambda > 0$ corresponds to discounted cost functionals while $\lambda = 0$ is related to long run average costs (see [19] and [6] for the case of finite Markov chains).

This problem has been studied by A. Bensoussan and J. L. Lions (see [4]) in the case of regular coefficients. Their main result is that the asymptotic behaviour of u_λ, as λ tends to zero, depends on the sign of the average value of f with respect to the measure $m\,dx$, m being the solution of the problem

$$\begin{cases} A^*m = 0 & \text{in } \Omega \\ B^*m = 0 & \text{on } \Gamma,\ m > 0\,, \quad \displaystyle\int_\Omega m\,dx = 1\,, \quad m \in C^2(\bar{\Omega})\,. \end{cases}$$

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In the general framework considered in this paper the main difficulties arise from the non-divergence structure of A. Actually, the fairly low regularity of the coefficients does not allow a Fredholm alternative approach by integration by parts. Nevertheless, using the Green's function $G(x, y, t)$ for the initial boundary value problem associated with A and B as constructed in ([13]), we prove the existence of a unique positive function $m \in L_\infty(\Omega)$ such that

$$m = \lambda J_\lambda^* m, \qquad \int_\Omega m\, dx = 1 .$$

Here, J_λ is the resolvant of the stationary problem

$$u \in W_p^2(\Omega), \quad Au + \lambda u = f \quad \text{in } \Omega, \qquad Bu = 0 \quad \text{on } \Gamma,$$

and J_λ^* its adjoint operator. Moreover, the identity

$$\int_\Omega \int_\Omega G(x, y, t) f(y)\, m(x)\, dx\, dy = \int_\Omega f(x)\, m(x)\, dx$$

holds for every $f \in L_p$; $1 \leqslant p \leqslant +\infty$.

The probability measure $d\mu = 1/|\Omega| m\, dx$ can be interpreted then as the invariant measure for a Feller process defined by the transition functions

$$P(x, t, E) = \int_E G(x, y, t)\, dy, \quad E \text{ Borel subset of } \Omega.$$

This, together with a crucial ergodic property proved in Theorem 3.1, is the key to prove that if the condition $\int_\Omega f d\mu > 0$ is satisfied, then the W_p^2 norms of u_λ are uniformly bounded with respect to λ and that $u = \lim u_\lambda$ in the weak topology of W_p^2 is the unique solution of

$$\operatorname{Max}[u - \psi; Au - f] = 0 \quad \text{in } \Omega, \qquad Bu = 0 \quad \text{on } \partial\Omega .$$

We send to Theorem 4.2 for this result and for the limiting behaviour in the case $\int_\Omega f\, d\mu \leqslant 0$.

To conclude this introduction we would like to point out that the analytical approach adopted in this paper to the stochastic control problem mentioned above yields a complete description of the asymptotic behaviour and of compatibility conditions for the solvability of the limit problem without the restrictions on the regularity of the data required by probabilistic methods (see [19]).

Let us also mention that Theorem 2.1 in Section 2 generalizes various known existence and uniqueness results (see for example [8], [23]) for the unconstrained problem

$$\begin{aligned} Au_\lambda + \lambda u_\lambda &= f \quad \text{in } \Omega\,, \\ Bu_\lambda &= 0 \quad \text{on } \Gamma\,. \end{aligned}$$

We finally mention that extensions of the results of this paper to integro-differential operators associated to diffusion processes with interior jumps are considered in a forthcoming paper by M. G. Garroni and J. L. Menaldi [12].

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1. – Basic assumptions and preliminary results.

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geqslant 2$, with boundary Γ of class C^2. We shall denote by Q_T the cylinder $\Omega \times]0, T[$, $0 < T < +\infty$, and by $\Sigma_T = \Gamma \times [0, T]$ its lateral boundary. Consider the operators

$$A = -\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{N} a_i(x)\frac{\partial}{\partial x_i} \tag{1.1}$$

$$B = \sum_{i=1}^{N} b_i(x)\frac{\partial}{\partial x_i}\,, \tag{1.2}$$

whose coefficients are assumed to satisfy

$$\begin{cases} a_{ij},\ a_i \in C^{0,\alpha}(\bar{\Omega}) \quad \text{for some } 0 < \alpha < 1 \\ \displaystyle\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geqslant \mu|\xi|^2\,, \qquad \mu > 0,\ \forall x \in \Omega,\ \xi \in \mathbb{R}^N \end{cases} \tag{1.3}$$

$$\begin{cases} b_i \in C^{0,\alpha}(\Gamma) \\ \left|\displaystyle\sum_{i=1}^{N} b_i(x)\nu_i(x)\right| \geqslant \beta\,, \qquad \beta > 0,\ \forall x \in \Gamma\,. \end{cases} \tag{1.4}$$

Here $\nu = (\nu_1, \dots, \nu_N)$ is the outward normal vector to Γ.

We shall denote by W_p^s, $W_p^{s,r}$ the usual Sobolev spaces for elliptic and parabolic problems with possibly non-integer s, r and by $\|\cdot\|_{s,p}$, $\|\cdot\|_{s,r,p}$ the corresponding norms (see [1]) and by $\|\cdot\|_p$ the usual $L_p(\Omega)$ norm.

We shall always assume

$$1 < p < \frac{1}{1-\alpha}. \tag{1.5}$$

This condition guarantees that $Bu \in W_p^{1-1/p}(\Gamma)$ for any $u \in W_p^2(\Omega)$.

We shall make use of regularized versions of operators A, B.

These are defined by

$$A^n = -\sum_{i,j=1}^{N} a_{ij}^n(x)\frac{\partial^2}{\partial x_i\,\partial x_j} + \sum_{i=1}^{N} a_i(x)\frac{\partial}{\partial x_i} \tag{1.6}$$

$$B^n = \sum_{i=1}^{N} b_i^n(x)\frac{\partial}{\partial x_i}, \tag{1.7}$$

where the coefficients a_{ij}^n, b_j^n are chosen in such a way that

$$\begin{cases} a_{ij}^n \in C^\infty(\bar\Omega), \quad a_{ij}^n \to a_{ij} \quad \text{in } C^{0,\alpha}(\bar\Omega) \quad \text{as } n \to +\infty, \\ \sum_{i,j=1}^{N} a_{ij}^n(x)\xi_i\xi_j \geq \mu|\xi|^2, \quad \mu > 0, \ \forall x \in \Omega, \ \xi \in \mathbf{R}^N, \end{cases} \tag{1.8}$$

$$\begin{cases} b_i^n \in C^\infty(\Gamma), \quad b_i^n \to b_i \quad \text{in } C^{0,\alpha}(\Gamma) \quad \text{as } n \to +\infty, \\ \left|\sum_{i=1}^{N} b_i^n(x)\nu_i(x)\right| \geq \beta > 0, \quad \forall x \in \Gamma, \end{cases} \tag{1.9}$$

with constants μ, β as in (1.3), (1.4). In this setting A^n can also be written in divergence form, namely

$$A^n = -\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(a_{ij}^n(x)\frac{\partial}{\partial x_j}\right) + \sum_{i=1}^{N}\left(a_i(x) + \sum_{k=1}^{N}\frac{\partial a_{ik}^n}{\partial x_k}(x)\right)\frac{\partial}{\partial x_i}. \tag{1.10}$$

The symbol $|E|$ stands for the Lebesgue measure of the subset $E \subseteq \Omega$, while χ_E denotes the characteristic function of E.

All equalities and inequalities between L_p functions will be meant to hold almost everywhere. Different constants occurring in various estimates will be denoted by the same letter C.

We are interested in the following oblique derivative problems

$$\text{(I)} \qquad \begin{cases} u \in W^2_p(\Omega),\ Au + \lambda u = f & \text{in } \Omega \\ Bu = 0 & \text{on } \Gamma \end{cases}$$

and

$$\text{(II)} \qquad \begin{cases} u \in W^2_p(\Omega),\ \operatorname{Max}\,[u - \psi;\ Au + \lambda u - f] = 0 & \text{in } \Omega\,, \\ Bu = 0 & \text{on } \Gamma\,. \end{cases}$$

Here, $\lambda \geq 0$ is a constant, f, ψ are given functions.

Our approach to problems (I), (II) relies on the use of the Green's function $G = G(x, y, t)$ for the parabolic initial-boundary value problem associated with A, B. The following results is contained, in a more general setting in Garroni-Solonnikov [13]:

THEOREM 1.1. *Under the assumptions* (1.3), (1.4), *there exists a function* $G(x, y, t)$ *such that*

$$\text{(1.11)} \qquad \begin{cases} \dfrac{\partial G}{\partial t} + A_x G = \delta_{x-y}\delta_t & \text{in } Q_T \\ G(x, y, 0) = 0 & \text{in } \Omega \times \Omega \\ B_x G = 0 & \text{on } \Sigma_T \end{cases}$$

and

$$\text{(1.12)} \qquad |G(x, y, t)| \leq C t^{-N/2} \exp\left[-C \frac{|x-y|^2}{t}\right],$$

$$\text{(1.13)} \qquad \left|\frac{\partial}{\partial x_i} G(x, y, t)\right| \leq C t^{-(N+1)/2} \exp\left[-C \frac{|x-y|^2}{t}\right],$$

$$\text{(1.14)} \qquad \left|\frac{\partial}{\partial t^s} \frac{\partial}{\partial x^r} G(x, y, t)\right| \leq C t^{-(N+1+\alpha)/2} \left(\varrho^{\alpha-1}(x) \vee t^{(\alpha-1)/2}\right) \exp\left[-C \frac{|x-y|^2}{t}\right],$$

$$\text{(1.15)} \qquad \left|\frac{\partial^2 G}{\partial x_i \partial x_j}(x, y, t) - \frac{\partial^2 G}{\partial x_i\, \partial x_j}(z, y, t)\right| \leq C|x-z|^\alpha t^{-(N+1+\alpha)/2} \left(\varrho^{-1}(x) \vee t^{-\frac{1}{2}}\right) \cdot \exp\left[-C \frac{|x-y|^2}{t}\right],$$

$$\text{(1.16)} \qquad \left|\frac{\partial G}{\partial t}(x, y, t) - \frac{\partial G}{\partial t'}(x, y, t')\right| \leq C|t-t'|^{\alpha/2} t^{-(N+1+\alpha)/2} \left(\varrho^{-1}(x) \vee (t')^{-\frac{1}{2}}\right) \cdot \exp\left[-C \frac{|x-y|^2}{t}\right].$$

Here above s, r are non negative integers such that $2s + r = 2$, $\varrho(x)$ denotes the distance of x from Γ, $a \vee b$ stands for the greatest between the numbers a, b and $t > t' > 0$.

Let us consider the parabolic mixed problem

$$\text{(1.17)} \qquad \begin{cases} \dfrac{\partial v}{\partial t} + Av = f & \text{in } Q_T \\ v(\cdot, 0) = \varphi & \text{in } \Omega \\ Bv = 0 & \text{on } \Sigma_T, \end{cases}$$

with $\varphi \in W_p^{2-2/p}(\Omega)$, $f \in L_p(Q_T)$ satisfying the compatibility conditions

$$\text{(1.18)} \quad \begin{cases} B\varphi = 0 & \text{on } \Gamma \ (p > 3) \\ F = \int\limits_0^T dt \int\limits_\Gamma d\sigma_x \int\limits_\Omega |B\varphi(y)|^3 \dfrac{dy}{(|x-y|^2 + t)^{(N+2)/2}} < +\infty, & (p = 3). \end{cases}$$

As a consequence of Theorem 1.1 and a result of Solonnikov ([21]) the following holds:

THEOREM 1.2. *Let us assume* (1.3), (1.4), (1.5). *Then, for any* $T > 0$, $f \in L_p(Q_T)$, $\varphi \in W_p^{2-2/p}(\Omega_T)$ *satysfying* (1.18), *there exists a unique solution* $v \in W_p^{2,1}(Q_T)$ *of* (1.17) *given by*

$$\text{(1.19)} \qquad v(x, t) = \int\limits_0^t \int\limits_\Omega G(x, y, t-\tau) f(y, \tau)\, dy\, d\tau + \int\limits_\Omega G(x, y, t)\varphi(y)\, dy .$$

Moreover,

$$\text{(1.20)} \qquad \|v\|_{2,1,p} \leq C[\|f\|_{L_p(Q_T)} + \|\varphi\|_{2-2/p,p} + F],$$

for some constant $C = C(T)$ *with polynomial growth in* T.

The next lemma exploits some properties of G that will be useful later.

LEMMA 1.1. *The Green's function* G *satisfies*

$$\text{(1.21)} \qquad G(x, y, t) \geq 0 ,$$

(1.22) for every ball $B \subset \Omega$ and $t > 0$ there exists $\gamma > 0$ such that $G(x, y, t) \geq \gamma$ for every $x \in \bar{\Omega}$, $y \in B$.

$$\text{(1.23)} \qquad \int\limits_\Omega G(x, y, t)\, dy \equiv 1 .$$

PROOF. By construction (see [13], Part 2) G is the sum $G_0 + G_1$, where

$$G_0(x, y, t) \simeq Ct^{-N/2} \exp\left[-C\frac{|x-y|^2}{t}\right], \quad \text{locally},$$
$$G_0(x, y, t) > 0$$
$$G_1(x, y, t) \simeq Ct^{-(N-\alpha)/2} \exp\left[-C\frac{|x-y|^2}{t}\right].$$

Hence,

$$G(x, y, t) \to +\infty \text{ if } t \to 0, \quad \text{with } |x-y|^2 \leqslant t. \tag{1.24}$$

Let us fix an arbitrary $y \in \Omega$. From (1.24) it follows that, for any sufficiently small positive ϱ,

$$\begin{cases} G(x, y, t) > 0 \\ \text{for} \quad (x, t) \in C_\varrho(y) = \{(x, t) \in Q_T\colon |x-y|^2 < t,\ 0 < t < \varrho^2\}. \end{cases} \tag{1.25}$$

Also,

$$\begin{cases} \dfrac{\partial G}{\partial t} + A_x G = 0 & \text{in } Q_T - C_\varrho(y) \\ G(x, y, 0) = 0 & \text{in } \Omega \times \Omega \\ B_x G = 0 & \text{on } \Sigma_T. \end{cases} \tag{1.26}$$

By the estimates (1.15), (1.16), G belongs to $C^{2+\alpha,(2+\alpha)/2}(Q_T - C_\varrho(y))$ and is $C^{1+\alpha,(1+\alpha)/2}$ up to the lateral boundary of $Q_T - C_\varrho(y)$.

Let us assume that $m = \inf\{G(x, y, t)\colon (x, t) \in \overline{Q_T - C_\varrho(y)}\}$ is attained at some point $(\bar{x}, \bar{t}) \in Q_T - C_\varrho(y)$.

Then the parabolic maximum principle (see [18]) yields $G \equiv m$ on $Q_{\bar{t}} - C_\varrho(y)$ and, in particular, on the lateral boundary of $C_\varrho(y)$.

Since ϱ was arbitrary, this contradicts (1.24).

On the other hand, m cannot be attained on Σ_T since this would contradict the oblique derivative conditon in (1.26). Therefore, m is attained at some point $(\bar{x}, 0)$. The initial condition in (1.26) then implies

$$G(x, y, t) \geqslant G(\bar{x}, y, 0) = 0 \quad \text{in } Q_T - C_\varrho(y).$$

This, together with (1.25), proves that (1.21) holds true.

Let us choose now $\tilde{t}$ such that $0 < \tilde{t} < \varrho^2$. The same argument as above shows that the infimum $\tilde{m}$ of G on $Q_T - Q_{\tilde{t}} - C_\varrho(y)$ is attained at some point $(\tilde{x}, y, \tilde{t})$. If it were $G(\tilde{x}, y, \tilde{t}) = 0$ then, as a consequence of the maximum principle when applied to $Q_T - C_\varrho(y)$, it would be

$$G(x, y, t) = 0 \quad \text{in } Q_{\tilde{t}} - C_\varrho(y) .$$

This is a contradiction, since G is strictly positive on $\partial C_\varrho(y)$, (see (1.25)).

Therefore, $\tilde{m} > 0$ and (1.22) is proved.

To prove (1.23), it is enough to combine the representation formula (1.19) with the observation that the solution of problem (1.17) with $f \equiv 0$ and $\varphi \equiv 1$ is $v \equiv 1$. □

2. - The case $\lambda > 0$.

In this section we shall deal with the case $\lambda > 0$ and prove existence, uniqueness and W_p^2 estimates for the solution of (I) and (II).

For more regular coefficients (that is a_{ij}, b_i Lipschitz continuous) problem (I) has been extensively studied (see, for example [2]). In the non-variational case (I) has been studied under various type of assumptions by C. Miranda [17], M. Chicco [8] and G. M. Troianiello [23].

The following theorem is related to the results mentioned above. Let us point out, however, that the boundary coefficients b_i are not required to be Lipschitz continuous and that exponents $1 < p < 2$ are allowed.

THEOREM 2.1. *Let us assume* (1.3), (1.4), (1.5). *Then, for any* $\lambda > 0$, $f \in L_p(\Omega)$, *problem* (I) *has a unique solution given by*

$$u_\lambda(x) = \lambda \int_0^\infty \exp[-\lambda t] \left[\int_0^t \int_\Omega G(x, y, t - \tau) f(y) \, dy \, d\tau \right] dt . \tag{2.1}$$

Moreover, the estimates

$$\|u_\lambda\|_p \leqslant \frac{C}{\lambda} \|f\|_p \tag{2.2}$$

$$\|u_\lambda\|_{2,p} \leqslant C[\|f\|_p + \|u_\lambda\|_p] \tag{2.3}$$

hold, where C *denotes different positive constants depending on* p, μ, β *and the Hölder norms of the coefficients, but independent of* $\lambda > 0$.

PROOF. Let $v = v(x, t)$ be the solution of problem (1.17) with $\varphi = 0$. Then (2.1) becomes

$$(2.4)\qquad u_\lambda(x) = \lambda\int_0^{+\infty}\exp[-\lambda t]\,v(x, t)\,dt\,.$$

Hence,

$$(2.5)\qquad \int_\Omega |u_\lambda(x)|^p\,dx \leqslant C(\lambda)\int_0^{+\infty}\int_\Omega \exp[-\lambda t]|\,v(x, t)|^p\,dx\,dt$$
$$= C(\lambda)\int_0^{+\infty}\exp[-\lambda t]\frac{d}{dt}\|v\|^p_{L_p(Q_t)}\,dt\,,$$

with $C(\lambda) = \lambda^{-1/p'}$.

Taking (1.20) into account, we obtain

$$(2.6)\qquad \|u_\lambda\|^p_p \leqslant C(\lambda)\,\lambda\,\|f\|^p_p\int_0^{+\infty}\exp[-\lambda t]t^{p+1}\,dt = \frac{C}{\lambda^p}\,\|f\|^p_p$$

and (2.2) is proved.

In a similar way,

$$(2.7)\qquad \int_\Omega\left|\frac{\partial^2 u_\lambda}{\partial x_i\,\partial x_j}(x)\right|^p dx \leqslant CC(\lambda)\,\lambda\,\|f\|^p_p\int_0^{+\infty}\exp[-\lambda t]t\,dt = C\|f\|^p_p\,.$$

By a standard interpolation argument, (2.7) and (2.2) give (2.3).

The next step is to show that u_λ is actually a solution of (I). To this purpose observe that

$$Au_\lambda = \lambda\int_0^{+\infty}\exp[-\lambda t]Av\,dt = \lambda\int_0^{+\infty}\exp[-\lambda t]\left(f - \frac{\partial v}{\partial t}\right)dt = f - \lambda\int_0^{+\infty}\exp[-\lambda t]\frac{\partial v}{\partial t}dt.$$

Hence, integrating by parts and taking the homogeneous initial condition on v into account,

$$Au_\lambda = f - \lambda u_\lambda \quad \text{in } \Omega\,.$$

On the other hand,

$$Bu_\lambda = \lambda\int_0^{+\infty}\exp[-\lambda t]Bv\,dt = 0 \quad \text{on } \Gamma\,,$$

since v satisfies the homogeneous boundary condition in (1.17).

Let us prove now that (I) has a unique solution. This amounts to show that if $z \in W_p^2(\Omega)$ is such that

$$\begin{aligned} Az + \lambda z &= 0 \qquad \text{in } \Omega\,, \\ Bz &= 0 \qquad \text{on } \Gamma \end{aligned}$$

then $z = 0$.

Let us denote to this purpose by v_1 and v_2, the solutions of

$$\begin{array}{lll} \dfrac{\partial v_1}{\partial t} + Av_1 = 0 & \dfrac{\partial v_2}{\partial t} + Av_2 = 0 & \text{in } Q_T \\ v_1(\cdot, 0) = z & v_2(\cdot, 0) = -Az = \lambda z & \text{in } \Omega \\ Bv_1 = 0 & Bv_2 = 0 & \text{on } \Sigma_T\,. \end{array}$$

It is easy to check that $v_1 - z$ and $\int_0^t v_2(x, s)\,ds$ satisfy the same mixed problem, namely

$$\begin{aligned} \frac{\partial v}{\partial t} + Av &= \lambda z \qquad \text{in } Q_T \\ v(\cdot, 0) &= 0 \qquad \text{in } \Omega \\ Bv &= 0 \qquad \text{on } \Sigma_T\,. \end{aligned}$$

By the uniqueness result in Theorem 1.2 it follows then that

$$\begin{aligned} v_1(x, t) - z(x) &= \int_0^t v_2(x, s)\,ds\,, \\ v_2(x, s) &= \lambda v_1(x, s)\,. \end{aligned}$$

These yield

$$\frac{\partial v_1}{\partial t} = \lambda v_1\,, \tag{2.8}$$

and, consequently,

$$\frac{\partial}{\partial t}(\exp[-\lambda t]v_1) = \exp[-\lambda t]\left(\frac{\partial v_1}{\partial t} - \lambda v_1\right) = 0\,.$$

Hence, $\exp[-\lambda t]v_1$ equals some function of x only, say $C(x)$. The limit relations,

$$\lim_{t\to\infty} \|\exp[-\lambda t]v_1\|_{L_p(Q_t)} = 0\,, \qquad \lim_{t\to 0} \|\exp[-\lambda t]v_1 - z\|_{L_p(Q_t)} = 0, \tag{2.9}$$

which follow from (1.20) and Theorem 1.2 imply $z = 0$. $\square$

REMARK 2.1. The case of an inhomogeneous boundary condition $Bu = Bh \in W_p^{1-1/p}(\Gamma)$ in (I) can be treated in a similar way, via a suitable change of unknown function.

COROLLARY 2.1. *The solution* u_λ *of* (I) *is also given by*

$$u_\lambda(x) = \int_0^\infty \exp[-\lambda t] \int_\Omega G(x, y, t) f(y)\, dy\, dt, \qquad f \in L_p(\Omega). \tag{2.10}$$

PROOF. Let us assume temporarily $f \in W_p^{2-2/p}(\Omega)$ and observe that

$$v(x, t) = \int_\Omega G(x, y, t) f(y)\, dy$$

is the unique solution of

$$\begin{aligned} &\frac{\partial v}{\partial t} + Av = 0 && \text{in } Q_T \\ &v(\cdot, 0) = f && \text{in } \Omega \\ &Bv = 0 && \text{on } \Sigma_T. \end{aligned}$$

A verification similar to the one in the proof of Theorem 2.1 shows that u_λ given by (2.10) is a solution of (I). By the uniqueness part of Theorem 2.1 the representations (2.1), (2.10) must coincide. The validity of (2.10) for general $f \in L_p(\Omega)$ follows by a density argument, taking (2.3) into account.

Let us consider now the unilateral problem (II). The following result holds:

THEOREM 2.2. *Let us assume* (1.3), (1.4), (1.5), $f \in L_p(\Omega)$ *and*

$$\psi \in W_p^2(\Omega), \qquad B\psi \geqslant 0 \quad \text{on } \Gamma. \tag{2.11}$$

Then, for any $\lambda > 0$ *there exists a unique solution* u *of* (II). *Moreover,*

$$u = \operatorname{Sup}\{v \in W_p^2(\Omega)/v \leqslant \psi,\ Av + \lambda v \leqslant f \quad \text{in } \Omega,\ Bv \leqslant 0 \text{ on } \Gamma\} \tag{2.12}$$

and u *satisfies the Lewy-Stampacchia inequality*

$$f \wedge (A\psi + \lambda\psi) \leqslant Au + \lambda u \leqslant f \quad \text{in } \Omega. \tag{2.13}$$

PROOF. Let us consider regularized operators $A^n, B^n (n = 1, 2, \dots)$ as described in § 1 and the auxiliary problems

$$\text{(2.14)} \qquad \begin{cases} u^n \in W^2_p(\Omega),\ \operatorname{Max}[u^n - \psi^n; A^n u^n + \lambda u^n - f] = 0 & \text{in } \Omega \\ B^n u^n = 0 & \text{on } \Gamma, \end{cases}$$

where ψ^n is the unique solution of

$$\begin{cases} \psi^n \in W^2_p(\Omega),\ A^n \psi^n + \lambda\psi^n = A\psi + \lambda\psi & \text{in } \Omega \\ B^n \psi^n = B\psi & \text{on } \Gamma. \end{cases}$$

The method of variational inequalities applies to problem (2.14) (see [3], [23]). Hence, (2.14) has a unique solution u^n and the Lewy-Stampacchia inequality holds;

$$\text{(2.15)} \qquad \begin{aligned} & f \wedge (A^n \psi^n + \lambda\psi^n) \leqslant A^n u^n + \lambda u^n \leqslant f \quad \text{in } \Omega, \\ & 0 = 0 \wedge B^n \psi^n \leqslant B^n u^n \leqslant 0 \quad \text{on } \Gamma(^*). \end{aligned}$$

From (2.15) it follows that

$$A^n u^n + \lambda u^n = g^n \quad \text{in } \Omega,$$

for some $g^n \in L_p(\Omega)$, $f \wedge (A^n \psi^n + \lambda\psi^n) \leqslant g^n \leqslant f$.

Hence, taking estimate (2.2), (2.3) into account,

$$\|u^n\|_{2,p} \leqslant \frac{C}{\lambda} \|g^n\|_p \leqslant C, \quad \text{independent of } n.$$

Therefore at least a subsequence of u^n converges weakly in $W^2_p(\Omega)$ as $n \to +\infty$ to some $u \in W^2_p(\Omega)$.

The passage to the limit in (2.14), (2.15) shows that u is a solution of (II) and (2.13) holds.

Let us prove now that any solution of (II) satisfies (2.12). Of course this will imply uniqueness of the solution.

Let v be an arbitrary function satisfying the inequalities in (2.12). Mimicking an argument due to Troianiello [24], let us define v^n as the solu-

(*) Inequality (2.15) can be proved in the case $1 < p < 2$ using the results of [11].

tion of

$$\text{(2.16)}\qquad \begin{cases} v^n \in W_p^2(\Omega)\,, \\ A^n v^n + \lambda v^n = Av + \lambda v - |(A^n - A)v| - |(A^n - A)u|\,, \\ B^n v^n = B^n v \wedge 0 \wedge (B^n - B)u\,, \end{cases}$$

where u is a solution of (II).

It is easy to check that

$$\begin{cases} A^n(v^n - v) + \lambda(v^n - v) \leqslant 0 & \text{in } \Omega \\ B^n(v^n - v) \leqslant 0 & \text{on } \Gamma\,, \end{cases}$$

from which it follows that $v^n \leqslant v \leqslant \psi$. It is also clear that $A^n v^n + \lambda v^n \leqslant f + (A^n - A)u$ in Ω, $B^n v^n \leqslant (B^n - B)u$ on Γ.

On the other hand any solution u of (II) satisfies

$$u \in W_p^2(\Omega),\ \max[u - \psi;\ A^n u + \lambda u - f - (A^n - A)u] = 0 \qquad \text{in } \Omega\,,$$

$$B^n u^n = (B^n - B)u \qquad \text{on } \Gamma\,.$$

Hence, known results in the variational case (see [23]) yield

$$v^n \leqslant u\,.$$

Since $v^n \to v$ in $W_p^2(\Omega)$, the statement (2.12) is proved. □

REMARK 2.2. The same result holds for less regular obstacle of the type $\psi = \bigwedge_{i=1}^{k} \psi_i$, $\psi_i \in W_p^2(\Omega)$, the inequality (2.12) being substituted by $f \wedge \left(\bigwedge_{i=1}^{k} (A\psi_i + \lambda\psi_i)\right) \leqslant u \leqslant f$.

Non-homogeneous boundary conditions can be treated by suitable change of variables.

3. – Fredholm alternative and invariant measures.

Let us denote by $J_\lambda (\lambda > 0)$ the Green operator of problem (I), which associates to any $f \in L_p(\Omega)$, $1 < p < 1/(1-\alpha)$ the unique solution $u_\lambda = J_\lambda f$ of problem (I).

LEMMA 3.1. *J_λ is a compact operator acting on $L_p(\Omega)$, $(1 < p < 1/1 - \alpha)$, and* $\operatorname{Ker}(I - \lambda J_\lambda) = \mathbb{R}$.

PROOF. The first statement is an immediate consequence of estimate (2.2) and the Rellich compactness theorem.

Since, as it is obvious, $\operatorname{Ker}(I - \lambda J_\lambda)$ is independent of λ we can take $\lambda = 1$. Taking (2.10) and Lemma 1.1 into account, we find that

$$J_1 f(x) = \int_0^{+\infty} \exp[-t]\,dt \int_\Omega G(x, y, t) f(y)\,dy > 0 \qquad \text{in } \Omega,$$

for any $f \in L_p(\Omega)$, $f \geqslant 0$, $f \not\equiv 0$.

Since $J_1 C = C$ for any real constant C, the theorem 6.6 of [20] applies, yielding that $\mu = 1$ is a simple eigenvalue of J_1. This proves the Lemma. □

Now, by the Fredholm alternative, the adjoint equation

$$m \in L_{p'}(\Omega), \quad (I - \lambda J_\lambda^*) m = 0 \left(\lambda > 0, \frac{1}{p'} + \frac{1}{p} = 1\right) \tag{3.1}$$

has a one dimensional space of solutions and

$$u \in L_p(\Omega), \quad (I - \lambda J_\lambda) u = g, \tag{3.2}$$

is solvable if and only if g satisfies the compatibility condition

$$\int_\Omega g m\,dx = 0. \tag{3.3}$$

The next theorem provides further information on $\operatorname{Ker}(I - \lambda J_\lambda^*)$, which will be of essential use for the result in § 4.

THEOREM 3.1. *Let us assume* (1.3), (1.4), (1.5).

Then, for every $\lambda > 0$, equation (3.1) *has a unique solution m such that*

$$m > 0, \qquad \frac{1}{|\Omega|}\int_\Omega m\,dx = 1, \qquad m \in L_\infty(\Omega). \tag{3.4}$$

Moreover, for any $t > 0$ *and* $f \in L_p(\Omega)$, $1 \leqslant p \leqslant \infty$,

$$\int_\Omega \left[\int_\Omega G(x, y, t) f(y)\, dy \right] m(x)\, dx = \int_\Omega f(x)\, m(x)\, dx, \tag{3.5}$$

$$\left\| \int_\Omega G(x, y, t) f(y)\, dy - \frac{1}{|\Omega|} \int_\Omega f(y) m(y)\, dy \right\|_p \leqslant K \|f\|_p \exp[-\nu t], \qquad \forall t > 0, \tag{3.6}$$

with K, ν *independent of* f (*).

PROOF. From (3.1) it follows that

$$m \in L_q(\Omega) \text{ for any } 1 < q < +\infty,$$

$$m(y) = \int_0^{+\infty} \exp[-t] \int_\Omega G(x, y, t)\, m(x)\, dx. \tag{3.7}$$

Then

$$|m(y)| \leqslant \|m\|_{q'} \int_0^\infty \exp[-t]\, dt \left(\int_\Omega t^{-(N/2)q} \exp\left[-c \frac{|x-y|^2}{t} q \right] dx \right)^{1/q}.$$

Therefore, the choice $q' > N/2$ yields

$$\|m\|_\infty \leqslant c \|m\|_{q'}.$$

Let us start by showing that any solution m of (3.1) has a constant sign almost everywhere in Ω.

At this purpose, set

$$\Omega^+ = \{x \in \Omega / m \geqslant 0\}, \qquad \Omega^- = \Omega - \Omega^+$$

and assume that $0 < |\Omega^+| < |\Omega|$.

Multiplying (3.1) by χ_{Ω^-} one finds that

$$\int_{\Omega^-} m\, dx = \lambda \int_{\Omega^+} m J_\lambda \chi_{\Omega^-} dx + \lambda \int_{\Omega^-} m J_\lambda \chi_{\Omega^-} dx. \tag{3.8}$$

(*) Further regularity properties of the function m will be the object of another paper (see [12]).

The representation (2.10), together with (1.22), (1.23) of Lemma 1.1, yields

$$J_\lambda \chi_{\Omega^-}(x) = \int_0^{+\infty} \exp[-\lambda t]\, dt \int_{\Omega^-} G(x, y, t)\, dy < \frac{1}{\lambda} \quad \text{in } \Omega .$$

Use this in (3.8) to get

$$\int_{\Omega^-} m\, dx \geqslant \lambda \int_{\Omega^-} m J_\lambda \chi_{\Omega^-} dx > \int_{\Omega^-} m\, dx ,$$

which is absurd. Hence $|\Omega^+| = |\Omega|$ and the statement is proved.

Since $\operatorname{Ker}(I - \lambda J_\lambda^*)$ is one-dimensional, this must contain some $\tilde{m} \geqslant 0$, $\tilde{m} \not\equiv 0$. Then, (3.4) is satisfied by $m = \tilde{m}/\left(|\Omega| \int_\Omega \tilde{m}\, dx\right)$.

Actually, $m > 0$ in Ω. Assume by contradiction that $\Omega^0 = \{x \in \Omega / m = 0\}$ has positive measure and multiply (3.1) by χ_{Ω^0}. Then,

$$0 = \int_\Omega m \chi_{\Omega^0}\, dx = \lambda \int_\Omega m J_\lambda \chi_{\Omega^0} dx = \lambda \int_\Omega m J_\lambda \chi_{\Omega^0}\, dx > 0 ,$$

since $J_\lambda \chi_{\Omega^0} > 0$ (see the proof of Lemma 3.1).

In order to prove (3.5), let us multiply (3.1) by an arbitrary $f \in L_p(\Omega)$ to obtain

$$\frac{1}{\lambda} \int_\Omega m f\, dx = \int_\Omega m J_\lambda f\, dx = \int_0^{+\infty} \exp[-\lambda t]\, dt \int_\Omega \left[\int_\Omega G(x, y, t) f(y)\, dy \right] m(x)\, dx .$$

Since $1/\lambda = \int_0^{+\infty} \exp[-\lambda t]\, dt$, the above yields

$$\int_0^{+\infty} \exp[-\lambda t]\, dt \int_\Omega \left[\int_\Omega G(x, y, t) f(y)\, dy - f(x) \right] m(x)\, dx = 0 . \tag{3.9}$$

The term in square brackets in (3.9) being a continuous function of t, (3.5) is proved.

Let us prove now the estimate (3.6). Following [5], Thm. 3.1 p. 365, we consider, for $n = 1, 2, \ldots$, and for any Borel subset E in Ω, the measures

$$m_n(E) = \inf_{x\in\Omega}\int_\Omega G(x, \xi, n)\chi_E(\xi)\, d\xi .$$

$$M_n(E) = \sup_{y\in\Omega}\int_\Omega G(y, \xi, n)\chi_E(\xi)\, d\xi .$$

Since

$$G(x, \xi, n+1) = \int_\Omega G(x, z, n)\, G(\xi, z, 1)\, dz ,$$

we obtain

$$\begin{aligned}(3.10)\quad & M_{n+1}(E) - m_{n+1}(E) \\ & = \sup_{x,y\in\Omega}\left|\int_\Omega [G(x, z, 1) - G(y, z, 1)]\, dz \int_\Omega G(z, \xi, n)\chi_E(\xi)\, d\xi\right| .\end{aligned}$$

If we define now the set $\Omega^+ = \Omega^+_{x,y}$ by

$$\Omega^+ = \{z \in \Omega | G(x, z, 1) > G(y, z, 1)\} ,$$

we have, taking (1.23) of Lemma 1.1, into account,

$$(3.11)\quad \int_{\Omega^+}[G(x, z, 1) - G(y, z, 1)]\, dz = 1 - \int_{\Omega^+} G(y, z, 1)\, dz - \int_{\Omega/\Omega^+} G(x, z, 1)\, dz ,$$

Hence, by (1.22) of Lemma 1.1,

$$(3.12)\quad \int_{\Omega^+}[G(x, z, 1) - G(y, z, 1)]\, dz \leqslant 1 - \gamma|B| .$$

Using this in (3.10), we obtain

$$\begin{aligned}(3.13)\quad M_{n+1}(E) - m_{n+1}(E) \leqslant \sup_{x,y} \bigg| & \int_{\Omega^+} M_n(E)\, [G(x, z, 1) - G(y, z, 1)]\, dz \\ & + \int_{\Omega/\Omega_+} m_n(E)\, [G(x, z, 1) - G(y, z, 1)]\bigg|\, dz \leqslant \sup_{x,y} \big(M_n(E) - m_n(E)\big)\big(1 - \gamma|B|\big) ,\end{aligned}$$

and, by iteration,

$$M_{n+1}(E) - m_{n+1}(E) \leqslant \big(M_1(E) - m_1(E)\big)\,(1 - \gamma|B|)^{n-1}\,. \tag{3.14}$$

Let us estimate now

$$\left|\int\limits_{\Omega} G(x, \xi, n+1)\,\chi_E(\xi)\,d\xi - \frac{1}{|\Omega|}\int\limits_{\Omega} \chi_E(\xi)\,m(\xi)\,d\xi\right|^p$$

or, which is the same, thanks to (3.5),

$$h(x) = \left|\int\limits_{\Omega} G(x, \xi, n+1)\,\chi_E(\xi)\,d\xi - \frac{1}{|\Omega|}\int\limits_{\Omega} m(\eta)\left[\int\limits_{\Omega} G(\eta, \xi, n+1)\,\chi_E(\xi)\,d\xi\right]d\eta\right|^p. \tag{3.15}$$

Now, the obvious inequalities

$$m_{n+1}(E) \leqslant \int\limits_{\Omega} G(x, \xi, n+1)\,\chi_E(\xi)\,d\xi \leqslant M_{n+1}(E)\,,$$

$$m_{n+1}(E)\,\frac{1}{|\Omega|}\int\limits_{\Omega} m(\eta)\,d\eta \leqslant \frac{1}{|\Omega|}\int\limits_{\Omega} m(\eta)\left[\int\limits_{\Omega} G(\eta, \xi, n+1)\,\chi_E(\xi)\,d\xi\right]d\eta \leqslant M_{n+1}(E)\,\frac{1}{|\Omega|}\int\limits_{\Omega} m(\eta)\,d\eta\,,$$

(3.4) and (3.14) yield

$$h(x) \leqslant |M_{n+1}(E) - m_{n+1}(E)|^p \leqslant |M_1(E) - m_1(E)|^p\,(1 - \gamma|B|)^{p(n-1)} \leqslant |M_1(E)|^p\,(1 - \gamma|B|)^{p(n-1)}\,.$$

At this point from (3.15) it follows that

$$\int\limits_{\Omega} h(x)\,dx \leqslant K(1 - \gamma|B|)^{p(n-1)}\|\chi_E\|_{p'}^p\,,$$

for some constant K depending only on Ω.

If we choose now $\nu = -p\log\big(1 - \gamma|B|\big)$, the inequality (3.6) is proved for positive integer $t = [t]$ and $f = \chi_E$.

Now for $t > [t]$, we have

$$\left\| \int G(x, y, t) f(y)\, dy - \frac{1}{|\Omega|} \int_{\Omega} f(y) m(y)\, dy \right\|_p$$
$$= \left\| \int_{\Omega} G(x, y, t - [t]) \int_{\Omega} [G(y, z, [t]) f(z)\, dz]\, dy - \frac{1}{|\Omega|} \int_{\Omega} f(y) m(y)\, dy \right\|_p$$
$$\leqslant K \|\chi_E\|_p \cdot \exp[-\nu[t] < K \|\chi_E\|_p \exp[-\nu t]\,.$$

Using a density argument we obtain (3.6). □

We want to give now a probabilistic interpretation of Theorem 3.1. By means of the Green's function one can define a family of probability measures $P(x, t, E)$ on Ω in the following standard way,

$$P(x, t, E) = \int_E G(x, y, t)\, dy,$$

where E is any Borel subset on $\bar{\Omega}$. To $P(x, t, E)$ it corresponds a linear operator $\Phi(t)$ defined by

$$\Phi(t) f(x) = \int_{\Omega} P(x, t, dy) f(y) = \int_{\Omega} G(x, y, t) f(y)\, dy\,. \tag{3.16}$$

The easy estimate

$$|\Phi(t) f(x)| \leqslant \sup_{y \in \bar{\Omega}} |f(y)| \int_{\Omega} G(x, y, t)\, dy\,, \tag{3.17}$$

together with the estimates of Theorem 1.1, implies that $\Phi(t)$ is a continuous operator on $C(\bar{\Omega})$. Taking the result of § 1 and (3.17) into account it is straightforward to check that

$$\begin{cases} \Phi(0) = I, \quad \Phi(t) \circ \Phi(s) = \Phi(t + s)\,, \\ \|\Phi(t)\| = 1\,, \\ \Phi(t) f \geqslant 0, \quad \forall f \geqslant 0, \quad \Phi(t)\, C = C, \quad \forall C \in \mathbb{R}\,, \\ \Phi(t) f \to f \text{ in } C(\bar{\Omega}), \quad \text{as } t \to 0^+\,. \end{cases} \tag{3.18}$$

In probabilistic terminology (see [9]) this means that $\Phi(t)$ is a Feller semigroup on $C(\bar{\Omega})$. Actually, $\Phi(t)$ is associated with a diffusion process in Ω, reflected at its boundary Γ according to the vector field $b = (b_1 \dots b_N)$, (see [22] and for recent results, [15]).

Let us recall for the convenience of the reader the notion of invariant probability measure for such a semigroup. A probability measure μ on $\bar{\Omega}$, endowed with the Borel σ-algebra, is invariant for $\Phi(t)$ if

$$\int_\Omega \Phi(t) f \, d\mu = \int_\Omega f \, d\mu \,, \qquad \forall f \in C(\bar{\Omega}) \,, \tag{3.19}$$

(see [9]). From Theorem 3.1 it follows then that

$$d\mu = \frac{1}{|\Omega|} m \, dx \tag{3.20}$$

can be interpreted as an invariant probability measure for the semigroup defined by (3.16). The estimate (3.6) still holds with respect to the norm in $C(\bar{\Omega})$ and this implies that μ is the unique invariant measure of $\Phi(t)$, (see [5]).

When the operator A has Lipschitz continuous coefficients and b is C^1, an integration by parts shows that (3.1) becomes the adjoint homogeneous differential problem of (I), namely

$$\begin{aligned} A^* m &= 0 \quad \text{in } \Omega \\ B^* m &= 0 \quad \text{on } \Gamma \,, \end{aligned} \tag{3.21}$$

(see [17], p. 14-15, for an explicit formula for the adjoint operator B^*).

We conclude this section by showing that even in the present framework the function m can be constructed purely by PDE methods. At this purpose, let us consider the problems

$$\left\{ \begin{aligned} & m_n \in W^2_p(\Omega) \,, \\ & A^*_n m_n = 0 \text{ in } \Omega \,, \\ & B^*_n m_n = 0 \text{ on } \Gamma , \qquad (n = 1, 2, \dots) \,, \end{aligned} \right. \tag{3.22}$$

where A^*_n, B^*_n are the adjoints of the regularized operators A_n, B_n given by (1.6), (1.7).

It is known (see for example [4]) that (3.22) has a unique solution m_n such that

$$m_n \in C^2(\bar{\Omega}) \,, \qquad m_n > 0 \,, \qquad \frac{1}{|\Omega|} \int_\Omega m_n \, dx = 1 \,, \qquad (n = 1, 2, \dots) \,.$$

Define now probability measures μ_n on $\bar{\Omega}$ by

$$d\mu_n = \frac{1}{|\Omega|} m_n \, dx \tag{3.23}$$

and denote by $\Phi_n(t)$ the Feller semigroup associated with A_n, B_n. Then, we have the following:

THEOREM 3.2. *Let us assume* (1.3), (1.4), (1.5) *and* $\alpha > (N+2)/2$. *Then, the measures* μ_n *defined by* (3.23) *are invariant for* $\Phi_n(t)$ *and converge weakly in the sense of measures, as* $n \to +\infty$, *to the invariant measure* μ *of* $\Phi(t)$, *defined by* (3.20).

PROOF. Consider the function

$$\theta_n(t) = \int_\Omega [\Phi_n(t) f(x) - f(x)] m_n(x) \, dx \qquad (n = 1, 2, \ldots),$$

where f is arbitrary in $W_p^{2-2/p}(\Omega)$, $p > (N+2)/2$. Then,

$$\theta_n'(t) = -\int_\Omega A_n \Phi_n(t) f(x) m_n(x) \, dx, \qquad \forall t > 0.$$

An integration by parts, taking (3.22) into account, shows that $\theta_n'(t) = 0$, $\forall t > 0$ hence $\theta_n(t) \equiv \theta_n(0) = 0$.

A standard density argument completes the proof of the first part of the theorem.

The weak convergence of μ_n to some probability measure $\tilde{\mu}$ being immediate, the next step is to identify $\tilde{\mu}$ with μ. To this end, let $f \in W_p^{2-2/p}(\Omega)$, $p > (N+2)/2$, and $z_n = (\Phi_n(t) - \Phi(t)) f$. Then, z_n satisfies

$$\begin{cases} z_n \in W_p^{2,1}(Q_T) & \\ \dfrac{\partial z_n}{\partial t} + A_n z_n = (A - A_n)\Phi(t) f & \text{in } Q_T \\ z_n(\cdot, 0) = 0 & \text{in } \Omega \\ B_n z_n = (B - B^n)\Phi(t) f & \text{on } \Sigma_T, \end{cases}$$

for any $T > 0$. By the parabolic estimates (see Theorem 1.2)

$$\|z_n\|_{2,1,p} \leqslant C[\|(A - A_n)\Phi(t) f\|_p + \|(B - B_n)\Phi(t) f\|_{1-1/p,p}], \tag{3.24}$$

with C independent of n.

The right-hand term of (3.24) tends to zero as $n \to +\infty$ and so

$$\Phi_n(t)f \to \Phi(t)f, \text{ uniformly in } x .$$

Hence,

$$\lim_{n\to+\infty} \int_\Omega \Phi_n(t)f\, d\mu_n = \int_\Omega \Phi(t)f\, d\tilde{\mu} , \qquad \forall f \in {}_{\scriptscriptstyle B}W_p^{2-2/p}(\Omega) ,$$

and by density the same holds true for $f \in C(\bar{\Omega})$.

Since, trivially,

$$\lim_{n\to\infty} \int_\Omega f\, d\mu_n = \int f\, d\tilde{\mu} ,$$

the measure $\tilde{\mu}$ is invariant for $\Phi(t)$. From (3.6) of Theorem 3.1 it follows that

$$\lim_{k\to+\infty} \Phi(t_k)f(x) = \frac{1}{|\Omega|}\int_\Omega f\, d\mu \qquad \text{a.e. in } \Omega ,$$

at least for a subsequence t_k. Therefore, by the Lebesgue dominated convergence theorem,

$$\lim_{k\to+\infty} \int_\Omega \Phi(t_k)f\, d\tilde{\mu} = \frac{1}{|\Omega|}\int_\Omega d\tilde{\mu} \int_\Omega f\, d\mu = \frac{1}{|\Omega|}\int_\Omega f\, d\mu . \tag{3.25}$$

Since $\tilde{\mu}$ is invariant for $\Phi(t)$, we have also

$$\int_\Omega \Phi(t)f\, d\tilde{\mu} = \int_\Omega f\, d\tilde{\mu} , \qquad \forall t > 0 , \qquad \forall f \in C(\bar{\Omega}) .$$

Hence, comparing this with (3.25), $\tilde{\mu} = \mu$. □

4. – The limiting case $\lambda = 0$.

This section is devoted to the study of problems (I) and (II) in the case $\lambda = 0$. Let us observe that

$$\begin{cases} u \in W_p^2(\Omega) , & Au = f \quad \text{in } \Omega \\ & Bu = 0 \quad \text{on } \Gamma \end{cases} \tag{4.1}$$

is equivalent to

$$u \in L_p(\Omega),\ (I - \lambda J_\lambda)u = J_\lambda f , \qquad (\lambda > 0) . \tag{4.2}$$

Hence, by the results of § 3, (4.1) is solvable if and only if $f \in L_p(\Omega)$ satisfies

$$\int_\Omega J_\lambda f m \, dx = 0 \tag{4.3}$$

for any solution m of (3.1) or, which is the same,

$$\int_\Omega f m \, dx = 0 \, . \tag{4.4}$$

An immediate consequence of the definition of J_λ and m which will be useful later on, is that the solution u_λ of (I) satisfies

$$\int u_\lambda m \, dx = \frac{1}{\lambda} \int_\Omega f m \, dx \, . \tag{4.5}$$

Let us state now the main result concerning problem (4.1) and the behaviour as $\lambda \to 0$ of the solutions u_λ of (I).

THEOREM 4.1. *Let us assume* (1.3), (1.4), (1.5), $f \in L_p(\Omega)$. *A necessary and sufficient condition for* (4.1) *to have a solution is that*

$$\int_\Omega f m \, dx = 0 \, . \tag{4.6}$$

If (4.6) *holds then,*

$$\|u_\lambda\|_{2,p} \leqslant C \|f\|_p \, , \qquad C \textit{ independent of } \lambda > 0 \tag{4.7}$$

and (4.1) *has a unique solution* u *such that*

$$\int_\Omega u m \, dx = 0$$

given by

$$u = \int_0^{+\infty} dt \int_\Omega G(x, y, t) f(y) \, dy = \lim_{\lambda \to 0} u_\lambda \, , \qquad \textit{weakly in } W_p^2(\Omega) \, . \tag{4.8}$$

PROOF. If (4.1) or, equivalently, (4.2) has a solution u for a given $f \in L_p(\Omega)$, then

$$\int_\Omega J_\lambda f m\, dx = \int_\Omega (I - \lambda J_\lambda)\, u m\, dx = \int_\Omega u(I - \lambda J_\lambda^*)\, m\, dx = 0$$

On the other hand,

$$\int_\Omega J_\lambda f m\, dx = \int_\Omega f J_\lambda^* m\, dx = \frac{1}{\lambda}\int_\Omega f m\, dx$$

and the necessity of condition (4.6) is established.

Let u_1, u_2 be two solutions of (4.1). Then,

$$u_1 - u_2 \in \operatorname{Ker}(I - \lambda J_\lambda)\,,$$

and so, by Lemma 3.1, $u_1 - u_2 = C$. If $\int_\Omega u_1 m\, dx = 0 = \int_\Omega u_2 m\, dx$, then $0 = C\int_\Omega m\, dx$, and therefore $C = 0$.

Let us prove now that if (4.6) is satisfied, then

$$\|u_\lambda\|_{2,p} \leqslant C\|f\|_p \qquad C \text{ independent of } \lambda > 0\,. \tag{4.9}$$

Using the Hölder inequality in the integral with respect to t in representation (2.10), one obtains

$$\int_\Omega |u_\lambda|^p\, dx \leqslant \left(\frac{1}{(\lambda + \nu/2)p'}\right)^{p/p'} \int_0^{+\infty} \exp\left[\frac{\nu p t}{2}\right] \left\| \int_\Omega G(x, y, t) f(y)\, dy \right\|_p dt\,. \tag{4.10}$$

Hence, taking (3.6) into account,

$$\|u_\lambda\|_p \leqslant \left(\frac{1}{(\lambda + \nu/2)p'}\right)^{1/p'} \left(\frac{2}{\nu p}\right)^{1/p} K\|f\|_p \leqslant C\|f\|_p \tag{4.11}$$

for some constant C depending on ν, p not on λ.

Therefore, the fundamental estimates (2.3) and (4.11) give (4.7). Now standard arguments show that u_λ converge weakly in $W_p^2(\Omega)$ as $\lambda \to 0$ to a solution u of (4.1) such that $\int_\Omega u m\, dx = 0$. This last fact follows from (4.5).

The estimate (3.6) guarantees that u is given by (4.8). □

REMARK 4.1. The uniform convergence of u_λ to u has been proved by Robin [19] in the case $f \in C(\bar{\Omega})$ and for Lipschitz continuous boundary vectorfield b. This result follows, of course, from Theorem 4.1, if $p > N$.

REMARK 4.2. As a consequence of Theorem 4.1

$$Av = h - \frac{1}{|\Omega|}\int_\Omega hm\, dx \qquad \text{in } \Omega,\ Bv = 0 \qquad \text{on } \Gamma$$

has a unique solution $\bar{v} \in W^2_p(\Omega)$ such that $\int_\Omega \bar{v} m\, dx = 0$, for any given $h \in L_p(\Omega)$.

From (4.8) it follows that $\bar{v}$ satisfies

$$\bar{v}(x) = \int_0^T dt \int_\Omega G(x, y, t)\left[h(y) - \frac{1}{|\Omega|}\int_\Omega h(\xi) m(\xi)\, d\xi\right] dy + \int_\Omega G(x, y, T)\bar{v}(y)\, dy$$

and the estimate (3.6) shows that

$$\frac{1}{T}\left\| \int_\Omega G(x, y, T)\bar{v}(y)\, dy - \bar{v}\right\|_p \leqslant \frac{\|v\|_p}{T}(1 + K\exp[-\nu T]) \to 0 \qquad \text{as } T \to +\infty\,.$$

Hence, the ergodic formula

$$\left\|\frac{1}{T}\int_0^T\int_\Omega G(x, y, t)h(y)\, dy\, dt - \frac{1}{|\Omega|}\int_\Omega hm\, dx\right\|_p \to 0 \qquad \text{as } T \to +\infty$$

holds for any $h \in L_p(\Omega)$.

We are interested now in the behaviour of the solution u_λ of the unilateral problem as $\lambda \to 0$.

Let us assume, for simplicity, $\psi \equiv 0$. For the case of $\psi \not\equiv 0$, see Remark 4.5 below. The next lemma contains two estimates which will be useful later.

LEMMA 4.1. *There exist constants C_1, C_2 independent of $\lambda > 0$ such that*

$$-C_1\|f\|_p \leqslant \lambda\int_\Omega u_\lambda m\, dx\,, \tag{4.12}$$

$$\left\|u_\lambda - \frac{1}{|\Omega|}\int_\Omega u_\lambda m\, dx\right\|_{2,p} \leqslant C_2\|f\|_p\,. \tag{4.13}$$

PROOF. From the Lewy-Stampacchia estimate (2.13) follows that for some $g_\lambda \in L_p(\Omega)$, $0 \leqslant g_\lambda \leqslant f^+$,

$$\begin{cases} Au_\lambda + \lambda u_\lambda = f - g_\lambda & \text{in } \Omega \\ Bu_\lambda = 0 & \text{on } \Gamma . \end{cases} \tag{4.14}$$

Hence (see 4.5),

$$0 \geqslant \lambda \int_\Omega u_\lambda m\, dx = \int_\Omega (f - g_\lambda) m\, dx \geqslant - C_1 \|f\|_p .$$

Set $C_\lambda = 1/|\Omega| \int_\Omega u_\lambda m\, dx$ and $w_\lambda = u_\lambda - C_\lambda$. From (4.14) it easily follows that w_λ satisfies

$$\begin{cases} w_\lambda \in W_p^2(\Omega),\ Aw_\lambda + \lambda w_\lambda = f - g_\lambda - \lambda C_\lambda, & \text{in } \Omega , \\ Bw_\lambda = 0, & \text{on } \Gamma , \\ \displaystyle\int_\Omega w_\lambda \mathrm{m}\, dx = 0 . \end{cases} \tag{4.15}$$

Taking into account (4.5) we have necessarily

$$\int_\Omega (f - g_\lambda - \lambda C_\lambda) m\, dx = 0 .$$

Hence, estimate (4.7) of Theorem 4.1 yields

$$\|w_\lambda\|_p \leqslant C \|f - g_\lambda - \lambda C_\lambda\|_p \leqslant C \|f\|_p ,$$

for some C independent of λ, and (4.13) follows through the basic estimate (2.3). $\square$

The next theorem describes the behaviour as $\lambda \to 0$ of the solutions u_λ of problem (II).

THEOREM 4.2. *Let us assume* (1.3), (1.4), (1.5), $f \in L_p(\Omega)$. *Then, as* $\lambda \to 0$,

1) *if* $\int_\Omega fm\, dx > 0$, *the solutions* u_λ *of* (II) *converge weakly in* $W_p^2(\Omega)$ *to the unique solution* u *of*

$$\begin{cases} u \in W_p^2(\Omega),\ \mathrm{Max}\,[u; Au - f] = 0 & \text{in } \Omega \\ Bu = 0 & \text{on } \Gamma ; \end{cases} \tag{4.16}$$

2) *if* $\int_\Omega fm\,dx < 0$, *then* $w_\lambda = u_\lambda - 1/|\Omega| \int_\Omega u_\lambda m\,dx$ *converge weakly in* $W_p^2(\Omega)$ *to the unique solution* w *of*

$$(4.17)\qquad \begin{cases} w \in W_p^2(\Omega),\ Aw = f - \dfrac{1}{|\Omega|}\displaystyle\int_\Omega fm\,dx & \text{in } \Omega, \\ Bw = 0 & \text{on } \Gamma, \\ \displaystyle\int_\Omega wm\,dx = 0\,; & \end{cases}$$

3) *if* $\int_\Omega fm\,dx = 0$, *and* $p > N/2$ *then* u_λ *converge weakly in* $W_p^2(\Omega)$ *to the maximum solution* u *of*

$$(4.18)\qquad \begin{cases} u \in W_p^2(\Omega),\ u \leqslant 0,\ Au = f & \text{in } \Omega \\ Bu = 0 & \text{on } \Gamma. \end{cases}$$

PROOF. Let us consider the functions $w_\lambda = u_\lambda - 1/|\Omega| \int_\Omega u_\lambda m\,dx = u_\lambda - C_\lambda$. Of course,

$$(4.19)\qquad \begin{cases} w_\lambda \in W_p^2(\Omega),\ \operatorname{Max}\,[w_\lambda + C_\lambda;\ Aw_\lambda + \lambda w_\lambda - f + \lambda C_\lambda] = 0 & \text{in } \Omega\,, \\ Bw_\lambda = 0 & \text{on } \Gamma. \end{cases}$$

By Lemma 4.1, at least a subsequence of w_λ converges weakly in $W_p^2(\Omega)$ as $\lambda \to 0$ to some w such that $\int_\Omega w_\lambda m\,dx = 0$ and $\lambda C_\lambda \to L \leqslant 0$.

Consider the sets $S_\lambda = \{x \in \Omega / w_\lambda(x) = -C_\lambda\}$. Then,

$$(4.20)\qquad \int_\Omega w_\lambda \chi_{S_\lambda} dx = -C_\lambda |S_\lambda|\,.$$

Since the left hand term has a finite limit as $\lambda \to 0$, (4.20) shows that if

$$(4.21)\qquad C_\lambda \to -\infty \text{ as } \lambda \to 0\,,$$

then $|S_\lambda| \to 0$ as $\lambda \to 0$.

From (4.19) it follows that

$$Aw_\lambda + \lambda w_\lambda = f - \lambda C_\lambda \quad \text{in } \Omega - S_\lambda\,.$$

Passing to the limit in the above equations we find that w satisfies

$$\left\{\begin{array}{lll} w \in W_p^2(\Omega), & Aw = f - L & \text{in } \Omega \\ \int_\Omega wm\, dx = 0,\ Bw = 0 & & \text{on } \Gamma, \end{array}\right. \tag{4.22}$$

$\Big($observe, to justify this, that $\int_{S_\lambda} |Aw_\lambda + \lambda w_\lambda - f + \lambda C_\lambda|\, dx \leqslant \|Aw_\lambda + \lambda w_\lambda - f + \lambda C_\lambda\|_p\, |S_\lambda|^{1/p'}\Big)$.

Let us assume now

$$\int_\Omega fm\, dx > 0\,. \tag{4.23}$$

From (4.22) it follows that $L\int_\Omega m\, dx = \int_\Omega fm\, dx$, which contradicts (4.23), since $L \leqslant 0$. Therefore, if (4.23) holds, then C_λ is bounded. This yields immediately the boundedness of u_λ in $W_p^2(\Omega)$, (see Lemma 4.1). A passage to the limit in (II) shows that the weak limit u of u_λ is a solution of (4.16). Let $\tilde{u}$ be another solution of (4.16). Then $\tilde{u}$ satisfies the inequalities $\tilde{u} \leqslant 0$, $A\tilde{u} + \lambda\tilde{u} \leqslant f$ in Ω and $B\tilde{u} = 0$ on Γ. Hence, by Theorem 2.2, $\tilde{u} \leqslant u_\lambda$ for every $\lambda > 0$, so that

$$\tilde{u} \leqslant u\,. \tag{4.24}$$

From (4.16) it follows that

$$A(u - \tilde{u})\cdot(u - \tilde{u}) \leqslant 0$$

and, taking (4.24) into account, this gives $A(u - \tilde{u}) \leqslant 0$. Hence $u - \tilde{u}$ satisfies

$$\begin{array}{ll} A(u - \tilde{u}) = g & \text{in } \Omega \\ B(u - \tilde{u}) = 0 & \text{on } \Gamma \end{array} \tag{4.25}$$

for some $g \leqslant 0$. The necessary condition of Theorem 4.1 implies that $g = 0$, and as a consequence of Lemma 3.1 one obtains $u - \tilde{u} = C$ for some constant C. If $C > 0$, then $\tilde{u} = u - C \leqslant -C < 0$ and therefore (4.16) reduces to

$$A\tilde{u} = f \quad \text{in } \Omega\,,$$

$$B\tilde{u} = 0 \quad \text{on } \Gamma\,.$$

By taking the necessary condition (4.6) into account we obtain $\int_\Omega fm\, dx = 0$ contradicting (4.23). Hence, by virtue of (4.24), $C = 0$ and $u = \tilde{u}$.

Assume now

$$\int_\Omega fm\,dx < 0 \tag{4.26}$$

and suppose that C_λ is bounded below. Then $C_\lambda \to \bar{C}$, $\lambda C_\lambda \to 0$ and, passing to the limit in (4.19) as $\lambda \to 0$ one finds that the weak limit w of w_λ satisfies

$$\begin{cases} \operatorname{Max}\,[w + \bar{C};\, Aw - f] = 0 & \text{in } \Omega \\ Bw = 0 & \text{on } \Gamma . \end{cases}$$

Hence, for some $g \geqslant 0$,

$$Aw = f - g, \qquad Bw = 0 .$$

So that necessarily, $\int_\Omega (f-g)m\,dx = 0$. But this contradicts (4.26) and therefore $C_\lambda \to -\infty$. Hence, (4.22) must hold and the necessary condition (4.6) yields $\int_\Omega fm\,dx = L|\Omega|$.

This proves the second part of the theorem.

Let us consider finally the case

$$\int_\Omega fm\,dx = 0 . \tag{4.27}$$

By Theorem 4.1 and the assumption $p > N/2$, the solutions z_λ of

$$\begin{aligned} Az_\lambda + \lambda z_\lambda &= f \quad \text{in } \Omega \\ Bz_\lambda &= 0 \quad \text{on } \Gamma \end{aligned} \tag{4.28}$$

are bounded in $W^2_p(\Omega)$ and $L_\infty(\Omega)$, uniformly in λ.

It is easy to check that $z_\lambda - \|z_\lambda\|_\infty$ is a subsolution of (II). Hence, taking (2.12) into account,

$$u_\lambda \geqslant z_\lambda - \|z_\lambda\|_\infty .$$

From Lemma 4.1 it follows that

$$\|u_\lambda\|_{2,p} \leqslant C_2\|f\|_p - \int_\Omega u_\lambda m\,dx \leqslant C_2\|f\|_p + \big\|z_\lambda - \|z_\lambda\|_\infty\big\|_\infty \|m\|_\infty ,$$

and therefore $\|u_\lambda\|_{2,p} \leqslant C$, for some constant C independent of λ. By the

Lewy-Stampacchia inequality (2.13), u_λ satisfies

$$\begin{cases} Au_\lambda + \lambda u_\lambda = f - g_\lambda & \text{in } \Omega\,, \\ Bu_\lambda = 0 & \text{on } \Gamma\,, \end{cases} \tag{4.29}$$

for some $g_\lambda \geqslant 0$ with $\|g_\lambda\|_p \leqslant C$. Let u, g be weak subsequential limits of u_λ and g_λ, respectively. Letting $\lambda \to 0$ in (4.29) one finds that u is a solution of

$$\begin{aligned} Au &= f - g \quad \text{in } \Omega\,, \\ Bu &= 0 \qquad\;\; \text{on } \Gamma\,. \end{aligned}$$

Then, necessarily,

$$\int_\Omega gm\,dx = \int_\Omega fm\,dx = 0\,,$$

so that $g = 0$. This proves that u satisfies (4.18).

To prove the maximality of $u = \lim_{\lambda\to 0} u_\lambda$, let $\tilde{u}$ be another solution of (4.18). Then, for every $\lambda > 0$,

$$\tilde{u} \leqslant 0\,, \; A\tilde{u} + \lambda\tilde{u} \leqslant f \quad \text{in } \Omega\,.$$

Hence by (2.12),

$$\tilde{u} \leqslant u_\lambda \quad \text{for every } \lambda > 0\,,$$

so that

$$\tilde{u} \leqslant \lim_{\lambda\to 0} u_\lambda = u\,.$$

This shows also that the whole family u_λ converges and the proof is complete. □

Remark 4.3. The proof of the theorem shows that also for $\int_\Omega fm\,dx = 0$, the functions $w_\lambda = u_\lambda - 1/|\Omega| \int_\Omega u_\lambda m\,dx$ converge weakly in $W^2_p(\Omega)$ to the unique solution of (4.17) without the restriction $p > N/2$.

Remark 4.4. The above theorem extends previous results of Bensoussan-Lions [4] and Robin [19].

Remark 4.5. The case of an obstacle $\psi \not\equiv 0$ with $B\psi = 0$ in (II) can be dealt with by means of the translation $z_\lambda = u_\lambda - \psi$. Then z_λ satisfies $\operatorname{Max}[z_\lambda; Az_\lambda + \lambda z_\lambda - f + A\psi + \lambda\psi] = 0$.

It is easy to check that if $\int_\Omega fm\,dx > 0$, then

$$a_\lambda = \int (f - A\psi - \lambda\psi) m\,dx = \int (f - \lambda\psi) m\,dx > 0$$

for sufficiently small λ. Hence by theorem 4.2, u_λ converges to the solution u of $\operatorname{Max}[u - \psi; Au - f] = 0$ in Ω, $Bu = 0$ on Γ.

Conversely, if $\int_\Omega fm\,dx < 0$, then $a_\lambda < 0$ for small λ and the result is that $w_\lambda = u_\lambda - C_\lambda$ converges to the unique solution w of

$$w \in W_p^2(\Omega),\ Aw = f - \frac{1}{|\Omega|}\int_\Omega fm\,dx \quad \text{in } \Omega,\ Bw = 0 \quad \text{on } \Gamma, \quad \int_\Omega wm\,dx = 0\,.$$

Finally, if $\int_\Omega fm\,dx = 0 = \int_\Omega \psi m\,dx$ and $p > N/2$, then u_λ converge weakly to the maximum solution u of

$$u \leqslant \psi\,, \qquad Au = f \quad \text{in } \Omega\,, \qquad Bu = 0 \quad \text{on } \Gamma\,.$$

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