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# Globally Regular Solutions to the $u^5$ Klein-Gordon Equation

MICHAEL STRUWE

## 1. - Introduction

Consider the non-linear wave equation

$$(1.1) \quad u_{tt} - \Delta u + u^p = 0 \text{ in } \mathbb{R}^3 \times \mathbb{R}_+$$

with initial data

$$(1.2) \quad u|_{t=0} = u_0, \quad u_t|_{t=0} = u_1.$$

In 1961 K. Jörgens proved that, for  $p < 5$ , equation (1.1) admits a unique regular solution  $u \in C^2$  for any Cauchy data  $u_0 \in C^3$ ,  $u_1 \in C^2$ , see [1, Satz 2, p. 298].

The case  $p = 5$  was later investigated by Rauch, who obtained global regularity for small initial energies

$$(1.3) \quad E_0 = \int_{\mathbb{R}^3} \left( \frac{1}{2} (|\nabla u_0|^2 + |u_1|^2) + \frac{1}{6} |u_0|^6 \right) dx < \frac{\pi}{\sqrt{3}},$$

see [3, Theorem, p. 347].

Rauch's approach, moreover, reveals that  $p = 5$  arises as a limiting exponent for a Sobolev embedding relevant for problem (1.1), see [3, estimate (14), p. 346]. This and recent progress in elliptic equations involving limiting non-linearities has been our motivation for studying problem (1.1-2).

The supercritical case  $p > 5$  seems to be open.

In this paper we show that Rauch's smallness assumption actually is unnecessary and that Jörgen's result continues to hold - at least for radially symmetric solutions - at the limiting exponent  $p = 5$ , which will be fixed from now on throughout this paper.

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**THEOREM 1.1.** *For any radially symmetric initial data  $u_0 \in C^3(\mathbb{R}^3)$ ,  $u_1 \in C^2(\mathbb{R}^3)$ ,  $u_0(x) = u_0(|x|)$ ,  $u_1(x) = u_1(|x|)$ , there exists a unique, global, radially symmetric solution  $u \in C^2(\mathbb{R}^3 \times [0, \infty[)$ ,  $u(x, t) = u(|x|, t)$  to the Cauchy problem (1.1-2), with  $p = 5$ .*

The proof involves a blow-up analysis of possible singularities of equation (1.1). Thereby we heavily exploit “conformal invariance” of (1.1), i.e. invariance of (1.1) under scaling

$$(1.5) \quad u \rightarrow u_R(x, t) = R^{1/2}u(Rx, Rt).$$

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## 2. - Some fundamental estimates

In this section we recall Rauch’s result for equation (1.1) and prove two basic integral estimates which result from testing (1.1) with suitable functions  $\varphi$ . Besides the standard choice  $\varphi = u_t$  - which gives rise to the well-known “energy inequality”, see Lemma 2.1 -, we will also use  $u$  and its radial derivative  $x \cdot \nabla u$  as testing functions: the remaining components of the generator

$$\frac{d}{dR} u_R|_{R=1} = tu_t + x \cdot \nabla u + \frac{1}{2} u$$

of the family (1.5). This will give rise to the crucial “Pohožaev-type identity” Lemma 2.2 (see [2] for a related result in an elliptic setting).

### 2.1 Notations

Denote  $z = (x, t)$  a generic point in space-time. The negative light-cone through  $z_0 = (x_0, t_0)$  is given by

$$C(z_0) = \{(x, t) \mid t \leq t_0, |x - x_0| \leq t_0 - t\}.$$

Its mantle and space-like sections are denoted by

$$M(z_0) = \{(x, t) \in C(z_0) \mid |x - x_0| = t_0 - t\},$$

resp. by

$$D(z_0, t) = C(z_0) \cap (\mathbb{R}^3 \times \{t\}).$$

If  $z_0 = (0, 0)$  the point  $z_0$  will be omitted from this notation.

Truncated cones will be denoted

$$C_s^t = C \cap (\mathbb{R}^3 \times [s, t]), \quad C_s^0 = C_s, \quad C_{-\infty}^t = C^t.$$

$B_R(x_0)$  denotes the Euclidean ball

$$B_R(x_0) = \{x \in \mathbb{R}^3 \mid |x - x_0| < R\}.$$

Again, if  $x_0 = 0$  we simply write  $B_R(0) = B_R$ .

Finally, for a  $C^1$ -function  $u$  and a space-like region  $\Omega(t) \subset (\mathbb{R}^3 \times \{t\})$ ,

$$E(u; \Omega(t)) = \int_{\Omega(t)} e(u) dx$$

denotes the energy of  $u$  in  $\Omega(t)$ , with density

$$e(u) = \frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6.$$

The letters  $c, C$  will denote generic positive constants, occasionally numbered for clarity.

### 2.2 The energy inequality

Let  $u \in C^2(\mathbb{R}^3 \times ]-\infty, 0])$  be a solution to (1.1). Actually, by finiteness of propagation speed, all estimates only require  $u$  to be  $C^2$  near suitable sections of cones.

Multiply (1.1) by  $u_t$ . This gives the identity

$$(2.1) \quad \left( \frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 \right)_t - \operatorname{div}(\nabla u \cdot u_t) = 0.$$

If we integrate this expression over a section  $C_s^t$  of the negative light-cone, we obtain the following result:

$$(2.2) \quad E(u; D(t)) - E(u; D(s)) + \frac{1}{\sqrt{2}} \int_{M_s^t} \left( \frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 - \frac{x}{|x|} \cdot \nabla u \, u_t \right) d\sigma = 0.$$

Note that the outward normal to  $M_s^t$  is given by  $n(x, t) = \frac{1}{\sqrt{2}} \left( \frac{x}{|x|}, 1 \right)$ ; moreover, we recognize the energy density  $e(u)$  inside the left bracket of (2.1).

Rauch [3, p. 345] interprets the boundary integrand as follows:

$$\begin{aligned}
 (2.3) \quad & \frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 - \frac{x}{|x|} \cdot \nabla u \cdot u_t \\
 &= \frac{1}{2} \left| \frac{x}{|x|} u_t - \nabla u \right|^2 + \frac{1}{6} |u|^6 = \frac{1}{2} |\nabla_y v|^2 + \frac{1}{6} |v|^6,
 \end{aligned}$$

where

$$(2.4) \quad v(y) = u(y, -|y|).$$

Thus we may state:

LEMMA 2.1. *For any  $s < t < 0$  there holds the energy estimate*

$$E(u; D(t)) + \int_{B_{|s|} \setminus B_{|t|}} \left\{ \frac{1}{2} |\nabla v|^2 + \frac{1}{6} |v|^6 \right\} dy = E(u; D(s)),$$

where  $v$  is given by (2.4).

### 2.3 A Pohožaev-type identity

The next result apparently is new. This and Lemma 3.3 are the crucial ingredients in the proof of Theorem 1.1.

LEMMA 2.2. *For  $u$  as above there holds*

$$\begin{aligned}
 & \frac{1}{3} \int_{C_{-1}} |u|^6 dx dt + E(u; D(-1)) \\
 & \leq \int_{D(-1)} u_t (x \cdot \nabla u + u) dx + \int_{B_1} \{ |y| |\nabla v|^2 + |\nabla v| |v| \} dy,
 \end{aligned}$$

where  $v$  is given by (2.4).

PROOF. Multiply (1.1) by  $tu_t + x \cdot \nabla u + u$ . By (2.1) the contribution from the first term is

$$\begin{aligned}
 0 &= \left( t \left[ \frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 \right] \right)_t - \operatorname{div}(\nabla u \cdot tu_t) \\
 &\quad - \left( \frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 \right).
 \end{aligned}$$

Similarly, we compute

$$\begin{aligned} 0 &= (u_{tt} - \Delta u + u^5)(x \cdot \nabla u) \\ &= \operatorname{div} \left( -x \frac{|u_t|^2}{2} - \nabla u(x \cdot \nabla u) + x \frac{|\nabla u|^2}{2} + x \frac{|u|^6}{6} \right) \\ &\quad + \{x \cdot \nabla u \ u_t\}_t + \frac{3}{2} |u_t|^2 - \frac{1}{2} |\nabla u|^2 - \frac{1}{2} |u|^6. \end{aligned}$$

Finally,

$$\begin{aligned} 0 &= (u_{tt} - \Delta u + u^5) u \\ &= \{u_t u\}_t - \operatorname{div}(\nabla u \cdot u) - |u_t|^2 + |\nabla u|^2 + |u|^6. \end{aligned}$$

Adding, we obtain that

$$\begin{aligned} 0 &= \frac{1}{3} |u|^6 + \left( t \left[ \frac{1}{2} (|u_t|^2 + |\nabla u|^2) + \frac{1}{6} |u|^6 \right] \right)_t + \left( (x \cdot \nabla u + u) u_t \right)_t \\ &\quad - \operatorname{div} \left( x \frac{|u_t|^2}{2} - x \frac{|\nabla u|^2}{2} + \nabla u(x \cdot \nabla u + u) + \nabla u \cdot t u_t - x \frac{|u|^6}{6} \right). \end{aligned}$$

Thus, when we integrate this expression over the cone  $C_{-1}^{-\varepsilon}$ , we obtain

$$\begin{aligned} (2.5) \quad &\frac{1}{3} \int_{C_{-1}^{-\varepsilon}} |u|^6 dx dt + E(u; D(-1)) - \varepsilon E(u; D(-\varepsilon)) \\ &= \int_{D(-1)} u_t(x \cdot \nabla u + u) dx - \int_{D(-\varepsilon)} u_t(x \cdot \nabla u + u) dx + BI, \end{aligned}$$

with  $BI$  denoting the following boundary integral

$$\begin{aligned} BI &= \frac{1}{\sqrt{2}} \int_{M_{-1}^{-\varepsilon}} \left[ |t| \left( |u_t|^2 + \frac{|x \cdot \nabla u|^2}{|x|^2} \right) - 2x \cdot \nabla u \ u_t - \left( u_t - \frac{x}{|x|} \cdot \nabla u \right) u \right] \, d\sigma \\ &= \frac{1}{\sqrt{2}} \int_{M_{-1}^{-\varepsilon}} \left[ |t| \left| u_t - \frac{x}{|x|} \cdot \nabla u \right|^2 - \left( u_t - \frac{x}{|x|} \cdot \nabla u \right) \dot{u} \right] \, d\sigma \\ &\leq \int_{\{y: |y| \leq 1\}} [ |y| |\nabla v|^2 + |\nabla v| |v| ] \, dy. \end{aligned}$$

By Lemma 2.1,

$$E(u; D(-\varepsilon)) \leq E(u; D(-1))$$

uniformly. Moreover, by Young's and Hölder's inequalities

$$\begin{aligned} \int_{D(-\varepsilon)} u_t(x \cdot \nabla u + u) dx &\leq \varepsilon \int_{D(-\varepsilon)} [ |u_t|^2 + |\nabla u|^2 + \varepsilon^{-2} |u|^2 ] dx \\ &\leq \varepsilon \int_{D(-\varepsilon)} ( |u_t|^2 + |\nabla u|^2 ) dx + \varepsilon \left( \int_{D(-\varepsilon)} |u|^6 dx \right)^{1/3} \leq C \varepsilon. \end{aligned}$$

Hence we may pass to the limit  $\varepsilon \rightarrow 0$  in (2.5) and the proof is complete.

*qed*

#### 2.4 Small energy

Finally, we recall the integral representation

$$(2.6) \quad u(0,0) = \underline{u}(0,0) - \frac{1}{4\pi} \int_{M_{t_0}} |t|^{-1} u^5(x,t) d\sigma$$

for the value  $u(0,0)$  of a solution  $u$  of (1.1) in terms of the solution  $\underline{u}$  of the homogeneous wave equation

$$(2.7) \quad \underline{u}_{tt} - \Delta \underline{u} = 0,$$

sharing the Cauchy data  $u_0, u_1$  of  $u$  at a time  $t_0 < 0$ :

$$(2.8) \quad \underline{u} = u_0 = u, \quad \frac{\partial}{\partial t} \underline{u} = u_1 = \frac{\partial}{\partial t} u \quad \text{at } t = t_0,$$

see [3, (10), p. 342].

We will only apply this formula for functions  $u$  which are of class  $C^2$  in a neighborhood of  $C_t$ , for some  $t < 0$ .

Following Rauch [3], we turn (2.6) into a linear inequality for  $\sup_{C_{t_0}^1} |u|$ :

Suppose

$$\sup_{C_{t_0}^1} |u| = |u(0,0)|$$

is achieved at the origin. Then, if we let

$$\mu(s) = \frac{1}{4\pi} \int_{M_s} |t|^{-1} u^4(x,t) d\sigma = \frac{\sqrt{2}}{4\pi} \int_{B_{|s|}} |y|^{-1} v^4(y) dy,$$

(2.6) implies the inequality, for any  $s > t_0$ ,

$$(2.9) \quad (1 - \mu(s)) \sup_{C_{t_0}^1} |u| \leq |\underline{u}(0,0)| - \frac{1}{4\pi} \int_{M_{t_0}^s} |t|^{-1} u^5(x,t) d\sigma.$$

By Hölder's inequality

$$(2.10) \quad \mu(s) = C \int_{B_{|s|}} |y|^{-1} v^4(y) dy \leq C \left( \int_{B_{|s|}} v^6 dy \right)^{1/2} \left( \int_{B_{|s|}} \frac{v^2(y)}{|y|^2} dy \right)^{1/2}.$$

Rauch now invokes Hardy's inequality

$$(2.11) \quad \int_{\mathbb{R}^3} \frac{\psi^2(y)}{|y|^2} dy \leq 4 \int_{\mathbb{R}^3} |\nabla \psi|^2 dy, \quad \forall \psi \in C_0^\infty(\mathbb{R}^3)$$

to estimate the last integral in (2.10).

If integration extends only over a bounded domain  $B_{2R}$ , (2.11) is not immediately applicable. However, if we truncate with a smooth localizing function  $\eta \in C_0^\infty(\mathbb{R}^3)$  satisfying the conditions:  $\eta \equiv 1$  on  $B_R$ ,  $\eta \equiv 0$  off  $B_{2R}$ ,  $0 \leq \eta \leq 1$ ,  $|\nabla \eta| \leq C/R$ , then with absolute constants  $C$  there holds

$$(2.12) \quad \begin{aligned} \int_{B_{2R}} \frac{\psi^2(y)}{|y|^2} dy &\leq \int_{B_{2R}} \frac{(\psi\eta)^2}{|y|^2} dy + \int_{B_{2R} \setminus B_R} \frac{\psi^2}{|y|^2} dy \\ &\leq 4 \int_{B_{2R}} |\nabla(\psi\eta)|^2 dy + R^{-2} \int_{B_{2R}} \psi^2 dy \\ &\leq C \int_{B_{2R}} |\nabla \psi|^2 dy + C R^{-2} \int_{B_{2R}} \psi^2 dy \\ &\leq C \int_{B_{2R}} |\nabla \psi|^2 dy + C \left( \int_{B_{2R}} \psi^6 dy \right)^{1/3}, \\ &\quad \forall \psi \in C^1(B_{2R}). \end{aligned}$$

Using (2.12) and the energy inequality, Lemma 2.1, the number  $\mu(s)$  from (2.10) may now be estimated

$$(2.13) \quad \begin{aligned} \mu(s) &\leq C \left( \int_{B_{|s|}} v^6 dy \right)^{1/2} \left( \int_{B_{|s|}} |\nabla v|^2 dy + \left( \int_{B_{|s|}} v^6 dy \right)^{1/3} \right)^{1/2} \\ &\leq C_0 [E(u; D(s)) + E(u; D(s))^{2/3}]. \end{aligned}$$

The remaining terms in (2.9) are easily bounded using Hölder’s inequality

$$\begin{aligned}
 \int_{M_{t_0}^s} |t|^{-1} |u|^5 d\sigma &\leq C \int_{B_{|t_0|} \setminus B_{|s|}} |y|^{-1} |v|^5 dy \\
 (2.14) \qquad &\leq C |s|^{-1/2} \left( \int_{B_{|t_0|}} v^6 dy \right)^{5/6} \leq C_1 |s|^{-1/2} E(u; D(t_0))^{5/6}.
 \end{aligned}$$

From (2.9), (2.13) we immediately obtain Rauch’s regularity result for small initial energies - while the more refined estimate (2.14) will be useful later on.

**THEOREM 2.3.** (Rauch [3]). *Suppose  $u$  is a  $C^2$ -solution of (1.1) in a neighborhood of  $C_{t_0}$  with initial data  $u = u_0$ ,  $u_t = u_1$  on  $D(t_0)$ . There exists an absolute constant  $\varepsilon_0 > 0$  with the property:*

*If  $E(u; D(t_0)) \leq \varepsilon_0$ , then*

$$|u(x, t)| \leq 2 \sup_{C_{t_0}} |u| \text{ for all } (x, t) \in C_{t_0},$$

where  $\underline{u}$  denotes the solution to the Cauchy-problem (2.7-8) for the homogeneous equation.

**PROOF.** By passing to a smaller cone  $\tilde{C} \subset C_{t_0}$ , if necessary, we may assume that

$$\sup_{C_{t_0}} |u| = |u(0, 0)|.$$

Determine  $\varepsilon_0 > 0$  such that  $C_0(\varepsilon_0 + \varepsilon_0^{2/3}) = \frac{1}{2}$ . Applying (2.9), (2.13) with  $s = t_0$  the Theorem follows.

*qed*

Note that there is a converse result to Theorem 2.3:

**PROPOSITION 2.4.** *Suppose  $u$  is a  $C^2$ -solution of (1.1) in a neighborhood of  $C_{t_0} \setminus \{0\}$  with initial data  $u = u_0 \in C^3$ ,  $u_t = u_1 \in C^2$  on  $D(t_0)$ , and suppose that  $|u(z)| \rightarrow \infty$  as  $z \rightarrow 0$ ,  $z \in C_{t_0} \setminus \{0\}$ . Then for any  $t \in [t_0, 0]$  there holds*

$$E(u; D(t)) \geq \varepsilon_0 > 0,$$

where  $\varepsilon_0$  is the constant of Theorem 2.3.

**PROOF.** Suppose  $E(u; D(t)) \leq \varepsilon_0$  for some  $t \in [t_0, 0]$ . Note that, since by assumption  $|u(z)| \rightarrow \infty$  as  $z \rightarrow 0$ , there exists a sequence  $\delta_m \rightarrow 0$ ,  $\delta_m > 0$ , such that  $\sup_{C_{t_0}^{-\delta_m}} |u|$  is attained in  $D(-\delta_m)$ . Hence (2.9), (2.13-14) are applicable

in suitable cones  $C_{t_0}(z_m) \subset C_{t_0}$ , with  $z_m \in D(-\delta_m)$ , and we obtain

$$(2.15) \quad \frac{1}{2} \sup_{C_{t_0}^{-\delta_m}} |u| \leq \sup_{C_{t_0}^+} |u| + C|t|^{-1/2}.$$

Since (2.15) holds for arbitrarily small  $\delta_m > 0$ , there results a contradiction, and the proof is complete.

*qed*

### 3. - Proof of Theorem 1.1

Let  $u_0 \in C^3(\mathbb{R}^3)$ ,  $u_1 \in C^2(\mathbb{R}^3)$  with  $u_0(x) = u_0(|x|)$ ,  $u_1(x) = u_1(|x|)$  be given radial functions. By [1; Satz 1, p. 297] the Cauchy problem (1.1-2) admits a unique (and hence radially symmetric)  $C^2$ -solution  $u(x, t) = u(|x|, t)$  locally, i.e. in a neighborhood of  $\mathbb{R}^3 \times \{0\}$ .

Suppose (by contradiction) that  $u$  is not globally regular. Then there is a singularity  $\bar{z} = (\bar{x}, \bar{t})$  such that  $|u(z)| \rightarrow \infty$  as  $z \rightarrow \bar{z}$ ,  $z \in C_0(\bar{z})$ , see [1, p. 301]. Replacing  $\bar{z}_0$  by another singular point in  $S = \{(x, t) : 0 \leq t < \bar{t}, |x| \leq |\bar{x}| + \bar{t} - t\}$ , if necessary, we may assume that  $u \in C^2$  in  $S$ .

Radial symmetry implies

LEMMA 3.1.  $\bar{x} = 0$ , in particular  $u \in C^2(\mathbb{R}^3 \times [0, \bar{t}])$ .

PROOF. By Proposition 2.4 and since  $u(x, t) = u(|x|, t)$ :

$$E(u; D(z, t)) \geq \varepsilon_0$$

for any  $z = (x, \bar{t})$ ,  $|x| = |\bar{x}|$ , and any  $t < \bar{t}$ .

Now, if  $|\bar{x}| > 0$ , for any given  $K \in \mathbb{N}$  we can choose points  $x_1, \dots, x_K$  satisfying  $|x_k| = |\bar{x}|$ ,  $1 \leq k \leq K$ , and  $t < \bar{t}$  such that with  $z_k = (x_k, \bar{t})$  we have:

$$D(z_j, \bar{t}) \cap D(z_k, \bar{t}) = \emptyset, \quad j \neq k.$$

But then, letting  $T = |\bar{x}| + \bar{t}$ ,  $Z = (0, T)$ , by Lemma 2.1:

$$\begin{aligned} K \varepsilon_0 &\leq \sum_{k=1}^K E(u; D(z_k, \bar{t})) = E\left(u; \bigcup_{k=1}^K D(z_k, \bar{t})\right) \\ &\leq E(u; D(Z, \bar{t})) \leq E(u; D(Z, 0)) < \infty, \end{aligned}$$

uniformly in  $K$ , and for large  $K$  we obtain a contradiction.

Hence a singularity is first encountered on the line  $\{x = 0\}$ .

*qed*

For convenience we shift coordinates such that  $\bar{z} = 0$  in our new coordinate frame, and denote  $-\bar{t} = t_0$ . Thus our solution  $u$  is transformed into a solution (indiscriminately denoted by  $u$ ) of (1.1), of class  $C^2$  in a neighborhood of  $C_{t_0} \setminus \{0\}$ , which becomes unbounded as  $z \rightarrow 0$ ,  $z \in C_{t_0} \setminus \{0\}$ .

Denote by  $\underline{u}$  the solution of the homogeneous wave equation (2.7) sharing the Cauchy data of  $u$  at  $t_0$ .  $\underline{u}$  is uniformly bounded in a closed neighborhood of  $C_{t_0}$ .

For  $R_m = 2^{-m}$ ,  $m \in \mathbb{N}$ , define the blown-up functions

$$u_m(x, t) = R_m^{1/2} u(R_m x, R_m t).$$

Each  $u_m$  is of class  $C^2$  in a neighborhood of a deleted cone  $C_{t_m} \setminus \{0\}$ ,  $t_m = t_0/R_m$ .

As in (2.4) we denote the trace of  $u_m$  on  $M_{t_m}$  by

$$v_m(y) = u_m(y, -|y|) = R_m^{1/2} v(R_m y).$$

Relabelling  $\{u_m\}$ , if necessary, we may assume that  $t_0 \leq -1$ .

Note that for any  $m$ , any  $t \in [t_m, 0]$ , by Lemma 2.1:

$$(3.1) \quad E(u_m; D(t)) \leq E(u_m; D(t_m)) = E(u; D(t_0)) =: E_0 < \infty.$$

On the other hand, since  $u_m$  becomes unbounded at 0, by Proposition 2.4:

$$(3.2) \quad E(u_m; D(t)) \geq \varepsilon_0, \quad \text{for all } t \in [t_m, 0],$$

for any  $m \in \mathbb{N}$ .

By Lemma 2.1 the energy  $E(u, D(t))$  is non-increasing in  $t$ , hence tends to a positive (by (3.2)) limit as  $t \rightarrow 0$ . But then, by Lemma 2.1 again,

$$(3.3) \quad \begin{aligned} & \int_{B_1 \setminus B_{|t|}} \left( \frac{1}{2} |\nabla v_m|^2 + \frac{1}{6} |v_m|^6 \right) dy \\ &= \int_{B_{R_m} \setminus B_{|t|R_m}} \left( \frac{1}{2} |\nabla v|^2 + \frac{1}{6} |v|^6 \right) dy \\ &\leq E(u; D(-R_m)) - E(u; D(tR_m)) \rightarrow 0, \end{aligned}$$

as  $m \rightarrow \infty$ , uniformly in  $t < 0$ .

LEMMA. 3.2. *There exists  $t < 0$  such that, for some  $s_m \in [-1, t]$ , there holds*

$$(3.4) \quad \int_{D(s_m)} (u_m)_t u_m \, dx \leq o(1),$$

where  $o(1) \rightarrow 0$  as  $m \rightarrow \infty$ .

PROOF. We may assume that

$$\int_{D(-1)} u_m^2 dx \geq C_2 > 0.$$

(Otherwise we can choose  $s_m = -1$  to achieve our claim). Choose  $t \in ]-1, 0[$  such that

$$\begin{aligned} \int_{D(t)} u_m^2 dx &\leq ct^2 \left( \int_{D(t)} u_m^6 dx \right)^{1/3} \\ &\leq c t^2 E(u_m; D(t))^{1/3} \leq c t^2 < C_2. \end{aligned}$$

Suppose by contradiction that

$$\int_{D(s)} (u_m)_t u_m dx \geq C_3 > 0$$

uniformly in  $m \in \mathbb{N}$ , for all  $s \in [-1, t]$ .

Then by (3.3) we obtain

$$\begin{aligned} &\int_{D(t)} u_m^2 dx - \int_{D(-1)} u_m^2 dx \\ &= 2 \int_{C_{-1}^t} (u_m)_t u_m dx dt - \frac{1}{\sqrt{2}} \int_{M_{-1}^t} u_m^2 dx \geq 2(1+t)C_3 - o(1), \end{aligned}$$

which, for large  $m$ , is in conflict with our choice of  $t$ .

*qed*

Since  $\{s_m\}$  is bounded away from 0, we may scale with  $s_m$  to achieve (3.4) with  $s_m = -1$  for all  $m$ . Note that with this change of scale the ratio  $R_m/R_{m+1}$  remains uniformly bounded, i.e. there exists  $R > 0$  such that

$$(3.5) \quad 0 < R^{-1} \leq |R_m/R_{m+1}| \leq R < \infty, \text{ for all } m.$$

Now apply Lemma 2.2:

$$\begin{aligned} &\frac{1}{3} \int_{C_{-1}} u_m^6 dx dt + E(u_m; D(-1)) \\ &\leq \int_{D(-1)} (u_m)_t x \cdot \nabla u_m dx + o(1). \end{aligned}$$

It follows that

$$(3.6) \quad \int_{C_{-1}} |u_m|^6 dx dt + \int_{D(-1)} \left\{ (1 - |x|) [|(u_m)_t|^2 + |\nabla u_m|^2] + |x| \left| \frac{x}{|x|} (u_m)_t - \nabla u_m \right|^2 + |u_m|^6 \right\} dx \rightarrow 0,$$

as  $m \rightarrow \infty$ .

LEMMA 3.3. *There exists a sequence  $\Lambda \subset \mathbb{N}$  such that*

$$\liminf_{m \rightarrow \infty, m \in \Lambda} \sup_{C_{t_m}^{-1}} |u_m| > 0.$$

PROOF. Suppose by contradiction that

$$\begin{aligned} \sup_{C_{t_m}^{-1}} |u_m| &= R_m^{1/2} \sup_{C_{t_0}^{-R_m}} |u| \\ &\geq R^{-1/2} \sup_{s \in [-R_{m-1}, -R_m]} \left( |s|^{1/2} \sup_{C_{t_0}^s} |u| \right) \rightarrow 0, \text{ as } m \rightarrow \infty. \end{aligned}$$

I.e. if we let

$$g(t) = |t|^{1/2} \sup_{C_{t_0}^t} |u|,$$

$g$  is continuous and satisfies

$$g(t) \rightarrow 0 \quad (t \rightarrow 0).$$

Also denote

$$h(t) = \sup_{s \geq \max\{t, -1\}} g(s).$$

Then  $h$  is continuous, non-increasing, and satisfies  $h(t) \equiv h(-1)$ , for  $t \leq -1$ , and  $h(t) \rightarrow 0$  ( $t \rightarrow 0$ ).

Now the proof proceeds as follows: first we establish that  $h(t)$  decays with a certain power of  $|t|$ :  $h(t) \leq c|t|^\epsilon$ , ( $t \rightarrow 0$ ).

In a second step we use this decay estimate to prove that  $u$  is uniformly bounded near 0 - which will yield the desired contradiction.

- i) Suppose  $h(t) = g(s)$  for some  $s \geq t \geq -1$  and that  $g(s)$  is attained at  $\tilde{z} = (\tilde{x}, \tilde{t})$ , where  $s \geq \tilde{t} = \lambda t$ ,  $|u(\tilde{z})| = \sup_{C_{t_0}^s} |u| = \sup_{C_{t_0}^t} |u|$ . Note that

if  $\lambda = \lambda(t) > 1$ :

$$h(t) = g(s) = |s|^{1/2} |u(\tilde{z})| \leq |t|^{1/2} |u(\tilde{z})| \leq g(t) \leq h(t)$$

i.e.

$$s = t; \quad h(t) = |t|^{1/2} |u(\tilde{z})|.$$

Similarly, for  $\bar{t} \in ]\tilde{t}, t]$ ,

$$h(\bar{t}) = \sup_{\tilde{t} \leq s} g(s) = |\bar{t}|^{1/2} |u(\tilde{z})| = \left| \frac{\bar{t}}{\tilde{t}} \right|^{1/2} h(\tilde{t}).$$

In particular,  $\lambda(\bar{t}) = \frac{\tilde{t}}{\bar{t}} > 1$ .

Denote

$$J = \{t \in ]-1, 0[ : \lambda(t) > 1\}.$$

Remark that  $J$  consists of a union of left open intervals  $I$  and, for any pair  $s \leq t$  belonging to such an interval  $I$ , there holds

$$h(t) = \left| \frac{t}{s} \right|^{1/2} h(s).$$

In particular, for any  $\varepsilon \in ]0, \mu]$ ,  $0 < \mu \leq \frac{1}{2}$ , there holds

$$(3.7) \quad (h(t) + |t|^\mu) \leq \left| \frac{t}{s} \right|^\varepsilon (h(s) + |s|^\mu), \quad \text{for all } s \leq t \in I.$$

On the other hand, if  $\lambda \leq 1$ , by (2.6)

$$(3.8) \quad \begin{aligned} h(t) &= g(s) = |s|^{1/2} |u(\tilde{z})| \\ &\leq |s|^{1/2} |\underline{u}(\tilde{z})| + c|s|^{1/2} \int_{M_{t_0}(\tilde{z})} |\tilde{t} - \tau|^{-1} |u(y, \tau)|^5 \, d\mathbf{o} \\ &\leq c|t|^{1/2} + c|\tilde{t}|^{1/2} \int_{-1}^{\tilde{t}} |\tilde{t} - \tau| \sup_{O_{t_0}^r} |u|^5 \, d\tau \\ &= c|t|^{1/2} + c|\tilde{t}|^{1/2} \int_{-1}^{\tilde{t}} |\tilde{t} - \tau| g^5(\tau) |\tau|^{-5/2} \, d\tau \\ &\leq c|t|^{1/2} + c|t|^{1/2} \int_{-1}^{\tilde{t}} \lambda^{1/2} g^5(\tau) |\tau|^{-3/2} \, d\tau \end{aligned}$$

$$\begin{aligned} &\leq c|t|^{1/2} + c|t|^{1/2} \int_{-\lambda^{-1}}^t g^5(\lambda\tau) |\tau|^{-3/2} d\tau \\ &\leq c|t|^{1/2} + c|t|^{1/2} \int_{-1}^t h^5(\tau) |\tau|^{-3/2} d\tau. \end{aligned}$$

Now choose  $\mu = \frac{1}{5}$ , and denote  $t_1 = t^\mu := t|t|^{\mu-1} < t$ . Since  $h$  is non-increasing and bounded we obtain from (3.8):

$$\begin{aligned} &(h(t) + |t|^\mu) \\ &\leq C|t|^{1/2} + |t|^\mu + C|t|^{1/2} \int_{t^\mu}^t h^5(\tau) |\tau|^{-3/2} d\tau + C|t|^{1/2} \int_{-1}^{t^\mu} h^5(\tau) |\tau|^{-3/2} d\tau \\ &\leq C|t|^{1/2} + |t|^\mu + C h^5(t^\mu) + C|t|^{1/2} \sup_{\tau < 0} h^5(\tau) (|t|^{-\mu/2} - 1) \\ &\leq C|t|^{1/2} + |t|^\mu + C|t|^{(1-\mu)/2} + C h^5(t^\mu) \\ &\leq C (|t|^\mu + h^5(t^\mu)) \leq C_4 (h(t_1) + |t_1|^\mu)^5. \end{aligned}$$

Iteratively define  $t_k = t_{k-1}^\mu < t_{k-1}$ ,  $k = 1, \dots, K$ . Suppose  $\lambda(t_k) \leq 1$  for all  $k = 1, \dots, K-1$ . Then

$$\begin{aligned} h(t) &\leq (h(t) + |t|^\mu) \leq C_4 (h(t_1) + |t_1|^\mu)^5 \\ &\leq C_4^6 (h(t_2) + |t_2|^\mu)^{25} \leq \dots \leq C_4^{\sum_{k=0}^{K-1} 5^k} (h(t_K) + |t_K|^\mu)^{5^K} \\ &= \left[ C_4^{1/4} (h(t_K) + |t_K|^\mu) \right]^{5^{K-1}} \cdot (h(t_K) + |t_K|^\mu). \end{aligned}$$

I.e., if for some  $\varepsilon > 0$ :

$$(3.9) \quad C_4^{1/4} (h(t_K) + |t_K|^\mu) \leq |t_K|^\varepsilon,$$

it follows that

$$\begin{aligned} (3.10) \quad h(t) + |t|^\mu &\leq \left( \left| \frac{t_K^{5^K}}{t_K} \right| \right)^\varepsilon (h(t_K) + |t_K|^\mu) \\ &= \left| \frac{t}{t_K} \right|^\varepsilon (h(t_K) + |t_K|^\mu). \end{aligned}$$

Note that, since  $h(t) \rightarrow 0$  ( $t \rightarrow 0$ ), there exist  $T \in ]-1, 0[$ ,  $\varepsilon \in ]0, \mu[$  such that (3.9) holds whenever  $t_K \in [T^\mu, T]$ . But then also (3.10) holds for all such  $t, t_K$ , provided  $\lambda(t)$ ,  $\lambda(t_k) \leq 1$ ,  $k = 1, \dots, K-1$ .

Now choose any  $\tau = \tau_0 > T$  and define a sequence  $\tau_1, \tau_2, \dots, \tau_K$  as follows:

$$\tau_{k+1} = \begin{cases} \tau_k^\mu, & \text{if } \lambda(\tau_k) \leq 1 \\ \tilde{\tau}_k, & \text{if } \lambda(\tau_k) > 1 \end{cases}, \quad k \in \mathbb{N}_0,$$

where  $\tilde{\tau}_k$  denotes the left end-point of the interval  $I \subset J$  containing  $\tau_k$ , if  $\lambda(\tau_k) > 1$ , and where

$$K = \sup \{k \in \mathbb{N} \mid \tau_{k-1} > T\}.$$

Note that

$$|\tau_{k+2}/\tau_k| \geq |\tau_k|^{\mu-1} \geq |T|^{\mu-1} > 1,$$

if  $\tau_k \geq \tau_{k+2} > T$ . Hence  $K$  exists and is finite, for every  $\tau < 0$ .

Combining (3.7) and (3.10) we see that

$$h(\tau) \leq (h(\tau) + |\tau|^\mu) \leq \left| \frac{\tau}{\tau_K} \right|^\epsilon (h(\tau_K) + |\tau_K|^\mu) \leq C|\tau|^\epsilon,$$

i.e.

$$\sup_{C_{t_0}^+} |u| = g(t) |t|^{-1/2} \leq h(t) |t|^{-1/2} \leq C|t|^{\epsilon - \frac{1}{2}}.$$

ii) Denote

$$\bar{\gamma} = \inf \left\{ \gamma > 0 : |t|^\gamma \sup_{C_{t_0}^+} |u| \leq C < \infty \text{ uniformly in } t \right\}.$$

By part i)  $\bar{\gamma} < \frac{1}{2}$  and we may choose  $\gamma > \bar{\gamma}$  such that  $\mu := 5\gamma - 2 < \bar{\gamma}$ ,  $\mu \neq 0$ .

Define

$$f(t) = |t|^\gamma \sup_{C_{t_0}^+} |u|.$$

Note that  $f(t)$  is uniformly bounded, continuous and satisfies  $f(t) \rightarrow 0$  as  $t \rightarrow 0$ .

By (2.6), for all  $z = (x, t) \in C_{t_0} \setminus \{0\}$ :

$$\begin{aligned} |u(z)| &\leq |\underline{u}(z)| + \frac{1}{4\pi} \int_{M_{t_0}(z)} |t - \tau|^{-1} |u(\xi, \tau)|^5 \, d\sigma(\xi, \tau) \\ (3.11) \quad &\leq C + C \int_{t_0}^t |t - \tau| f^5(\tau) |\tau|^{-5\gamma} \, d\tau \\ &\leq C + C \sup_{\tau < 0} f^5(\tau) \int_{t_0}^t |\tau|^{1-5\gamma} \, d\tau \leq C + C|t|^{2-5\gamma}. \end{aligned}$$

First suppose  $\mu > 0$ . Then from (3.11) we obtain

$$|t|^\mu |u(x, t)| \leq C|t|^\mu + C \leq C,$$

uniformly for all  $z = (x, t) \in C_{t_0} \setminus \{0\}$ , which contradicts the definition of  $\bar{\gamma}$ .

Thus  $\mu \leq 0$ . But then by (3.11)  $u$  is uniformly bounded in  $C_{t_0} \setminus \{0\}$ , contrary to hypothesis.

*qed*

To proceed with the proof of Theorem 1.1, let  $z_m = (x_m, s_m) \in C_{t_m}^{-1}$ ,  $m \in \Lambda$ , satisfy

$$(3.12) \quad |u_m(z_m)| = \sup_{C_{t_m}^{s_m}} |u_m| = \min\{1, \sup_{C_{t_m}^{-1}} |u_m|\} = R_m^{1/2} u(R_m x_m, R_m s_m).$$

Note that by Lemma 3.3

$$(3.13) \quad \liminf_{m \rightarrow \infty, m \in \Lambda} |u_m(z_m)| = 2 c_5 > 0;$$

in particular, by (3.12)

$$(3.14) \quad R_m s_m \rightarrow 0, \quad s_m \leq -1.$$

Now by (2.9), (2.13), (2.14), if  $E(u_m; D(z_m, s)) < \varepsilon_0$  for some  $s < s_m$ :

$$\begin{aligned} c_5 &\leq |u_m(z_m)| \leq 2 \left( R_m^{1/2} u(R_m z_m) + c|s - s_m|^{-1/2} \right) \\ &\leq o(1) + 2 c|s - s_m|^{-1/2} \leq o(1) + \frac{1}{2} c_5, \end{aligned}$$

provided  $s \leq s_m - c_6$  for some  $c_6 > 0$ .

Since  $c_5 > 0$ , this is impossible for large  $m$ , and it follows that

$$E(u_m; D(z_m, s)) \geq \varepsilon_0,$$

for  $s \in [t_m, s_m - c_6]$ ,  $m \geq m_0$ .

By radial symmetry

$$(3.15) \quad E(u_m; D(z, s)) \geq \varepsilon_0$$

for such  $s, m \geq m_0$ , for all  $z = (x, s_m)$  with  $|x| = |x_m|$ .

LEMMA 3.4. *For any  $c > 0$ , any family  $\{x^k\}_{1 \leq k \leq K}$  in  $\mathbb{R}^3$ , with  $|x^k| = r \geq 0$ ,  $|x^j - x^k| \geq c^{-1}r$ ,  $j \neq k$ , there exists  $\sigma_m \in [t_m, s_m - c_6]$  such that*

$$(3.16) \quad E \left( u_m; \bigcup_{j \neq k} D(z^j, \sigma_m) \cap D(z^k, \sigma_m) \right) \rightarrow 0$$

as  $m \rightarrow \infty$ ,  $m \in \Lambda$ , where  $z^i = (x^i, s_m)$ .

PROOF. Since  $|s_m| \geq 1$ , by uniform convexity of balls in  $\mathbb{R}^3$ , there exists  $\varepsilon > 0$  such that

$$D(z^j, s) \cap D(z^k, s) \subset \{x \in D(s) : |x| < (1 - \varepsilon)|s|\} =: D^\varepsilon(s)$$

for all  $j \neq k$ ,  $s \in [R(s_m - c_6), s_m - c_6] =: I_m$ .

(Note that

$$|Rs_m| \leq \frac{R}{R_m} |R_m s_m| \leq c|t_m| |R_m s_m| = o(|t_m|),$$

by (3.14). Hence  $R(s_m - c_6) \geq t_m$ , for  $m \geq m_0$ ).

Now by (3.5), there exists  $k(m)$  such that

$$\sigma_m := -\frac{R_{k(m)}}{R_m} \in I_m.$$

Observe that by (3.14) again:

$$R_{k(m)} = -\sigma_m R_m \leq c|s_m|R_m \rightarrow 0,$$

hence  $k(m) \rightarrow \infty$ , ( $m \rightarrow \infty, m \in \Lambda$ ).

But then by (3.6)

$$\begin{aligned} E(u_m; D^\varepsilon(\sigma_m)) &= E(u_{k(m)}; D^\varepsilon(-1)) \\ &\leq c \cdot \varepsilon^{-1} \int_{D(-1)} \left\{ (1 - |x|) (|(u_{k(m)})_t|^2 + |\nabla u_{k(m)}|^2) \right. \\ &\quad \left. + |x| \left| \frac{x}{|x|} (u_{k(m)})_t - \nabla u_{k(m)} \right|^2 + |u_{k(m)}|^6 \right\} dx \rightarrow 0 \end{aligned}$$

as  $m \rightarrow \infty$ ,  $m \in \Lambda$ .

This proves the claim.

*qed*

We can now complete the *proof of Theorem 1.1*.

Given  $K \in \mathbb{N}$ , we can find  $c > 0$  such that, for any  $m \in \mathbb{N}$ , there are  $K$  points  $x_m^j$ ,  $1 \leq j \leq K$  such that  $|x_m^j| = |x_m|$ ,  $|x_m^j - x_m^k| \geq c^{-1} |x_m|$  for all  $1 \leq j \neq k \leq K$ . Denote  $z_m^j = (x_m^j, s_m)$ .

Let  $\sigma_m \in [t_m, s_m]$  denote the number determined in Lemma 3.4 for the family  $\{x_m^j\}$ .

By (3.15-16) and (3.1)

$$\begin{aligned}
 K\varepsilon_0 &\leq \sum_{j=1}^K E(u_m; D(z_m^j, \sigma_m)) \\
 &\leq E(u_m; \bigcup_{j=1}^K D(z_m^j, \sigma_m)) \\
 &\quad + \sum_{j \neq k} E(u_m; D(z_m^j, \sigma_m) \cap D(z_m^k, \sigma_m)) \\
 &\leq E(u_m; D(\sigma_m)) + o(1) \leq E_0 + o(1),
 \end{aligned}$$

where  $o(1) \rightarrow 0$  as  $m \rightarrow \infty$ ,  $m \in \Lambda$ .

For  $K$  sufficiently large we obtain a contradiction, and the proof is complete.

*qed*

#### 4. - A Remark on the non-symmetric case

Estimate (3.6) suggests that, also in the non-symmetric case, singularities tend to build up in a rotationally symmetric pattern. Using this observation, it is possible to extend our results to arbitrary initial data  $u \in C^3$ ,  $u_1 \in C^2$ , provided the modulus of continuity of the blow-up functions  $u_m$ , restricted to  $C_{t_m}^s$  (where  $u_m$  is uniformly bounded by 1), can be uniformly bounded.

#### Added in proof

Generalizations of (1.1-2) to higher dimensions were studied for instance by Brenner and von Wahl [4] or Pecher [5], where results analogous to those found by Jörgens in dimension 3 were obtained. See [4] for further references.

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