ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 14, nº 2 (1987), p. 199-215

<http://www.numdam.org/item?id=ASNSP_1987_4_14_2_199_0>

Holomorphic Generators of Some Ideals in $C^\infty(\overline{D})$

PAOLO DE BARTOLOMEIS

dedicated to B.V. Shabat

0. Introduction, notations and statement of the main results

Let $D \subset \mathbb{C}^{n+1}$ be a bounded domain with C^∞-smooth boundary, V a complex submanifold of a neighbourhood of $\overline{D}$ such that $\overline{D \cap V} = \overline{D} \cap V \neq \emptyset$, $\mathcal{I}_V$ the sheaf of ideals of V and set:

$$\mathfrak{I}^\infty(V) = \{f \in C^\infty(\overline{D}) | f|_V = 0\},$$
$$I^\infty(V) = \{f \in A^\infty(D) = \mathcal{O}(D) \cap C^\infty(\overline{D}) | f|_V = 0\}.$$

It is well known (see e.g. [7]) that if $g_1, \ldots, g_k \in \mathcal{O}(\overline{D})$ $g_j|_D \in I^\infty(V)$, $1 \leq j \leq k$, represent a complete defining system for V (i.e. for every $x \in \overline{D}$, $g_{1,x}, \ldots, g_{k,x}$ generates $\mathcal{I}_{V,x}$ over $\mathcal{O}_x$), then $g_1, \ldots, g_k, \overline{g}_1, \ldots, \overline{g}_k$ generate $\mathfrak{I}^\infty(V)$ over $C^\infty(\overline{D})$ if and only if $\overline{D}$ and V are regularly separated in the sense of -Lojasiewicz, i.e. there exist $h \in \mathbb{Z}^+$ and $C > 0$ such that for every $x \in \overline{D}$ we

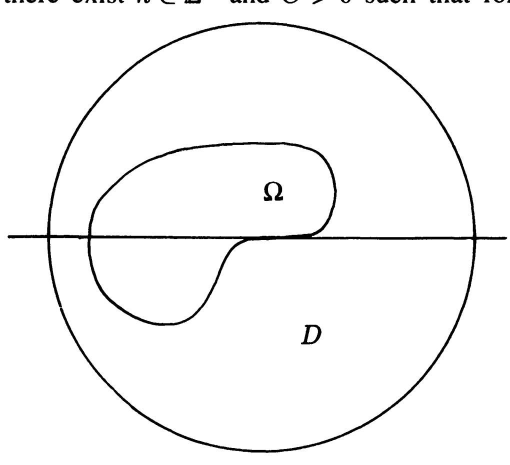

Fig. 1

Pervenuto alla Redazione il 20 Gennaio 1986.

have:

$$\operatorname{dist}^h(x, V \cap \overline{D}) \leq C \operatorname{dist}(x, V)$$

It is a natural question to ask under which assumptions, more in general, $I^\infty(V) \cup \overline{I^\infty(V)}$ generates $\Im^\infty(V)$ over $C^\infty(\overline{D})$.

It is clear that is not always the case:

take e.g.: $V = L = \{z_{n+1} = 0\}$, Ω any bounded domain with C^∞-smooth boundary such that $\overline{\Omega \cap L} = \overline{\Omega} \cap L \neq \emptyset$ and $\overline{\Omega}$ and L are not regularly separated somewhere; let B a ball containing $\overline{\Omega}$ and let finally $D = B \setminus \overline{\Omega}$. Obviously we have $A^\infty(D) = A^\infty(B)$, so $I^\infty(V)$ is generated by z_{n+1} (cf. [1] [4]), while $(z_{n+1}, \overline{z_{n+1}})C^\infty(\overline{D}) \underset{\neq}{\subset} \Im^\infty(V)$.

Of course, pseudoconcavity of D plays an essential role in this example.

The main result of this paper is the following:

THEOREM. *Let $D \subset \mathbb{C}^{n+1}$ be a bounded strictly pseudoconvex domain with C^∞-smooth boundary, let V be a complex submanifold of a neighbourhood of $\overline{D}$ such that $\overline{D \cap V} = \overline{D} \cap V \neq \emptyset$, and let $g_1, \dots, g_k$ be a complete defining system for V.*

Then there exists $m \in \mathbb{Z}^+$ such that for every $f \in \Im^\infty(V)$ one can find $\lambda_1, \dots, \lambda_m \in I^\infty(V)$, $a_1, \dots, a_k$, $b_1, \dots, b_k$, $c_1, \dots, c_m$, $d_1, \dots, d_m \in C^\infty(\overline{D})$ in such a way that:

$$f = \sum_{j=1}^{k} (a_j g_j + b_j \overline{g}_j) + \sum_{h=1}^{m} (c_h \lambda_h + d_h \overline{\lambda}_h).$$

Note that no requirement other than $\overline{D \cap V} = \overline{D} \cap V \neq \emptyset$ is made about the mutual position of D and V.

The general ideas of the proof are the following:

1. Investigating the geometry of $D \cap V$ (Lemmas 1.1 and 1.2) we prove that, in the strictly pseudoconvex case, the area of bad contact (i.e. non regular separation)
between D and V, can be locally included in a totally real submanifold Σ of bD
2. Since Σ is totally real, functions in $I^\infty(V)$ are (relatively) flabby on Σ and so, in some sense, they can be deformed on Σ (Proposition 2.1) in order to reproduce locally any (possibly bad) behaviour of functions in $\Im^\infty(V)$.
3. Using some arguments from [4], we pass from the local result to the Theorem (Lemma 3.1 and proposition 3.2).

As a corollary of the main Theorem, we obtain (Corollary 3.3) that regular separation is necessary and sufficient condition for $I^\infty(V)$ to be generated over $A^\infty(D)$ by $g_1, \dots, g_k$.

The result of Corollary 3.3 can be found in the paper by E. Amar [2], which represented one of the starting points of the present investigation.

Some of the results presented in this paper where announced in [3].

1. - The geometrical situation.

The first step of the proof of the Theorem is to investigate the local geometry of $D \cap V$, especially at those points where V and bD meet non-transversally.

In order to perform this investigation, let $D \subset \mathbb{C}^{n+1}$ be a strictly pseudoconvex domain with C^∞-smooth boundary and let L be a complex hyperplane such that $\overline{L \cap D} = L \cap \overline{D} \neq \emptyset$ and L and bD are not transversal at $x \in L \cap bD$; then it is possible to choose local complex coordinates (z, z_{n+1}), $z = (z_1, \ldots, z_n)$ in a neighbourhood N of x in such a way that

$$\text{i)} \quad T_x^{\mathbb{C}} bD = \{z_{n+1} = 0\} = L, \quad T_x^{\mathbb{R}} bD = \{\text{Re } z_{n+1} = 0\}$$
$$\text{ii)} \quad D \cap N = \{\text{Re } z_{n+1} > r(z, \text{Im } z_{n+1})\}$$

where:

$$r(z, \text{Im } z_{n+1}) = p(z) + \varphi(z) + \psi(z, \text{Im } z_{n+1}),$$

with

a) $p(z) = \bar{z}A^t z + \text{Re } zB^t z$ with $A, B \in M_{n,n}(\mathbb{C})$, $A = A^* > 0$, $B = {}^tB$

b) $\varphi(z) = o(|z|^2)$ for $z \to 0$

c) $\psi(z, \text{Im } z_{n+1}) = O(|\text{Im } z_{n+1}|^2)$ for $\text{Im } z_{n+1} \to 0$.

Let $h(z) = p(z) + \varphi(z)$.

LEMMA 1.1. *Up to complex linear changes of coordinates, we can assume there exist $k, r \in \mathbb{Z}^+$, $0 \le k \le n$, $0 \le r \le n-k$, such that setting $z_j = x_j + iy_j$ and $T = (x_{k+1}, \ldots, x_n, y_{k+1}, \ldots, y_n)$ we have*

$$p(z) = p(x_1, \ldots, x_n, y_1, \ldots, y_n) = 2\sum_{j=1}^{k} y_j^2 + TP^tT,$$

where P is a non-singular symmetric element of $M_{2(n-k),2(n-k)}(\mathbb{R})$ such that: P is positive definite on

$$V^+ = \{x_j = 0, \quad k+1 \le j \le k+r\}$$

and negative definite on

$$V^- = \{z_j = 0, \ y_i = 0, \quad k+r+1 \le j \le n, \ k+1 \le i \le k+r\}.$$

PROOF.

1. Up to an obvious complex linear change of coordinates (c.l.c.c.) we can assume $p(z) = \bar{z}\,{}^tz + \text{Re } zB^tz$.

2. The space of degeneracy of p is given by $W = \{dp = 0\} = \{{}^t\bar{z} + B^tz = 0\}$ and thus it is totally real: up to another c.l.c.c. we can assume there exists $k \in \mathbb{Z}^+$, $0 \le k \le n$ such that

$$W = \{z_{k+1} = \ldots = z_n = 0, \quad y_1 = \ldots = y_k = 0\}.$$

This is equivalent to say

$$B = \begin{pmatrix} -I_k & 0 \\ 0 & A \end{pmatrix} \qquad A = R + iS$$

and so we obtain the description of p we are looking for, setting:

$$P = \begin{pmatrix} I+R & -S \\ -S & I-R \end{pmatrix}.$$

3. By means of the ordinary spectral theorem, we can find an Euclidean-orthonormal, P-orthogonal basis $\mathcal{B} = \{v_1, \dots, v_{2(n-k)}\}$of $\mathbb{C}^{n-k}_{z_{k+1} \dots z_n}$; assume the index of negativity of P is r and ${}^t v_j P v_j < 0$, $1 \le j \le r$; thus P is positive definite on $V^+ = [v_{r+1}, \dots, v_{2(n-k)}]$, which is the Euclidean-orthogonal complement of $V^- = [v_1, \dots, v_r]$; since p is strictly subharmonic when restricted to any complex direction in $\mathbb{C}^{n-k}_{z_{n+1}, \dots, z_n}$, then V^- is totally real and so with a final orthogonal c.l.c.c., we can assume

$$V^- = \{z_j = 0 \;\; y_i = 0 \quad k+r+1 \le j \le n, \;\; k+1 \le i \le k+r\}$$

and consequently:

$$V^+ = \{x_j = 0 \quad k+1 \le j \le k+r\}.$$

LEMMA 1.2. *Assume complex coordinates are chosen in such a way that* p *appears in the normalized form given by Lemma* 1.1*; thus:*

a) *if* $k = 0$, *then there exist a neighbourhood* U *of* 0 *and* $K > 0$ *such that if* $x \in U \cap \overline{D}$ *then*

$$(\#_a): \qquad \operatorname{dist}^2(x, L \cap \overline{D}) \le K \operatorname{dist}(x, L)$$

and so, in particular L *and* $\overline{D}$ *are regularly separated at* 0;

b) *if* $k > 0$, *then there exists a totally real* $(k+r)$*-dimensional* C^∞*-submanifold* S *of* L, *passing through* 0 *for which there exist a neighbourhood* U *of* 0 *and* $K > 0$ *such that if* $\Sigma = (S \times \operatorname{Re} \mathbb{C}_{z_{n+1}}) \cap bD$ *and* $Z = L \cup \Sigma$ *then for every* $x \in U \cap \overline{D}$ *we have*

$$(\#_b): \qquad \operatorname{dist}^2(x, Z \cap \overline{D}) \le K \operatorname{dist}(x, Z)$$

and so, in particular Z *and* $\overline{D}$ *are regularly separated at* 0.

PROOF. First of all note that if $x = (z, z_{n+1}) \in \overline{D}$ then we have

$$\operatorname{Re} z_{n+1} \ge r(z, \operatorname{Im} z_{n+1}) = h(z) + O(|\operatorname{Im} z_{n+1}|^2)$$

and so

$$h(z) \leq \operatorname{Re} z_{n+1} + O(|\operatorname{Im} z_{n+1}|^2) \leq c'(|\operatorname{Re} z_{n+1}| + |\operatorname{Im} z_{n+1}|) \leq c|z_{n+1}|.$$

a) Assume $k = 0$.

1. Since we are interested only in those points $x = (z, z_{n+1}) \in \overline{D}$ where $h(z) > 0$, in order to get (#a), it is enough to prove

$$\operatorname{dist}^2(z, \overline{D} \cap L) \leq c|h(z)| \text{ for } z \in L \text{ near } 0$$

and this condition, of course has nothing to do with the complex structure.

2. Up to a real linear change of coordinates, we can assume

$$p(z) = p(u, v) = |u|^2 - |v|^2$$

where $u = (u_1, \ldots, u_p)$, $v = (v_1, \ldots, v_q)$, $p + q = 2n$.

Recall that $h(u, v) = p(u, v) + \varphi(u, v)$ and $\varphi(u, v) = o(|u|^2 + |v|^2)$ and so, given $\lambda > 0$, let $\rho > 0$ such that, if $|u|^2 + |v|^2 \leq \rho^2$ then $|\varphi(u, v)| < \frac{\lambda}{2}(|u|^2 + |v|^2)$; setting

$$p_\lambda = p + \lambda(|u|^2 + |v|^2) \quad H_\lambda = \{p_\lambda < 0\} \quad A_\lambda = \complement H_{-\lambda},$$

in the ball $B(0, \rho)$ we have:

$$p_{-\lambda} < h < p_\lambda$$

and therefore

i) if $x \in H_\lambda$, then $x \in L \cap \overline{D}$ i.e. $H_\lambda \subset \overline{D} \cap L$

ii) if $x = (u, v) \in A_\lambda$ then $p(u, v) \geq \lambda(|u|^2 + |v|^2)$ and

$$h(x) > p(u, v) - \frac{\lambda}{2}(|u|^2 + |v|^2) \geq \frac{\lambda}{2}(|u|^2 + |v|^2) \geq c \operatorname{dist}^2(x, L \cap \overline{D}),$$

so we have to consider only

$$x \in C_\lambda = \complement(H_\lambda \cup A_\lambda) = \left\{(u, v) \in \mathbb{R}^p \times \mathbb{R}^q; \frac{1 - \lambda}{1 + \lambda}|v|^2 \leq |u|^2 \leq \frac{1 + \lambda}{1 - \lambda}|v|^2\right\}.$$

Let $C = \{p = 0\}$ and let ν be the outward pointing normal unit vector field to $C - \{0\}$, extended to $C_\lambda - \{0\}$; for a fixed small λ, ν defines a projection $\pi: C_\lambda - \{0\} \to C - \{0\}$ thus, for $x = (u, v) \in C_\lambda$, we have

$$\frac{\partial h}{\partial \nu}(x) = \frac{\partial p}{\partial \nu} + o(|x|) \geq c|\pi(x)|;$$

so if $\hat{x} \in C_\lambda \cap L \cap bD$ is a point on the line from x parallel to $\nu(\pi(x))$, we have

$$|h(x)| = |h(x) - h(\hat{x})| \geq c|\pi(x)||x - \hat{x}|$$

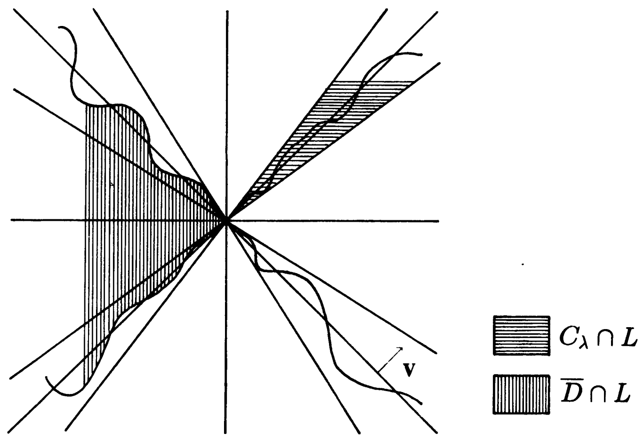

Fig. 2

and since $|\pi(x)| \geq |x - \hat{x}|$, we obtain

$$|h(x)| \geq c|x - \hat{x}| \geq c \operatorname{dist}^2(x, L \cap \overline{D}).$$

b) Assume $k > 0$.

1. Let

$$S = \left\{ (x_1, \ldots, x_n, y_1, \ldots, y_n) \in L \,\middle|\, \frac{\partial h}{\partial x_l} = 0, \right.$$
$$\left. \frac{\partial h}{\partial y_m} = 0, \;\; k + r + 1 \leq l \leq n, \;\; 1 \leq m \leq n \right\}$$

we have $0 \in S$ and so, in virtue of the implicit functions theorem, there exists a neighbourhood U of 0 such that in $L \cap U$:

$$S = \{(x_1, \ldots, x_n, y_1, \ldots, y_n) \in L | x_l = \eta_l(x_1, \ldots, x_{k+r}),$$
$$y_m = \alpha_m(x_1, \ldots, x_{k+r}), k + r + 1 \leq l \leq n, \;\; 1 \leq m \leq n\}$$

for C^∞-smooth functions η_l, α_m: so S is totally real (cf. e.g. [5]); set $\Sigma = (S \times \operatorname{Re} \mathbb{C}_{z_{n+1}}) \cap bD$ and $Z = L \cup \Sigma$.

2. Write $\overline{D \cap U} = \hat{M}_K \cup \hat{N}_K$ where:

$$\hat{M}_K = \{x \in \overline{D \cap U} | \operatorname{dist}^2(x, \Sigma) \leq K \operatorname{dist}(x, L)\} \text{ and } \hat{N}_K = \overline{D \cap U} - \hat{M}_K$$

if $x \in \hat{M}_K$ then

$$\begin{aligned}\operatorname{dist}^2(x, Z \cap \overline{D}) &= \min\{\operatorname{dist}^2(x,\Sigma), \operatorname{dist}^2(x, L \cap \overline{D})\} \leq \operatorname{dist}^2(x,\Sigma) \\ &\leq \begin{Bmatrix} C\operatorname{dist}(x,\Sigma) \\ K\operatorname{dist}(x,L) \end{Bmatrix} \\ &\leq c'\min\{\operatorname{dist}(x,\Sigma), \operatorname{dist}(x,L)\} = c'\operatorname{dist}(x,Z).\end{aligned}$$

3. We have the following

CLAIM 1. *Let*

$$Q = \{(x_1,\dots,x_n,y_1,\dots,y_n) \in L \cap U \\ |h(x_1,\dots,x_{k+r},\eta_{k+r+1},\dots,\eta_n,\alpha_1,\dots,\alpha_n) \geq 0\};$$

if $\pi: \mathbb{C}^{n+1} \to L$ *is the natural projection, then there exists* $K > 0$ *such that if* $x \in \overline{D \cap U}$ *and* $\pi(x) \in Q$, *then* $x \in M_K$.

PROOF OF CLAIM 1. Let $x \in \overline{D}$, $x = (z, z_{n+1})$ with

$$z = (x_1,\dots,x_n,y_1,\dots,y_n) \in Q$$

let $x' = (z,0)$, $x'' = (\hat{z},0)$ where $\hat{z} = (x_1,\dots,x_{k+r},\eta_{k+r+1},\dots,\eta_n,\alpha_1,\dots,\alpha_n)$; of course $\hat{z} \in Q \cap S$; then

$$h(z) = h(\hat{z}) + \frac{1}{2}\operatorname{Hess}(h)(\hat{z})[z-\hat{z}] + O(|z-\hat{z}|^3)$$

where $\operatorname{Hess}(h)(\hat{z})$ is the Hessian quadratic form of h at $\hat{z}$: we have $\operatorname{Hess}(h)$ =$\operatorname{Hess}(p)$+$\operatorname{Hess}(\varphi)$ and, since p is positive definite on $L^+ = \{z \in L | x_j = 0,\quad 1 \leq j \leq k+r\}$, $z - \hat{z} \in L$ and $\varphi(z) = o(|z|^2)$, we obtain

$$h(z) \geq h(\hat{z}) + c|z-\hat{z}|^2 \geq h(\hat{z}) + c'\operatorname{dist}(z,S);$$

so

$$\begin{aligned}\operatorname{dist}(x,\Sigma) \leq \operatorname{dist}(x,x') + \operatorname{dist}(x',\Sigma) &= |z_{n+1}| + \operatorname{dist}(x',\Sigma) \\ &\leq |z_{n+1}| + \operatorname{dist}(x',x'') + \operatorname{dist}(x'',\Sigma).\end{aligned}$$

Now we have:

i) $\operatorname{dist}(x',x'') \leq c_2 \operatorname{dist}(x',S)$

ii) since $(\hat{z}, h(z)) \in \Sigma$:

$$\operatorname{dist}(x'',\Sigma) \leq \operatorname{dist}(x'',(\hat{z},h(\hat{z}))) = h(\hat{z}) < h(z);$$

so:

$$\begin{aligned}\operatorname{dist}^2(x,\Sigma) &\leq c_3(|z_{n+1}|^2 + \operatorname{dist}^2(z,S) + h^2(z)) \\ &\leq c_4(|z_{n+1}|^2 + h(z)) \leq K|z_{n+1}| = K\operatorname{dist}(x,L)\end{aligned}$$

and the proof of claim 1 is complete.
4. Next step is the following:

CLAIM 2. *If* $x \in \overline{D \cap U}$ *and* $\pi(x) \notin Q$, *then there exists* $K > 0$ *such that*

$$\operatorname{dist}^2(x, L \cap \overline{D}) \leq K \operatorname{dist}(x, L).$$

PROOF OF CLAIM 2. It is enough to show that if $x = (z, z_{n+1}) \in \overline{D \cap U}$ and $z \notin Q \cup (L \cap \overline{D})$ then $h(z) \geq c \operatorname{dist}^2(z, L \cap \overline{D})$; now for such an x we have $h(z) > 0$ while $h(\hat{z}) = h(x_1, \dots, x_{k+r}, \eta_{k+r+1}, \dots, \eta_n, \alpha_1, \dots, \alpha_n) < 0$; in the segment $[\hat{z}, z]$, consider the last point $\tilde{z}$ such that $h(\tilde{z}) = 0$ and let $f(t) = h((1-t)\tilde{z} + tz)$ since $f''(t) = \operatorname{Hess}(h)((1-t)\hat{z} + tz)[z - \hat{z}] \geq c|z - \hat{z}|^2$, then $f(t)$ is a convex increasing function in $[0, 1]$; moreover we have:

$$h(z) = f(1) = f(0) + f'(0) + \frac{1}{2} f''(\hat{t}) \text{ for } \hat{t} \in [0, 1];$$

since $f(0) = h(\tilde{z}) = 0$, $f'(0) \geq 0$, we obtain precisely

$$h(z) \geq c \operatorname{dist}^2(x, L \cap \overline{D}).$$

5. Summing up:
given $x \in \overline{D \cap U}$, if $\pi(x) \in Q$, then by claim 1, $x \in M_K$ and so $\operatorname{dist}^2(x, Z \cap \overline{D}) \leq c_1 \operatorname{dist}(x, Z)$; if $\pi(x) \notin Q$, then by claim 2, $\operatorname{dist}^2(x, L \cap \overline{D}) \leq c_2 \operatorname{dist}(x, L)$ and so

$$\begin{aligned}
\operatorname{dist}^2(x, Z \cap \overline{D}) &= \min\{\operatorname{dist}^2(x, \Sigma), \operatorname{dist}^2(x, L \cap \overline{D})\} \\
&\leq c_2 \min\{\operatorname{dist}(x, \Sigma), \operatorname{dist}(x, L)\} \\
&= c_2 \operatorname{dist}(x, Z)
\end{aligned}$$

and the proof of Lemma 1.2 is complete.

REMARK 1.3. a) lemma 1.2 asserts essentially that if D is strictly pseudoconvex, then $\overline{D}$ and L are not regularly separated at most "along" a totally real submanifold Σ of bD (see [2] for some partial results in this direction);

b) it follows from Lemma 1.2 and Whitney extension theorems (cf. e.g. [7]) that if $f \in \Im^\infty(L)$ and f is infinitely flat on Σ then it is possible to find a C^∞-smooth extension F of f around $\overline{D \cap U}$, vanishing on $L \cap U$.

2. - The semi-local case.

Lemma 1.2 enables us to prove the following semi-local version of the main Theorem:

PROPOSITION 2.1. *Let* $D \in \mathbb{C}^{n+1}$ *be a bounded strictly pseudoconvex domain with* C^∞*-smooth boundary and let* $g \in \mathcal{O}(D')$, *where* $D \subset\subset D'$, *such that, if*

$V = \{g = 0\}$, then $\overline{V \cap D} = V \cap \overline{D} \neq \emptyset$; let $x \in \overline{D}$ such that $\partial g(x) \neq 0$: then for every neighbourhood U of x, there exists another neighbourhood W of x such that if $f \in C^\infty(\overline{U})$ and $f|_{U\cap D\cap V} \equiv 0$ then for every pseudoconvex domain $\tilde{D}$ with C^∞-smooth boundary such that $D \subset \tilde{D} \subset\subset D'$ and $D \cap W = \tilde{D} \cap W$, we can find $\lambda \in A^\infty(\tilde{D})$ such that $\lambda|_D \in I^\infty(V)$, and $a_1, \ldots, a_4 \in C^\infty(\overline{D})$, in such a way that on $\overline{W \cap D}$ we have

$$f = a_1 g + a_2 \overline{g} + a_3 \lambda + a_4 \overline{\lambda}.$$

PROOF. 1. We can assume $x \in bD \cap V$ otherwise there is almost nothing to prove.

2. If V and bD are transversal at x, we obtain the result with $\lambda \equiv 0$, using the well-known techniques for the regularly separated case.

3. If V and bD are not transversal at x, then we can choose complex coordinates near x in such a way that $z_{n+1} = g$ (and so we can identify near x, V with $L = \{z_{n+1} = 0\} = T_x^{\mathbb{C}} bD$); performing the c.l.c.c. as in Lemma 1.1, again we can assume $k > 0$ and construct S, Σ, Z as in Lemma 1.2 b), in a neighbourhood $W' \subset U$ of O.

4. Let $f \in C^\infty(\overline{U})$ such that $f|_{U\cap D\cap V} \equiv 0$; choose $j \in \mathbb{Z}^+$ in such a way that if $\tilde{f} = f + jg$ then

$$\left|\frac{\partial \tilde{f}}{\partial z_{n+1}}\right| - \left|\frac{\partial \tilde{f}}{\partial \overline{z}_{n+1}}\right| \neq 0$$

in W'; let $M = \{x \in W' | \tilde{f} = 0\}$: then it is possible to find $\varphi \in C^\infty(L, \mathbb{C})$ such that $\varphi|_{L\cap\overline{D}} \equiv 0$ and

$$M = \{\varphi(z_1, \ldots, z_n) = z_{n+1}\} \cap W'$$

then (cf. e.g. [7]) in $W' \cap D$ we have

$$\tilde{f} = a(\varphi - z_{n+1}) + b(\overline{\varphi - z_{n+1}}) \text{ for } a, b \in C^\infty(\overline{D});$$

we want to factorize φ.

We need two preliminary lemmas; first of all let

$$\mathcal{E} = \{\sigma \in C^\infty(\mathbb{R}^+, \mathbb{R}^+) | \text{ for every } k \in \mathbb{Z}^+ \sigma^{(k)}(o) = 0, \sigma'(x) > 0 \text{ if } x > 0\}$$

then we have:

LEMMA 2.2 *Given $\varphi \in C^\infty(L, \mathbb{C})$ such that $\varphi|_{L\cap\overline{D}} \equiv 0$, it is possible to find $\hat{\varphi} \in C^\infty(L, \mathbb{R})$ such that $\{\hat{\varphi} = 0\} = L \cap \overline{D}$ and $\sigma \in \mathcal{E}$ in such a way that*

$$\sigma(\hat{\varphi}(z)) \geq |\varphi(z)|$$

PROOF. For any $\varepsilon > 0$, let $K_\varepsilon = \{z \in L | \operatorname{dist}(z, L \cap \overline{D}) \leq \varepsilon\}$ and let $\lambda(\varepsilon) = \sup_{K_\varepsilon} |\varphi(z)|$ thus we have: $\lambda(\varepsilon) \searrow 0$ if $\varepsilon \searrow 0$ and $\lambda(\varepsilon) = o(\varepsilon^k)$ for every $k \in \mathbb{Z}^+$; so it is possible to find $\hat{\lambda}$, $\hat{\mu} \in \mathcal{E}$ such that:
i) $\hat{\lambda} > \lambda$,
ii) $\hat{\lambda} = o(\hat{\mu}^k)$ for every $k \in \mathbb{Z}^+$ and so $\hat{\lambda} = \sigma \circ \hat{\mu}$ for $\sigma \in \mathcal{E}$.
Let now $\rho \in C^\infty(L \setminus \overline{D})$ such that for $z \in L \setminus \overline{D}$

$$\operatorname{dist}(z, L \cap \overline{D}) \leq \rho(z) \leq 2 \operatorname{dist}(z, L \cap \overline{D})$$

and set

$$\hat{\varphi}(z) = \begin{cases} \hat{\mu}(\rho(z)) \text{ on } L \setminus \overline{D} \\ \quad 0 \text{ on } L \cap \overline{D} \end{cases}$$

thus $\hat{\varphi} \in C^\infty(L, \mathbb{R})$, $\{\hat{\varphi} = 0\} = L \cap \overline{D}$ and

$$\begin{aligned} \sigma(\hat{\varphi}(z)) = \sigma \circ \hat{\mu}(\rho(z)) &\geq \sigma \circ \hat{\mu}(\operatorname{dist}(z, L \cap \overline{D})) \\ &= \hat{\lambda}(\operatorname{dist}(z, L \cap \overline{D})) \geq \lambda(\operatorname{dist}(z, L \cap \overline{D})) \geq |\varphi(z)|. \end{aligned}$$

LEMMA 2.3. *Let $a \in C^\infty(L, \mathbb{C})$ such that $a|_{L \cap D} \equiv 0$; set $A(z_1, \ldots, z_n, z_{n+1}) = a(z_1, \ldots, z_n)$: then the following facts are equivalent:*
i) $a(z) = o(|h(z)|^k)$ for $z \to L \cap \overline{D \cap W'}$ and every $k \in \mathbb{Z}^+$
ii) $A|_{\overline{D \cap W'}}$ admits a C^∞-smooth extension around $\overline{D \cap W'}$ vanishing on $L \cap W'$.

PROOF. i)⇒ii) we claim that, if $\alpha = (\alpha_1, \ldots, \alpha_{n+1}, \alpha_{\overline{1}}, \ldots, \alpha_{\overline{n+1}}) \in (\mathbb{Z}^+)^{2n+2}$, setting

$$f_\alpha(x) = \begin{cases} 0 \text{ if } \alpha_{n+1} + \alpha_{\overline{n+1}} > 0 \\ \begin{cases} D^\alpha A(x) \text{ if } x \in \overline{D \cap W'} \\ 0 \qquad \text{ if } L \setminus \overline{D \cap W'} \end{cases} \end{cases}$$

then the $(f_\alpha)_{\alpha \in (\mathbb{Z}^+)^{2n+2}}$ are, under assumption i), Whitney data on $\overline{(D \cap L) \cap W'}$ i.e. for any $\alpha \in (\mathbb{Z}^+)^{2n+2}$, any $m \in \mathbb{Z}^+$

$$f_\alpha(x) = \sum_{|\beta| \leq m} \frac{1}{\beta!} f_{\alpha+\beta}(y)(x-y)^\beta + o(|x-y|^m)$$

uniformly for $|x - y| \to 0$; in fact:
1) if $x, y \in \overline{D \cap W'}$ or $x, y \in L \cap W'$, we have nothing to prove;
2) if $x \in \overline{D \cap W'} \setminus L$, $y \in L \cap W'$, from i) it follows that, for any $\alpha \in (\mathbb{Z}^+)^{2n+2}$ such that $\alpha_{n+1} + \alpha_{n+1} = 0$ and any $m \in \mathbb{Z}^+$, setting $x = (z, z_{n+1})$, we have:

$$f_\alpha(x) = D^\alpha a(z) = o(|h(z)|^m)$$

and $|h(z)| \leq c(|z_{n+1}| + |z-y|) \leq c'|x-y|$;

3) if $x \in L \cap W'$, $y \in \overline{D \cap W'} \setminus L$, $y = (z, z_{n+1})$ then for any $\alpha \in (\mathbb{Z}^+)^{2n+2}$, any $m \in \mathbb{Z}^+$

$$f_\alpha(x) - \sum_{|\beta| \leq m} \frac{1}{\beta!} D^{\alpha+\beta} A(y)(x-y)^\beta$$
$$= -D^\alpha a(x) + o(|x-y|^m) = o(|x-y|^m)$$

and so ii) follows from Whitney extension theorems (cf. e.g. [7]).

ii)$\Rightarrow$i) let F be the extension in assumption ii); if $z \in L \cap W'$, let $x = (z, h(z))$, $y = (z, 0)$: if $\alpha = (\alpha_1, \dots, \alpha_n, 0, \alpha_{\overline{1}}, \dots, \alpha_{\overline{n}}, 0) \in (\mathbb{Z}^+)^{2n+2}$ then we have:

$$D^\alpha a(z) = D^\alpha F(z) = \sum_{|\beta| \leq m} \frac{1}{\beta!} D^{\alpha+\beta} F(y)(x-y)^\beta + o(|x-y|^m)$$
$$= o(|x-y|^m) = o(|h(z)|^m).$$

Going back to the proof of Proposition 2.1, using Lemma 2.2, we can find $\hat{\varphi} \in C^\infty(L, \mathbb{R})$ and $\sigma \in \mathcal{E}$ such that $\{\hat{\varphi} = 0\} = L \cap \overline{D}$ and $\sigma(\hat{\varphi}(z)) \geq |\varphi(z)|$.

We can find also $\omega, q, \alpha \in \mathcal{E}$ such that

$$\omega \circ q \circ \alpha = \sigma$$

and so setting $s = \alpha \circ \hat{\varphi}$ we obtain

$$\varphi(z) = o(|q(s)(z)|^k)$$

for $z \to L \cap \overline{D \cap W'}$ and every $k \in \mathbb{Z}^+$; since $\varphi \equiv 0$ when $h(z) \leq 0$, we have also

$$(\#) \qquad \varphi(z) = o(|h(z) + q(s)(z)|^k)$$

for $z \to L \cap \overline{D \cap W'}$ and every $k \in \mathbb{Z}^+$.

Let now $F: \mathbb{C}^{n+1}_z \to \mathbb{C}^{n+1}_w$ defined by

$$\begin{cases} w_j = z_j & 1 \leq j \leq n \\ w_{n+1} = q(s)(z_1, \dots, z_n) + z_{n+1} \end{cases}$$

and $G = F^{-1}: \mathbb{C}^{n+1}_w \to \mathbb{C}^{n+1}_z$

$$\begin{cases} z_j = w_j & 1 \leq j \leq n \\ z_{n+1} = w_{n+1} - q(s)(w_1, \dots, w_n) \end{cases}$$

be C^∞-smooth changes of coordinates: then

$$F(D \cap W') = \{\operatorname{Re} w_{n+1} > r'(w_1, \dots, w_n, \operatorname{Im} w_{n+1})\}$$

where

$$r'(w_1,\dots,w_n,\mathrm{Im}\ w_{n+1}) = r(w_1,\dots,w_n,\mathrm{Im}\ w_{n+1}) + q(s)(w_1,\dots,w_n)$$

and so

$$h'(w_1,\dots,w_n) = h(w_1,\dots,w_n) + q(s)(w_1,\dots,w_n).$$

Setting

$$\Phi(w_1,\dots,w_n,w_{n+1}) = \varphi(w_1,\dots,w_n),$$

using (#) and Lemma 2.3, we obtain that $\Phi|_{F(D\cap W')}$ admits an extension which is C^∞-smooth around $\overline{F(D\cap W')}$ and vanishes on $M = \{w_{n+1} = 0\}$ and so $\Phi|_{D\cap W'}$ admits an extension which is C^∞-smooth around $\overline{D\cap W'}$ and vanishes on

$$(G(M) = \{q(s)(z_1,\dots,z_n) + z_{n+1} = 0\}) \cap W';$$

since Φ is $\overline{n+1}$-flat on $L\cap D\cap W'$, this implies (cf. [4]) that it is possible to find $c \in C^\infty(\overline{D})$ such that on $\overline{D\cap W'}$ we have

$$\varphi(z) = c(z,z_{n+1})(q(s)(z_1,\dots,z_n) + z_{n+1}).$$

We want to factorize $q(s)$.

5. Let $W \subset B_{n+1}(0,\varepsilon/2) \subset B_{n+1}(0,\varepsilon) \subset W'$ be a neighbourhood of O and let $\chi \in C_0^\infty(W'\cap L)$, $\chi \equiv 1$ on $W\cap L$; set $\hat{s} = \chi\cdot s$. Since S is totally real we can find (cf. [5]) $\tilde{s} \in C^\infty(L,\mathbb{C})$ such that

1) $\tilde{s}|_{S\cap W'} = \hat{s}|_{S\cap W'}$
2) $\overline{\partial}\tilde{s}|_{S\cap W'} = 0$ up to infinite order
3) $\mathrm{supp}\tilde{s} \subset \mathrm{supp}\hat{s}$;

let $\beta \in C_0^\infty(\mathbb{C})$ such that $\mathrm{supp}\beta \subset B(0,\varepsilon)$, $\beta \equiv 1$ on $B(0,\varepsilon/2)$: thus setting

$$\check{s}(z_1,\dots,z_{n+1}) = \beta(z_{n+1})\tilde{s}(z_1,\dots,z_n)$$

we have that $\overline{\partial}\check{s}$, as element of $C^\infty_{(0,1)}(\overline{D\cap W})$, is infinitely flat on Σ and since $Z = L\cup\Sigma$ and $\overline{D}$ are, by Lemma 1.2 b), regularly separated at O, then the data

$$\begin{cases} D^\alpha\overline{\partial}\check{s} & \text{on } \overline{D\cap W} \\ 0 & \text{on } \overline{Z\cap W} \end{cases}$$

as Whitney data coinciding on the intersection, are Whitney data on $\overline{(D\cup Z)\cap W}$ (cf. e.g. [7]) i.e. $\overline{\partial}\check{s}|_{D\cap W}$ admits an extension C^∞-smooth around $\overline{D\cap W}$ vanishing on $L\cap W$, and so

$$\alpha = \frac{\overline{\partial}\check{s}}{z_{n+1}} \in C^\infty_{(0,1)}(\overline{D\cap W});$$

since, for a suitable ε, supp $\overline{\partial}\check{s} \subset W'$, we have

$$\alpha = \frac{\overline{\partial}\check{s}}{g} \in C^{\infty}_{(0,1)}(\overline{\tilde{D}})$$

for any domain $\tilde{D}$ as in the statement of Proposition 2.1; thus, following [6], it is possible to find $u \in C^{\infty}(\overline{\tilde{D}})$ such that $\overline{\partial}u = \alpha$ on $\tilde{D}$ and

$$\lambda = gu - \check{s} \in A^{\infty}(\tilde{D}), \quad \lambda|_{\overline{D}} \in I^{\infty}(V).$$

6. Extend now q to $\mathbb{C}_{\varsigma}$ in the obvious way: $q(\varsigma) = q(|\varsigma|)$; then we have

$$q(\varsigma + \eta) = q(\varsigma) + \hat{a}\eta + \hat{b}\overline{\eta} \quad \text{for} \quad \hat{a}, \hat{b} \in C^{\infty}(\mathbb{C});$$

we obtain on $W \cap D$

$$s = s - \check{s} + \check{s} = s - \check{s} + gu - \lambda$$

and

$$q(s) = q(s - \check{s}) + \hat{a}\cdot(gu - \lambda) + \hat{b}(\overline{gu - \lambda})$$

where $q(s - \check{s})$ as element of $C^{\infty}(\overline{D \cap W})$ is infinitely flat on Σ and, by the same argument as before,

$$q(s - \check{s}) = d \cdot g \text{ for } d \in C^{\infty}(\overline{D});$$

thus we have on $W \cap D$

$$q(s) = d \cdot g + \hat{a}\cdot(gu - \lambda) + \hat{b}\cdot(\overline{gu - \lambda})$$
$$\varphi = c \cdot [(d + \hat{a}u + 1)\cdot g + \hat{b}\overline{ug} - \hat{a}\lambda - \hat{b}\overline{\lambda}]$$

and, putting everything together, we obtain finally:

$$f = a_1 g + a_2\overline{g} + a_3\lambda + a_4\overline{\lambda}$$

with $a_1, a_2, a_3, a_4 \in C^{\infty}(\overline{D})$.

REMARK 2.4. In general it is not possible to simplify the representation of a C^{∞}-smooth function by means of holomorphic functions, given in Proposition 2.1, i.e., given $f \in \Im^{\infty}(V)$, in general it is not possible to find a single $\lambda \in I^{\infty}(V)$ such that, at least locally

$$f = a\lambda + b\overline{\lambda} \text{ for } a, b \in C^{\infty}(\overline{D}).$$

In fact, let $V = L = \{z_{n+1} = 0\}$ and $f \in \Im^{\infty}(L)$ such that:

$$\text{i)} \ \left|\frac{\partial f}{\partial z_{n+1}}\right| - \left|\frac{\partial f}{\partial \overline{z}_{n+1}}\right| \not\equiv 0$$

$$\text{ii)} \ \{f = 0\} \cap D \supsetneq L \cap D$$

(and this is possible whenever L has an infinite order of contact with bD along some real direction); if $f = a\lambda + b\overline{\lambda}$ with $\lambda \in I^{\infty}(L)$ and $a, b \in C^{\infty}(\overline{D})$, from i) we obtain

$$(|a|^2 - |b|^2)\left|\frac{\partial \lambda}{\partial z_{n+1}}\right|^2 \not\equiv 0$$

and

$$\lambda = (\overline{a}f - b\overline{f})(|a|^2 - |b|^2)^{-1};$$

thus $\{\lambda = 0\}$ is a complex submanifold of D containing $\{f = 0\}$: contradiction.

3. - The general case.

Our next step is to extend Proposition 2.1 to the case of arbitrary codimension.

Consider first the case V is a linear submanifold; in this direction, we have the following

LEMMA 3.1. *Let $D \subset \mathbb{C}^{n+1}$ be a bounded strictly pseudoconvex domain with C^{∞}-smooth boundary and let $V = \{z_{k+1} = \cdots = z_{n+1} = 0\}$; assume*

$$\overline{D \cap V} = \overline{D} \cap V \not\equiv \emptyset;$$

let $x \in \overline{D}$: then for every neighbourhood U of x, there exists another neighbourhood W of x such that, if $f \in C^{\infty}(\overline{U})$ and $f|_{U \cap D \cap V} \equiv 0$, then it is possible to find $\lambda \in I^{\infty}(V)$ and $a, b, a_{k+1}, \dots, a_{n+1}, b_{k+1}, \dots, b_{n+1} \in C^{\infty}(\overline{D})$ in such a way that on $\overline{W \cap D}$ we have

$$f = \sum_{j=k+1}^{n+1} (a_j z_j + b_j \overline{z}_j) + a\lambda + b\overline{\lambda}.$$

PROOF. 1. We can assume $x \in bD \cap V$, V and bD are not transversal at x and therefore, e.g. $T_x^{\mathbb{C}} bD = L = \{z_{n+1} = 0\}$.

2. Let $M = \{z_{k+1} = \cdots = z_n = 0\}$: thus bD and M are transversal at x and therefore in a neighbourhood $W \subset U$ of x: thus we can find another strictly pseudoconvex domain $\tilde{D} \supset D$ such that $D \cap W = \tilde{D} \cap W$ and M and $b\overline{D}$ are transversal everywhere, so $\tilde{D}^{(1)} = M \cap \tilde{D}$ is a strictly pseudoconvex $(k+1)$-dimensional domain with C^{∞}-smooth boundary.

Let $f \in C^{\infty}(\overline{U})$ such that $f|_{D \cap U \cap V} \equiv 0$; since V is 1-codimensional in $\tilde{D}^{(1)}$, applying *proposition 2.1.* to $\tilde{D}^{(1)}$ and $f|_{U \cap M}$, we can find $a_{n+1}, b_{n+1}, a, b \in C^{\infty}(\overline{\tilde{D}})$, $\mu \in A^{\infty}(\tilde{D}^{(1)})$, $\mu|_{D^{(1)} \cap V} \equiv 0$ such that, on $\overline{\tilde{D}^{(1)} \cap W}$

$$f = a_{n+1} z_{n+1} + b_{n+1} \overline{z}_{n+1} + a\mu + b\overline{\mu}$$

Now, since M and $b\tilde{D}$ are transversal, by [4] (Lemma 2 ii)), it is possible to find $\lambda \in A^{\infty}(\tilde{D})$ such that $\lambda|_{\tilde{D}^{(1)}} = \mu$, so if

$$F = a_{n+1}z_{n+1} + b_{n+1}\bar{z}_{n+1} + a\lambda + b\bar{\lambda}$$

we have $(F - f)|_{(D\cap W)\cap M} = 0$ and again on $\overline{D \cap W}$

$$F - f = \sum_{j=k+1}^{n} (a_j z_j + b_j \bar{z}_j)$$

for $a_j,\ b_j \in C^{\infty}(\overline{D})$, $1 \leq j \leq n$, so the proof of Lemma 3.1 is complete.

We have now the following

PROPOSITION 3.2. *Let D, V, $g_1, \dots, g_k$ as in the main Theorem and assume $g_j \in \mathcal{O}(D')$ $1 \leq j \leq k$, where $D' \supset \overline{D}$; then, for every neighbourhood U of x there exists another neighbourhood W of x such that for every function $f \in C^{\infty}(\overline{U})$ such that $f|_{D\cap U\cap V} \equiv 0$, it is possible to find $\lambda \in I^{\infty}(V)$ and a, b, $a_1, \dots, a_k$, $b_1, \dots, b_k \in C^{\infty}(\overline{D})$ in such a way that in $\overline{W \cap D}$ we have*

$$f = \sum_{j=1}^{k} (a_j g_j + b_j \bar{g}_j) + a\lambda + b\bar{\lambda}.$$

PROOF 1. As usual, we can assume $x \in V \cap bD$; let $G : D' \to \mathbb{C}^k$ be the holomorphic map given by $G(z) = (g_1(z), \dots, g_k(z))$ and let Γ be its graph.

2. Let $f \in C^{\infty}(\overline{U})$ such that $f|_{D\cap U\cap V} \equiv 0$; since $(g_1, \dots, g_k)$ is a complete defining system for V, we can find (cf. [4], Lemma 5) a neighbourhood A of x in $\mathbb{C}^{n+1} \times \mathbb{C}^k$ and complex coordinates $v_1, \dots, v_q$, $q = n + 1 + k$, in such a way that

$$A \cap \mathbb{C}^{n+1} = \{v_{n+2} = \cdots = v_q = 0\}$$
$$A \cap \Gamma = \{v_{n+2-d} = \cdots = v_{n+1-d+k} = 0\}$$

where $d = n + 1 - \dim_{\mathbb{C}} V \leq k$, thus, since $\Gamma \cap D' = V$,

$$V \cap A = \{v_{n+2-d} = \cdot = v_q = 0\}.$$

3. Let now $W \subset\subset W' \subset U$ be two neighbourhoods of x in $\mathbb{C}^{n+1}$ such that $A \cap \mathbb{C}^{n+1} \supset W'$ and let $\rho = C_0^{\infty}(W')$ such that $\rho \equiv 1$ on W; set $\tilde{f} = \rho f$; setting

$$\tilde{F}(v_1, \dots, v_q) = \tilde{f}(v_1, \dots, v_{n+1}) \quad \text{for} \quad (v_1, \dots, v_q) \in [(W' \cap D) \times \mathbb{C}^k] \cap A$$

we obtain $\tilde{F}|_{\Gamma\cap[(W'\cap D)\times\mathbb{C}^k]\cap A} = 0$ so we can construct in $D' \times \mathbb{C}^k$ a strictly pseudoconvex domain B with C^{∞}-smooth boundary such that

i) $B \cap (D' \times \{0\}) = D$

ii) $B \cap A \subset [(W' \cap D) \times \mathbb{C}^k] \cap A$

and we can extend $\tilde{F}$ to an element F of $C^\infty(\overline{B})$ in such a way that $F|_{\Gamma\cap B} \equiv 0$ and $F|_{D\cap W} = f$.

4. Now $\Gamma \cap B$ is holomorphically equivalent to a plane section, thus, using Lemma 3.1., we can find a neighbourhood $\tilde{W}$ of x in $\mathbb{C}^{n+1} \times \mathbb{C}^k$, $\Lambda \in A^\infty(B)$ such that $\Lambda|_{\Gamma\cap B} \equiv 0$, $\tilde{a}, \tilde{b}$, $\tilde{a}_1, \ldots, \tilde{a}_k$, $\tilde{b}_1, \ldots, \tilde{b}_k \in C^\infty(\overline{B})$ in such a way that on $\overline{B \cap \tilde{W}}$

$$F = \sum_{j=1}^{k} [a_j \cdot (g_j - w_j) + b_j \cdot (\overline{g_j - w_j})] + \tilde{a}\Lambda + \tilde{b}\overline{\Lambda}$$

and therefore, setting

$$a_j = \tilde{a}_j|_{\overline{D}}, \quad b_j = \tilde{b}_j|_{\overline{D}}, \quad 1 \leq j \leq k,$$
$$a = \hat{a}|_{\overline{D}}, \quad b = \hat{b}|_{\overline{D}}, \quad \lambda = \Lambda|_{\overline{D}} \in I^\infty(V),$$

we obtain precisely

$$f = \sum_{j=1}^{k} (a_j g_j + b_j \overline{g}_j) + a\lambda + b\overline{\lambda}.$$

We are now in the position to prove our main Theorem: using Proposition 3.2, we can construct an open cover $\mathcal{U} = (W^{(h)})_{1\leq h\leq m}$ of $\overline{D}$ in such a way that, for every $f \in \mathfrak{I}^\infty(V)$ one can find $\lambda_1, \ldots, \lambda_m \in I^\infty(V)$, $a_1^{(h)}, \ldots, a_k^{(h)}$, $b_1^{(h)}, \ldots, b_k^{(h)}$, $c^{(h)}, d^{(h)} \in C^\infty(\overline{D})$ $1 \leq h \leq m$ such that on $\overline{D \cap W^{(h)}}$

$$f = \sum_{j=1}^{k} (a_j^{(h)} g_j + b_j^{(h)} \overline{g}_j) + c^{(h)}\lambda_h + d^{(h)}\overline{\lambda}_h.$$

Let $\mathcal{A}$ be the sheaf on $\overline{D}$ of germs of functions C^∞-smooth up to bD and let

$$\mathcal{B} = (g_1, \ldots, g_k, \overline{g}_1, \ldots, \overline{g}_k, \lambda_1, \ldots, \lambda_m, \overline{\lambda}_1, \ldots, \overline{\lambda}_m)\mathcal{A}$$

thus $f \in H^\circ(\overline{D}, \mathcal{B})$.

Consider the exact sequence of sheaves

$$O \longrightarrow \mathcal{R} \longrightarrow \mathcal{A}^{\oplus 2(k+m)} \xrightarrow{\mu} \mathcal{B} \rightarrow O$$

where:

$$\mu(a_1, \ldots a_k, b_1 \ldots b_k, c_1 \ldots c_m, d_1, \ldots d_m) = \sum_{j=1}^{k} (a_j g_j + b_j \overline{g}_j) + \sum_{h=1}^{m} (c_h \lambda_h + d_h \overline{\lambda}_h)$$

and $\mathcal{R}$ is the sheaf of relations C^∞-smooth up to bD between $g_1, \ldots, g_k$, $\overline{g}_1, \ldots, \overline{g}_k, \lambda_1, \ldots, \lambda_m, \overline{\lambda}_1, \ldots, \overline{\lambda}_m$; since $\mathcal{R}$ is a fine sheaf, passing to the cohomology sequence, we obtain:

$$O \longrightarrow H^\circ(\overline{D}, \mathcal{R}) \longrightarrow [H^\circ(\overline{D}, \mathcal{A})]^{\oplus 2(k+m)} \xrightarrow{\mu} H^\circ(\overline{D}, \mathcal{B}) \longrightarrow O$$

is exact and this concludes the proof of the main Theorem.

From the main Theorem we can deduce the following (cf. also [2]).

COROLLARY 3.3. *Let D, V, $g_1, \dots, g_k$ as in the main Theorem; then the following statements are equivalent:*
i) *$\overline{D}$ and V are regularly separated;*
ii) *$g_1, \dots, g_k$ generate $I^\infty(V)$ over $A^\infty(D)$.*

PROOF. i)⇒ii): see [1] and [4].
ii)⇒i) if $g_1, \dots, g_k$ generate $I^\infty(V)$ over $A^\infty(D)$, from the main Theorem it follows that $g_1, \dots, g_k$, $\overline{g}_1, \dots, \overline{g}_k$ generate $\Im^\infty(V)$ over $C^\infty(\overline{D})$, so (see introduction) $\overline{D}$ and V are regularly separated.

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