ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

W. M. TULCZYJEW

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Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 14, nº 2 (1987), p. 217-227

<http://www.numdam.org/item?id=ASNSP_1987_4_14_2_217_0>

Cohomology of the Lagrange Complex

W.M. TULCZYJEW

Introduction

The Lagrange complex was introduced in connection with the inverse problem of the calculus of variations of curves in a differential manifold [6], [7]. Only the Poincaré lemma was proved. The cohomology of the complex was not studied. Similar complexes were introduced later by other authors in the calculus of variations of sections of differential fibrations and cohomology theorems were proved [1], [5], [9]. These results are not applicable to the calculus of variations of curves. Cohomology of the Lagrange complex is studied in the present note. A theorem relating the cohomology of the complex to de Rham cohomology of the manifold is proved. Bundles of k-velocities introduced by Ehresmann [2] are used together with the theory of derivations formulated by Frölicher and Nijenhuis [3] and adapted to the present application in [4]. The relation between the Lagrange complex and the Euler-Lagrange equations of the calculus of variations was discussed in [8].

1. - Bundles of k-velocities

Let M be a differential manifold. In the set $C^\infty(\mathbb{R}, M)$ of smooth curves in M we introduce an equivalence relation for each nonnegative integer k. Two curves $\gamma:\mathbb{R} \to M$ and $\gamma':\mathbb{R} \to M$ are k-equivalent if $D_0^j(f \circ \gamma') = D_0^j(f \circ \gamma)$ for each smooth function f on M and $j = 0, \dots, k$). The symbol $D_0^j F$ denotes the derivative of order j of a function $F:\mathbb{R} \to \mathbb{R}$ at 0. The k-equivalence class of a curve γ will be denoted by $t_0^k\gamma$. The set of all k-equivalence classes of smooth curves is denoted by T^kM and called the bundle of k-velocities in M. The bundle projection $\tau_M^k: T^kM \to M$ is defined by $\tau_M^k(t_0^k\gamma) = \gamma(0)$.

Pervenuto alla Redazione il 6 Febbraio 1986.

For $k' \geq k$ a mapping $\tau_M^{k,k'}: T^{k'}M \to T^kM$ is defined by $\tau_M^{k,k'}(t_0^{k'}\gamma) = t_0^k\gamma$. Relations $\tau_M^{k,k'} \circ \tau_M^{k',k''} = \tau_M^{k,k''}$ hold for $k'' \geq k' \geq k$. The mapping τ_M^0 is bijective. Consequently we indentify T^0M with M and $\tau_M^{0,k}$ with τ_M^k. The 1-velocity bundle T^1M is the tangent bundle TM of M.

Let f be a smooth function on M. For $j = 0, \ldots, k$ we define functions f_j on T^kM by $f_j(t_0^k\gamma) = D_0^j(f \circ \gamma)$. For $j < 0$ we set $f_j = 0$. The bundle T^kM can be given a unique structure of a differential manifold such that if $(x^k)_{k=1,\ldots,m}$ are coordinates in an open submanifold $U \subset M$ then $(x_j^k)_{k=1,\ldots,m,\, j=0,\ldots,k}$ are coordinates in $T^kU = (\tau_M^k)^{-1}(U) \subset T^kM$. Mappings $\tau_M^{k,k'}$ are differential fibrations. The fibration $\tau_M^k: T^kM \to M$ is of fibre-type $\mathbb{R}^{km}$ if m is the dimension of M.

2. - Forms and derivations

Let $\Phi_k = \bigoplus_{q=0}^{\infty} \Phi_k^q$ be the exterior algebra of differential forms on T^kM. For each k, Φ_k is a differential graded algebra with the exterior product denoted by $\wedge$ and the exterior differential denoted by d. Mappings

$$\rho_{k'}^k: \Phi_k \to \Phi_{k'}$$

defined by $\rho_{k'}^k(\mu) = (\tau_M^{k,k'})^*\mu$ are injective homomorphisms and satisfy relations

$$\rho_{k''}^{k'} \circ \rho_{k'}^k = \rho_{k''}^k$$

for $k'' \geq k' \geq k$. It follows that the system $(\Phi_k, \rho_{k'}^k)$ is a direct system. Let $\Phi = \bigoplus_{q=0}^{\infty} \Phi^q$ be the direct limit of this system. We use a canonical contruction of the direct limit. Each space Φ^q is defined as the quotient of the disjoint union $\bigcup_k \Phi_k^q$ by the equivalence relation according to which two elements μ and μ' of Φ_k^q and $\Phi_{k'}^q$ respectively are equivalent if there is an integer k'' such that $k'' \geq k$, $k'' \geq k'$ and $\rho_{k''}^{k'}(\mu) = \rho_{k''}^k(\mu)$. The space Φ is defined as the quotient of the disjoint union $\bigcup_k \Phi_k$ by the same equivalence relation. It is canonically identified with the product of the spaces Φ^q. Each mapping ρ^k defined as the restriction to Φ_k of the canonical projection of $\bigcup_k \Phi_k$ onto Φ is injective. Spaces Φ^q have unique vector space structures and Φ has a unique structure of a differential graded algebra such that mappings ρ^k are differential graded algebra homomorphisms. The exterior product in Φ is denoted by $\wedge$ and the exterior differential is denoted by d. We identify elements of Φ_k with their images in Φ by ρ^k.

Let $\varepsilon : \mathbb{R} :\longrightarrow \Phi^0_0$ be the mapping which turns numbers into constant functions. The sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\varepsilon} \Phi^0_0 \xrightarrow{\mathrm{d}} \Phi^1_0 \xrightarrow{\mathrm{d}} \Phi^2_0 \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Phi^q_0 \xrightarrow{\mathrm{d}} \dots$$

is a cochain complex. For $q > 0$ the quotient space Ker $(\mathrm{d} : \Phi^q_0 \to \Phi^{q+1}_0)/\mathrm{Im}\ (\mathrm{d} : \Phi^{q-1}_0 \to \Phi^q_0)$ is denoted by $H^q(M;\mathbb{R})$ and called a *de Rham cohomology space*. We will denote by $H^0(M;\mathbb{R})$ the quotient space Ker $(\mathrm{d} : \Phi^0_0 \to \Phi^1_0)/\mathrm{Im}\ \varepsilon$. Since T^kM is a bundle with Euclidean fibres the cohomology spaces for the chain complex

$$0 \longrightarrow \mathbb{R} \xrightarrow{\varepsilon} \Phi^0_k \xrightarrow{\mathrm{d}} \Phi^1_k \xrightarrow{\mathrm{d}} \Phi^2_k \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Phi^q_k \xrightarrow{\mathrm{d}} \dots$$

are isomorphic to the de Rham cohomology spaces. The same is true of the direct limit complex

$$0 \longrightarrow \mathbb{R} \xrightarrow{\varepsilon} \Phi^0 \xrightarrow{\mathrm{d}} \Phi^1 \xrightarrow{\mathrm{d}} \Phi^2 \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Phi^q \xrightarrow{\mathrm{d}} \dots$$

A derivation of Φ of degree r is a linear operator $a : \Phi \to \Phi$ such that

$$a\mu \in \Phi^{q+r}$$

and

$$a(\mu \wedge \nu) = a\mu \wedge \nu + (-1)^{rq} \mu \wedge a\nu$$

if $\mu \in \Phi^q$. The exterior differential d is a derivation of degree 1. We denote by i the derivation of degree 0 defined by $i\mu = q\mu$ if $\mu \in \Phi^q$. If a is a derivation of degree r then

$$[i, a] = ra.$$

The commutator

$$[a, b] = ab - (-1)^{rs} ba$$

of derivations a and b of degrees r and s rispectively is a derivation of degree $r + s$.

A derivation is said to be of type i_* if it acts trivially in Φ^0. A derivation a is said to be of type d_* if $[a, \mathrm{d}] = 0$. If i_A is a derivation of type i_* then $\mathrm{d}_A = [i_A, \mathrm{d}]$ is a derivation of type d_*. All derivations of type d_* can be constructed in this way from derivations of type i_* [4]. A derivation of type i_* is completely characterized by its action in Φ^1. Derivations are local operators and for each k an element of Φ^1_k is locally expressed as a sum of differentials of coordinates multiplied by functions on T^kM. If follows that to define a derivation of type i_* it is enough to specify its action on the differentials $\mathrm{d}f_j$ of functions f_j defined in Section 1.

A derivation i_T of type i_* and degree -1 is defined by

$$i_T \mathrm{d}f_j = f_{j+1}$$

for $j \geq 0$. The corresponding derivation

$$\mathrm{d}_T = [i_T, \mathrm{d}]$$

is of degree 0.

For each nonnegative integer n a derivation $i_{F(n)}$ of type $i_\star$ and degree 0 is defined by

$$i_{F(n)}\mathrm{d}f_j = \frac{j!}{(j-n)!}\mathrm{d}f_{j-n}.$$

The derivation $i_{F(0)}$ is the derivation i defined above. For $n < 0$ we set $i_{F(n)} = 0$.

Relations

$$[i_{F(n)}, \mathrm{d}_T] = n i_{F(n-1)}$$

are easily verified.

3. - The Lagrange complex

Let a linear operator $\tau: \Phi \to \Phi$ be defined by

$$\tau = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{d}_T^n i_{F(n)}.$$

The series converges since for each μ in Φ, $i_{F(n)}\mu = 0$ for sufficiently large n.

PROPOSITION 3.1. $\tau \mathrm{d}_T = 0$, $\tau\tau = \tau i$ *and* $\tau \mathrm{d}\tau = \tau \mathrm{d} i$.

PROOF. The first equality is proved by

$$\begin{aligned}
\tau \mathrm{d}_T &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{d}_T^n i_{F(n)} \mathrm{d}_T \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{d}_T^n \left([i_{F(n)}, \mathrm{d}_T] + \mathrm{d}_T i_{F(n)} \right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} \mathrm{d}_T^n i_{F(n-1)} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mathrm{d}_T^{n+1} i_{F(n)} \\
&= 0.
\end{aligned}$$

The remaining two equalities follow immediately from the first.

Q.E.D.

We define a sequence of operators $\tau^q: \Phi^q \to \Phi^q$ by $\tau^0\mu = \mu$ and $\tau^q\mu = \frac{1}{q}\tau\mu$ if $q > 0$. It follows from Proposition 3.1 that $\tau^q \mathrm{d}_T 0 = 0$ for $q > 0$. Also $\tau^q\tau^q = \tau^q$ and $\tau^{q+1}\mathrm{d}\tau^q = \tau^{q+1}\mathrm{d}$ for $q \geq 0$. We introduce a sequence of subspaces $\Lambda^q \subset \Phi^q$ defined by $\Lambda^q = \tau^q\Phi^q$, and a sequence of linear operators $\delta^q: \Lambda^q \to \Lambda^{q+1}$

defined by $\delta^q \mu = \tau^{q+1} \mathrm{d}\mu$. We denote by Λ^{-1} the direct sum $\Phi^0 \bigoplus \mathbb{R}$. We define operators $\delta^{-2} : \mathbb{R} \to \Lambda^{-1}$ and $\delta^{-1} : \Lambda^{-1} \to \Lambda^0$ by $\delta^{-2}\mu = \varepsilon\mu \bigoplus 0$ and $\delta^{-1}(\mu \bigoplus \nu) = \mathrm{d}_T \mu + \varepsilon\nu$.

PROPOSITION 3.2. *The sequence*

$$0 \longrightarrow \mathbb{R} \xrightarrow{\delta^{-2}} \Lambda^{-1} \xrightarrow{\delta^{-1}} \Lambda^0 \xrightarrow{\delta^0} \Lambda^1 \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{q-1}} \Lambda^q \xrightarrow{\delta^q} \ldots$$

is a cochain complex.

PROOF. From $\mathrm{d}\varepsilon = 0$ it follows that $\mathrm{d}_T \varepsilon = i_T \mathrm{d}\varepsilon = 0$ and $\delta^0 \varepsilon = \tau^1 \mathrm{d}\varepsilon = 0$. Also $\delta^0 \mathrm{d}_T = \tau^1 \mathrm{d}\mathrm{d}_T = \tau^1 \mathrm{d}_T \mathrm{d} = 0$. Hence $\delta^{-1}\delta^{-2} = 0$ and $\delta^0 \delta^{-1} = 0$. For $q \geq 0$ we have $\delta^{q+1}\delta^q = \tau^{q+2} \mathrm{d} \tau^{q+1} \mathrm{d} = \tau^{q+2} \mathrm{d}\mathrm{d} = 0$.

Q.E.D.

DEFINITION 3.1. The sequence

$$0 \longrightarrow \mathbb{R} \xrightarrow{\delta^{-2}} \Lambda^{-1} \xrightarrow{\delta^{-1}} \Lambda^0 \xrightarrow{\delta^0} \Lambda^1 \xrightarrow{\delta^1} \ldots \xrightarrow{\delta^{q-1}} \Lambda^q \xrightarrow{\delta^q} \ldots$$

is called the *Lagrange complex.*

Local exactness of the Lagrange complex (the Poincaré lemma) was proved in an earlier publication [6]. The following theorem relates the cohomology of the Lagrange comples to the de Rham cohomology of the manifold M.

THEOREM 3.1. *For $q \geq 0$ the Lagrange cohomology space $L^q(M) = \operatorname{Ker} \delta^q / \operatorname{Im} \delta^{q-1}$ is isomorphic to the direct sum $H^{q+1}(M;\mathbb{R}) \bigoplus H^q(M;\mathbb{R})$ of de Rham cohomology spaces, and $L^{-1}(M)$ is isomorphic to $H^0(M;\mathbb{R})$.*

4. - Proof of the cohomology theorem

Let a linear operator $\sigma : \Phi \to \Phi$ be defined by

$$\sigma = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mathrm{d}_T^{n-1} i_{F(n)}.$$

This operator is related to the operator τ by $\tau + \mathrm{d}_T \sigma = i$.

PROPOSITION 4.1. $\sigma \mathrm{d}_T = \imath$, $\sigma\tau = 0$ *and* $\sigma \mathrm{d}\tau = \sigma \mathrm{d}\imath - \imath \mathrm{d}\sigma$.

PROOF.

$$\begin{aligned}\sigma \mathrm{d}_T &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mathrm{d}_T^{n-1} \imath_{F(n)} \mathrm{d}_T \\ &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mathrm{d}_T^{n-1} \left([\imath_{F(n)}, \mathrm{d}_T] + \mathrm{d}_T \imath_{F(n)} \right) \\ &= -\sum_{n=1}^{\infty} \frac{(-1)^n}{(n-1)!} \mathrm{d}_T^{n-1} \imath_{F(n-1)} - \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \mathrm{d}_T^{n} \imath_{F(n)} \\ &= \imath.\end{aligned}$$

The remaining two equalities are easily verified.

Q.E.D.

For $q > 0$ we define operators $\sigma^q : \Phi^q \to \Phi^q$ by $\sigma^q \mu = \frac{1}{q} \sigma \mu$. Relations $\tau^q + \mathrm{d}_T \sigma^q = 1$, $\sigma^q \mathrm{d}_T = 1$, $\sigma^q \tau^q = 0$ and $\sigma^{q+1} \mathrm{d}\tau^q = \sigma^{q+1}\mathrm{d} - \mathrm{d}\sigma^q$ follow from Proposition 4.1.

We will denote the direct sum $\Gamma \bigoplus \Delta$ of vector spaces Γ and Δ by $\binom{\Gamma}{\Delta}$ and adopt matrix notation for linear mappings between direct sums of vector spaces. In terms of this notation we write $\Lambda^{-1} = \binom{\Phi^0}{\mathbb{R}}$, $\delta^{-2} = \binom{\varepsilon}{0}$ and $\delta^{-1} = (\mathrm{d}_T, \varepsilon)$.

A cochain complex

$$0 \longrightarrow \mathbb{R} \xrightarrow{\partial^{-2}} \Sigma^{-1} \xrightarrow{\partial^{-1}} \Sigma^0 \xrightarrow{\partial^0} \Sigma^1 \xrightarrow{\partial^1} \ldots \xrightarrow{\partial^{q-1}} \Sigma^q \xrightarrow{\partial^q} \ldots$$

is defined by

$$\sigma^{-1} = \begin{pmatrix} \Phi^0 \\ \mathbb{R} \end{pmatrix}, \ \Sigma^0 = \begin{pmatrix} \Phi^1 \\ \Phi^0 \end{pmatrix}, \ \Sigma^1 = \begin{pmatrix} \Phi^2 \\ \Phi^1 \end{pmatrix}, \ldots, \Sigma^q = \begin{pmatrix} \Phi^{q+1} \\ \Phi^q \end{pmatrix}, \ldots$$

and

$$\partial^{-2} = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}, \ \delta^{-1} = \begin{pmatrix} \mathrm{d} & 0 \\ 0 & \varepsilon \end{pmatrix}, \ \partial^0 = \begin{pmatrix} \mathrm{d} & 0 \\ 0 & \mathrm{d} \end{pmatrix}, \ldots, \partial^q = \begin{pmatrix} \mathrm{d} & 0 \\ 0 & \mathrm{d} \end{pmatrix}, \ldots$$

We introduce a sequence

$$\kappa^{-2} : \mathbb{R} \to \mathbb{R}, \ \kappa^{-1} : \Lambda^{-1} \to \Sigma^{-1}, \ \kappa^0 : \Lambda^0 \to \Sigma^0, \ldots, \kappa^q : \Lambda^q \to \Sigma^q, \ldots$$

of linear operators defined by

$$\kappa^{-2} = 1, \ \kappa^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \kappa^0 = \begin{pmatrix} \sigma^1 \mathrm{d} \\ 1 - \imath_T \sigma^1 \mathrm{d} \end{pmatrix}, \ldots$$

$$\ldots, \kappa^q = \begin{pmatrix} (-1)^q \sigma^{q+1} \partial \\ 1 - \imath_T \sigma^{q+1} \mathrm{d} \end{pmatrix}, \ldots$$

LEMMA 4.1. *The diagram*

$$\begin{array}{ccccccccccccc}
0 \to & \mathbb{R} & \overset{\delta^{-2}}{\to} & \Lambda^{-1} & \overset{\delta^{-1}}{\to} & \Lambda^{0} & \overset{\delta^{0}}{\to} & \Lambda^{1} & \overset{\delta^{1}}{\to}\dots\overset{\delta^{q-1}}{\to} & \Lambda^{q} & \overset{\delta^{q}}{\to}\dots \\
& \downarrow{\scriptstyle \kappa^{-2}} & & \downarrow{\scriptstyle \kappa^{-1}} & & \downarrow{\scriptstyle \kappa^{0}} & & \downarrow{\scriptstyle \kappa^{1}} & & \downarrow{\scriptstyle \kappa^{q}} & \\
0 \to & \mathbb{R} & \overset{\partial^{-2}}{\to} & \Sigma^{-1} & \overset{\partial^{-1}}{\to} & \Sigma^{0} & \overset{\partial^{0}}{\to} & \Sigma^{1} & \overset{\partial^{1}}{\to}\dots\overset{\partial^{q-1}}{\to} & \Sigma^{q} & \overset{\partial^{q}}{\to}\dots
\end{array}$$

is commutative.

PROOF.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} 1$$

is obviously true. Hence, $\kappa^{-1}\delta^{-2} = \partial^{-1}\kappa^{-2}$.

From

$$\sigma^{1}\mathrm{d}\mathrm{d}_T = \sigma^{1}\mathrm{d}_T\mathrm{d} = \mathrm{d},$$
$$\sigma^{1}\mathrm{d}\varepsilon = 0,$$
$$(1 - i_T\sigma^{1}\mathrm{d})\mathrm{d}_T = \mathrm{d}_T - i_T\mathrm{d} = \mathrm{d}i_T = 0 \quad (\text{on } \Phi^0)$$

and

$$(1 - i_T\sigma^{1}\mathrm{d})\varepsilon = \varepsilon$$

it follows that

$$\begin{pmatrix} \sigma^{1}\mathrm{d} \\ 1 - i_T\sigma^{1}\mathrm{d} \end{pmatrix}(\mathrm{d}_T, \varepsilon) = \begin{pmatrix} \mathrm{d} & 0 \\ 0 & \varepsilon \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, $\kappa^{0}\delta^{-1} = \partial^{-1}\kappa^{-1}$.

From

$$(-1)^{q+1}\sigma^{q+2}\mathrm{d}\tau^{q+1}\mathrm{d} = (-1)^{q}\mathrm{d}\sigma^{q+1}\mathrm{d}$$

and

$$\begin{aligned}(1 - i_T\sigma^{q+2}\mathrm{d})\tau^{q+1}\mathrm{d} &= \tau^{q+1}\mathrm{d} + i_T\mathrm{d}\sigma^{q+1}\mathrm{d} \\ &= \mathrm{d} - \mathrm{d}_T\sigma^{q+1}\mathrm{d} + i_T\mathrm{d}\sigma^{q+1}\mathrm{d} \\ &= \mathrm{d} - \mathrm{d}i_T\sigma^{q+1}\mathrm{d} \\ &= \mathrm{d}(1 - i_T\sigma^{q+1}\mathrm{d})\end{aligned}$$

for $q \geq 0$ it follows that

$$\begin{pmatrix} (-1)^{q+1}\sigma^{q+2}\mathrm{d} \\ 1 - i_T\sigma^{q+2}\mathrm{d} \end{pmatrix}\tau^{q+1}\mathrm{d} = \begin{pmatrix} \mathrm{d} & 0 \\ 0 & \mathrm{d} \end{pmatrix}\begin{pmatrix} (-1)^{q}\sigma^{q+1}\mathrm{d} \\ 1 - i_T\sigma^{q+1}\mathrm{d} \end{pmatrix}.$$

Hence, $\kappa^{q+1}\delta^{q} = \partial^{q}\kappa^{q}$.

Q.E.D.

Let a sequence

$$\lambda^{-2}: \mathbb{R} \to \mathbb{R},\ \ \lambda^{-1}: \Sigma^{-1} \to \Lambda^{-1},\ \ \lambda^{0}: \Sigma^{0} \to \Lambda^{0}, \ldots, \lambda^{q}: \Sigma^{q} \to \Lambda^{q}, \ldots$$

of linear operators be defined by

$$\lambda^{-2} = 1,\ \ \lambda^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \ \lambda^{0} = (\tau^{0} i_T, \tau^{0}), \ldots, \lambda^{q} = ((-1)^{q} \tau^{q} i_T, \tau^{q}), \ldots$$

LEMMA 4.2. *The diagram*

$$\begin{array}{ccccccccccccc} 0 \to & \mathbb{R} & \overset{\partial^{-2}}{\to} & \Sigma^{-1} & \overset{\partial^{-1}}{\to} & \Sigma^{0} & \overset{\partial^{0}}{\to} & \Sigma^{1} & \overset{\partial^{1}}{\to} \cdots \overset{\partial^{q-1}}{\to} & \Sigma^{q} & \overset{\partial^{q}}{\to} \cdots \\ & \downarrow \lambda^{-2} & & \downarrow \lambda^{-1} & & \downarrow \lambda^{0} & & \downarrow \lambda^{0} & & \downarrow \lambda^{q} & \\ 0 \to & \mathbb{R} & \overset{\delta^{-2}}{\to} & \Lambda^{-1} & \overset{\delta^{-1}}{\to} & \Lambda^{0} & \overset{\delta^{0}}{\to} & \Lambda^{1} & \overset{\delta^{1}}{\to} \cdots \overset{\delta^{q-1}}{\to} & \Lambda^{q} & \overset{\delta^{q}}{\to} \cdots \end{array}$$

is commutative.

PROOF.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix} 1$$

is obviously satisfied. Hence, $\lambda^{-1}\partial^{-2} = \delta^{-2}\lambda^{-2}$.

From

$$\tau^{0} i_T \mathrm{d} = i_T \mathrm{d} = \mathrm{d}_T \ \ (\text{on } \Phi^{0})$$

and

$$\tau^{0}\varepsilon = \varepsilon$$

it follows that

$$(\tau^{0} i_T, \tau^{0}) \begin{pmatrix} \mathrm{d} & 0 \\ 0 & \varepsilon \end{pmatrix} = (\mathrm{d}_T, \varepsilon) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence, $\lambda^{0}\partial^{-1} = \delta^{-1}\lambda^{-1}$.

From

$$\begin{aligned} (-1)^{q+1}\tau^{q+1} i_T \mathrm{d} &= (-1)^{q+1}(\tau^{q+1}\mathrm{d}_T - \tau^{q+1}\mathrm{d} i_T) \\ &= (-1)^{q}\tau^{q+1}\mathrm{d} i_T \\ &= (-1)^{q}\tau^{q+1}\mathrm{d}\tau^{q} i_T \\ &= (-1)^{q}\delta^{q}\tau^{q} i_T \end{aligned}$$

and

$$\tau^{q+1}\mathrm{d} = \tau^{q+1}\mathrm{d}\tau^{q} = \delta^{q}\tau^{q}$$

for $q \geq 0$ it follows that

$$((-1)^{q+1}\tau^{q+1}, \tau^{q+1}) \begin{pmatrix} \mathrm{d} & 0 \\ 0 & \mathrm{d} \end{pmatrix} = \tau^{q+1}\mathrm{d}((-1)^{q}\tau^{q} i_T, \tau^{q}).$$

Hence, $\delta^{q+1}\partial^q = \delta^q\lambda^q$.

Q.E.D.

LEMMA 4.3.

$$\lambda^{-2}\kappa^{-2} = 1, \quad \lambda^{-1}\kappa^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} and \lambda^q\kappa^q = 1$$

for $q \geq 0$.

PROOF. The first two equalities are obvious. For $q \geq 0$ we have

$$((-1)^q\tau^q i_T, \tau^q)\begin{pmatrix} (-1)^q\sigma^{q+1}\mathrm{d} \\ 1 - i_T\sigma^{q+1}\mathrm{d} \end{pmatrix} = \tau^q i_T\sigma^{q+1}\mathrm{d} + \tau^q - \tau^q i_T\sigma^{q+1}\mathrm{d}$$

$$= \tau^q = 1 \text{ in } \Lambda^q$$

Hence $\lambda^q\kappa^q = 1$.

Q.E.D.

For $q > 0$ we define linear operators $\eta^q: \Sigma^q \to \Sigma^{q-1}$ by

$$\eta^q = \begin{pmatrix} \sigma^q i_T & (-1)^q\sigma^q \\ (-1)^q i_T\sigma^q i_T & i_T\sigma^q \end{pmatrix}.$$

LEMMA 4.4.

$$\kappa^{-2}\lambda^{-2} = 1, \quad \kappa^{-1}\lambda^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \kappa^0\lambda^0 = 1$$

and

$$\kappa^q\lambda^q = 1 - \eta^{q+1}\partial^q - \partial^{q-1}\eta^q$$

for $q > 0$.

PROOF. The first two equalities are obvious. For $q \geq 0$ we have

$$\begin{aligned} &\begin{pmatrix} (-1)^q\sigma^{q+1}\mathrm{d} \\ 1 - i_T\sigma^{q+1}\mathrm{d} \end{pmatrix}((-1)^q\tau^q i_T, \tau^q) = \\ &= \begin{pmatrix} \sigma^{q+1}\mathrm{d}\tau^q i_T & (-1)^q\sigma^{q+1}\mathrm{d}\tau^q \\ (-1)^q(1 - i_T\sigma^{q+1}\mathrm{d})\tau^q i_T & (1 - i_T\sigma^{q+1}\mathrm{d})\tau^q \end{pmatrix} \\ &= \begin{pmatrix} 1 - \sigma^{q+1}i_T\mathrm{d} - \mathrm{d}\sigma^q i_T & (-1)^q(\sigma^{q+1}\mathrm{d} - \mathrm{d}\sigma^q) \\ (-1)^q(i_T\sigma^{q+1}i_T\mathrm{d} - \mathrm{d}i_T\sigma^q i_T) & 1 - i_T\sigma^{q+1}\mathrm{d} - \mathrm{d}i_T\sigma^q \end{pmatrix} \\ &= 1 - \eta^{q+1}\partial^q - \partial^{q-1}\eta^q. \end{aligned}$$

Q.E.D.

PROOF OF THEOREM 3.1. It follows from Lemma 4.1 and Lemma 4.2 that sequences

$$\kappa^{-2} : \mathbb{R} \to \mathbb{R},\ \kappa^{-1} : \Lambda^{-1} \to \sigma^{-1},\ \kappa^{0} : \Lambda^{0} \to \Sigma^{0}, \dots, \kappa^{q} : \Lambda^{q} \to \Sigma^{q}, \dots$$

and

$$\lambda^{-2} : \mathbb{R} \to \mathbb{R},\ \lambda^{-1} : \Sigma^{-1} \to \Lambda^{-1},\ \lambda^{0} : \Lambda^{0} \to \Sigma^{0}, \dots, \lambda^{q} : \Sigma^{q} \to \Lambda^{q}, \dots$$

define cochain morphisms κ and λ between the Lagrange complex

$$0 \longrightarrow \mathbb{R} \xrightarrow{\delta^{-2}} \Lambda^{-1} \xrightarrow{\delta^{-1}} \Lambda^{0} \xrightarrow{\delta^{0}} \Lambda^{1} \xrightarrow{\delta^{1}} \dots \xrightarrow{\delta^{q-1}} \Lambda^{q} \xrightarrow{\delta^{q}} \dots$$

and the complex

$$0 \longrightarrow \mathbb{R} \xrightarrow{\partial^{-2}} \Sigma^{-1} \xrightarrow{\partial^{-1}} \Sigma^{0} \xrightarrow{\partial^{0}} \Sigma^{1} \xrightarrow{\partial^{1}} \dots \xrightarrow{\partial^{q-1}} \Sigma^{q} \xrightarrow{\partial^{q}} \dots$$

These morphisms induce linear mappings $\kappa^{q}_{\star}: L^{q}(M) \to S^{q}(M)$ and $\lambda^{q}_{\star}: S^{q}(M) \to L^{q}(M)$ between the Lagrange cohomology spaces $L^{q}(M) = \operatorname{Ker} \delta^{q}/\operatorname{Im} \delta^{q-1}$ and the cohomology spaces $S^{q}(M) = \operatorname{Ker} \delta^{q}/\operatorname{Im} \delta^{q-1}$ for $q \geq -1$. Lemma 4.3 and Lemma 4.4 imply that the mappings $\kappa^{q}_{\star}$ and $\lambda^{q}_{\star}$ are isomorphisms. Lagrange cohomology spaces $L^{q}(M)$ are thus canonically isomorphic to the cohomology spaces $S^{q}(M)$, and each space $S^{q}(M)$ is in turn isomorphic to $H^{q+1}(M;\mathbb{R}) \oplus H^{q}(M;\mathbb{R})$.

Q.E.D.

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Department of Mathematics
The University of Calgary
Calgary, Alberta T2N IN4, Canada