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A Bochner-Martinelli Formula for Vector Fields which Satisfy the Generalized Cauchy-Riemann Equations

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We prove a Bochner-Martinelli type formula for vector fields u on $\mathbb{R}^n$ which satisfy the system of generalized Cauchy-Riemann equations proposed by M. Riesz: curl $u = 0$, div $u = 0$.

As an application it is shown that, for smoothly bounded open Ω in $\mathbb{R}^n$ such that all components of $\mathbb{R}^n \backslash \Omega$ are simply connected, the tangential part of the boundary values of such "Cauchy-Riemann vector fields" is characterized by the condition $d(i^* u) = 0$. Here u is considered as a 1-form and $i : \partial\Omega \to \mathbb{R}^n$ denotes the inclusion. This result extends work of Korányi and Vági for the unit ball in $\mathbb{R}^n$.

0. - Introduction

Let Ω be an open subset of $\mathbb{R}^n$. A vector field $u = (u_1, ..., u_n)$ on Ω, $u \in C^1(\Omega; \mathbb{R}^n)$, is said to satisfy the generalized Cauchy-Riemann equations (abbreviated: GCRE) if

(i) $$\frac{\partial u_j}{\partial x_k}(x) = \frac{\partial u_k}{\partial x_j}(x), \qquad j \neq k, x \in \Omega,$$

(ii) $$\text{div} u := \sum_{j=1}^{n} \frac{\partial u_j}{\partial x_j}(x) = 0, \qquad x \in \Omega,$$

(cf. [5], p. 234). We will also say, in this case, that u is a Cauchy-Riemann vector field on Ω.

The system (i)+(ii) is called the M. Riesz system in [5]. Note that if $n = 2$, (u_1, u_2) satisfies (i)+(ii) iff $u_1 - iu_2$ is holomorphic.

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If we introduce the one-form $u = u_1 dx_1 + \ldots + u_n dx_n$ on Ω, (i) is the same as

$$\text{(i)}' \qquad du = 0.$$

Depending on which of the two points of view seems expedient, we will alternately consider a $u \in C^1(\Omega; \mathbb{R}^n)$ to be a vector field or a one-form.

Note that a $u \in C^1(\Omega; \mathbb{R}^n)$ satisfies the GCRE iff, locally, it is the gradient of a harmonic function. In particular, the u_k's are harmonic then.

In this paper we prove a Bochner-Martinelli type formula for Cauchy-Riemann vector fields, see theorem 1.1 below.

As an application, we characterize (theorem 4.5) the tangential parts of the boundary values of a Cauchy-Riemann vector field u on a smoothly bounded domain Ω in $\mathbb{R}^n$ by the condition $d(i^* u) = 0$, where $i : \partial\Omega \to \mathbb{R}^n$ is the inclusion, and $i^* u$ is the pullback of u considered as a one-form. In the special case of a ball in $\mathbb{R}^n$ this result is due to Korányi and Vági [4], who used the representation theory of SO(n).

1. - A Bochner-Martinelli type formula for Cauchy-Riemann vector fields.

Let $E_x(y)$ be a fundamental solution with pole in x of the Laplacian $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial y_j^2}$.

For instance, we can take

$$(1.1) \qquad E_x(y) = E(x,y) = \begin{cases} -\dfrac{1}{(n-2)\omega_n} \dfrac{1}{|x-y|^{n-2}}, & n > 2, \\ \dfrac{1}{2\pi} \log|x-y|, & n = 2, \end{cases}$$

the standard fundamental solution, where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$. In this section we will derive the following Bochner-Martinelli type formula.

THEOREM 1.1. Let $\Omega \subseteq \mathbb{R}^n$ *be open, bounded, with* C^1*-boundary* $\partial\Omega$. *Let* $u = (u_1, \ldots u_n) \in C^1(\overline{\Omega}; \mathbb{R}^n)$ *satisfy the generalized Cauchy-Riemann equations. Then for* $x \in \Omega, 1 \leq k \leq n$,

$$(1.2) \quad u_k(x) = \int_{\partial\Omega} \left\{ (-1)^{k-1} \left(u_k \frac{\partial E_x}{\partial y_k} - \sum_{l: l \neq k} u_l \frac{\partial E_x}{\partial y_l} \right) dy_1 \wedge \ldots [k] \ldots \wedge dy_n \right.$$
$$\left. + \sum_{j: j \neq k} (-1)^{j-1} \left(u_k \frac{\partial E_x}{\partial y_j} + u_j \frac{\partial E_x}{\partial y_k} \right) dy_1 \wedge \ldots [j] .. \wedge dy_n \right\}.$$

(The symbol $[j]$ means, as usual, that the term dy_j is omitted.)

For a given vector field u on $\partial\Omega, u = (u_1, ... u_n) \in C(\partial\Omega; \mathbb{R}^n)$, say, we let $T(u)$ denote the vector field (or one-form) on $\mathbb{R}^n \backslash \partial\Omega$ whose k-th component is defined by the right-hand side of (1.2). For the sake of concreteness, we will assume that $E_x(y)$ is given by (1.1), but much of the following remains true with other fundamental solutions.

If we substitute formula (1.1) for $E_x(y)$ in (1.2), we obtain

$$(1.3) \quad (Tu)_k(x) = \frac{1}{\omega_n} \int_{\partial\Omega} \frac{1}{|y-x|^n} \{(-1)^{k-1} [(y_k - x_k)u_k(y) - \sum_{l:l\neq k} (y_l - x_l)u_l(y)] dy_1 \wedge ... [k] ... \wedge dy_n + \sum_{j:j\neq k} (-1)^{j-1} [(y_j - x_j)u_k(y) + (y_k - x_k)u_j(y)] \, dy_1 \wedge ... [j] ... \wedge dy_n \}.$$

REMARK 1.2. In case $n = 2$, (1.3) is the Cauchy integral formula (in real variable notation).

Note that, unlike the Poisson formula for u_k and like the Bochner-Martinelli formula in several complex variables (cf. [1]), (1.3) is universal: the kernel is the same for all Ω.

Also, unlike the representation of u_k as a sum of two surface potentials (cf. (1.5) below), (1.3) does not involve derivatives of the u_j's.

The derivation of (1.2) starts with one of Green's formulas which we now recall in differential form notation. Let $\omega = dx_1 \wedge ... \wedge dx_n$ be the volume form on $\mathbb{R}^n$. If $V = (V_1, ..., V_n)$ is a C^1-vector field, let

$$V \lfloor \omega := \sum_{j=1}^{n} (-1)^{j-1} V_j dx_1 \wedge ... [j] ... \wedge dx_n.$$

Then $d(V \lfloor \omega) = \left(\sum_{j=1}^{n} \frac{\partial V_j}{\partial x_j} \right) \omega = (\text{div} V)\omega.$

Let $u, v \in C^1(\overline{\Omega})$. Let ∇ denote the gradient operator: $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n} \right)$. Then

$$(1.4) \qquad \int_{\Omega} (u\Delta v - v\Delta u)\omega = \int_{\partial\Omega} u(\nabla v \lfloor \omega) - v(\nabla u \lfloor \omega).$$

PROOF OF THEOREM 1.1. Let $u \in C^1(\overline{\Omega}; \mathbb{R}^n)$ satisfy the GCRE in Ω and let $x \in \Omega$. From (1.4), applied to u and $v = E_x$, and $\Delta u_k = 0$ we obtain

$$(1.5) \qquad u_k(x) = \int_{\partial\Omega} u_k(\nabla E_x \lfloor \omega) - E_x(\nabla u_k \lfloor \omega).$$

Now it turns out that $\nabla u_k \lfloor \omega$ is the d of an $(n-2)$-form.

LEMMA 1.3. *Suppose that* $u = (u_1, ..., u_n)$ *satisfies the* GCRE. *Let*

$$\beta_k = \beta_k(u) := \sum_{j=1, j\neq k}^{n} (-1)^{j+k} \operatorname{sgn}(j-k) u_j dx_1 \wedge ...[k]...[j]... \wedge dx_n$$

(*where* sgn $(x) := \frac{x}{|x|}, x \in \mathbb{R}, x \neq 0$). *Then* $d\beta_k = \nabla u_k \lfloor \omega, 1 \leq k \leq n$.

PROOF.

$$\begin{aligned} d\beta_k &= \sum_{j:j\neq k} (-1)^{j-1}(-1)^{k-1} \operatorname{sgn}(j-k) \frac{\partial u_j}{\partial x_k} dx_k \wedge dx_1 \wedge ...[k]...[j]... \wedge dx_n \\ &+ \sum_{j:j\neq k} (-1)^{j-1}(-1)^{k-1} \operatorname{sgn}(j-k) \frac{\partial u_j}{\partial x_j} dx_j \wedge dx_1 \wedge ...[k]...[j]... \wedge dx_n \\ &= \sum_{j:j\neq k} (-1)^{j-1} \frac{\partial u_j}{\partial x_k} dx_1 \wedge ...[j]... \wedge dx_n \\ &- (-1)^{k-1} \left(\sum_{j:j\neq k} \frac{\partial u_j}{\partial x_j} \right) dx_1 \wedge ...[k]... \wedge dx_n \\ &= \sum_{j=1}^{n} (-1)^{j-1} \frac{\partial u_j}{\partial x_k} dx_1 \wedge ...[j]... \wedge dx_n, \text{ since div } u = 0, \\ &= \nabla u_k \lfloor \omega, \text{ since } du = 0. \end{aligned}$$

We continue with the proof of theorem 1.1. By the previous lemma and Stokes' theorem, formula (1.5) becomes

$$u_k(x) = \int_{\partial\Omega} u_k(\nabla E_x \lfloor \omega) + dE_x \wedge \beta_k. \tag{1.6}$$

A straightforward computation now shows that

$$\begin{aligned} u_k(\nabla E_x \lfloor \omega) + dE_x \wedge \beta_k &= (-1)^{k-1} \left(u_k \frac{\partial E_x}{\partial y_k} - \sum_{l:l\neq k} u_l \frac{\partial E_x}{\partial y_l} \right) \\ dy_1 \wedge ...[k]... \wedge dy_n &+ \sum_{j:j\neq k} (-1)^{j-1} \left(u_k \frac{\partial E_x}{\partial y_j} + u_j \frac{\partial E_x}{\partial y_k} \right) \\ dy_1 \wedge ...[j]... \wedge dy_n. \end{aligned} \tag{1.7}$$

This proves theorem 1.1.

We now derive a Pompeiu-type formula for arbitrary C^1-vector fields on $\overline{\Omega}$, by applying Stokes' theorem to the $(n-1)$-form on the right-hand side of (1.2).

THEOREM 1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be open, bounded, with C^1-boundary. Let $u = (u_1, ... u_n) \in C^1(\overline{\Omega}; \mathbb{R}^n)$. Then for $x \in \Omega, 1 \le k \le n$,*

$$(1.8)\ u_k(x) = -\int_{\Omega} \left(\sum_{j:j\neq k} \left(\frac{\partial u_k}{\partial y_j} - \frac{\partial u_j}{\partial y_k} \right) \frac{\partial E_x}{\partial y_j} + \operatorname{div} u \frac{\partial E_x}{\partial u_k} \right) \omega(y)$$

$$+ \int_{\partial\Omega} \left\{ (-1)^{k-1} \left(u_k \frac{\partial E_x}{\partial y_k} - \sum_{l:l\neq k} u_l \frac{\partial E_x}{\partial y_l} \right) dy_1 \wedge ...[k]... \wedge dy_n \right.$$

$$\left. + \sum_{j:j\neq k} (-1)^{j-1} \left(u_k \frac{\partial E_x}{\partial y_j} + u_j \frac{\partial E_x}{\partial y_k} \right) dy_1 \wedge ...[j]... \wedge dy_n \right\}.$$

PROOF. Let $x \in \Omega$, and let $\overline{B}(x, \varepsilon)$ denote the closed ball around x with radius ε. Apply Stokes' theorem on $\Omega \backslash \overline{B}(x, \varepsilon)$, ε sufficiently small, to the $(n-1)$-form $u_k(\nabla E_x \lfloor \omega) + dE_x \wedge \beta_k$, $\beta_k = \beta_k(u)$ as in lemma 1.3 (cf. formula (1.7)).

Now

$$d(u_k(\nabla E_x \lfloor \omega)) = du_k \wedge (\nabla E_x \lfloor \omega) + u_k d(\nabla E_x \lfloor \omega)$$
$$= du_k \wedge (\nabla E_x \lfloor \omega) \text{ on } \Omega \backslash B(x, \varepsilon),$$

since $d(\nabla E_x \lfloor \omega) = \operatorname{div} (\nabla E_x)\omega = (\Delta E_x)\omega = 0$ on $\Omega \backslash \overline{B}(x, \varepsilon)$, and

$d(dE_x \wedge \beta_k) = -dE_x \wedge d\beta_k$.

Hence,

$$(1.9) \int_{\partial B(x,\varepsilon)} u_k(\nabla E_x \lfloor \omega) + dE_x \wedge \beta_k = -\int_{\Omega \backslash \overline{B}(x,\varepsilon)} du_k \wedge (\nabla E_x \lfloor \omega) - dE_x \wedge d\beta_k$$
$$+ \int_{\partial\Omega} u_k(\nabla E_x \lfloor \omega) + dE_x \wedge \beta_k.$$

One easily shows that the left-hand side of (1.9) converges to $u_k(x)$ if $\varepsilon \to 0$. For instance, one can expand the u_j's to first order in a Taylor series around x, and then use theorem 1.1 with $\Omega = B(x, \varepsilon)$ and u_j identically equal to $u_j(x), 1 \le j \le n$.

To finish the proof we must show that

$$du_k \wedge (\nabla E_x \lfloor \omega) - dE_x \wedge d\beta_k = \left(\sum_{j:j\neq k} \left(\frac{\partial u_k}{\partial y_j} - \frac{\partial u_j}{\partial y_k} \right) \frac{\partial E_x}{\partial y_j} + \text{ div } u \frac{\partial E_x}{\partial y_k} \right) \omega(y).$$

This is again a straightforward computation, left to the reader.

2. - The case $n = 3$

In the special case $n = 3$ it is possible to re-interpret formula (1.8) in terms of the operations $\bullet, \times$ (inner and exterior product) and curl( ) $= \nabla \times$ ( ) from vector calculus. This may be of interest in applications.

Let $\Omega \subseteq \mathbb{R}^3, u = (u^1, u^2, u^3)$ be as in theorem 1.4. Let $n(y)$ denote the outward unit normal to $\partial\Omega$ at $y \in \partial\Omega$, and let σ denote the volume-form on $\partial\Omega$ induced by ω (i.e. σ is the surface measure).

Formula (1.8) can be shown to be equivalent to

$$\begin{aligned} u(x) = &-\int_\Omega (\text{curl } u \times \nabla E_x + (\text{div } u)\nabla E_x)\omega \\ &+ \int_{\partial\Omega} \{((u - 2(u \bullet n)n) \times \nabla E_x) \times n \\ &+ ((u - 2(u \bullet n)n) \bullet \nabla E_x)n\}\sigma \end{aligned} \tag{2.1}$$

One can prove (2.1) by checking that the integrands of (2.1) agree pointwise on $\Omega, \partial\Omega$, respectively, with those of (1.8). For the integrand of the volume integral in (2.1) this is straightforward.

For the boundary term it is probably easiest to introduce suitable coordinates depending on the point of $\partial\Omega$: if $y_0 \in \partial\Omega$, choose Euclidean coordinates (x_1, x_2, x_3) of $\mathbb{R}^3$ such that $y_0 = 0, n(y_0) = (0, 0, -1)$. Then $\sigma(y_0) = dy_1 \wedge dy_2|y_0$, and dy_3, considered as a one-form on $\partial\Omega$, is 0 in y_0 (locally around y_0, $\partial\Omega$ is given by an equation $y_3 = h(y_1, y_2)$, so that

$$dy_3 = \frac{\partial h}{\partial y_1} dy_1 + \frac{\partial h}{\partial y_2} dy_2 \text{ on } \partial\Omega).$$

It is now again straightforward to check that the integrands of the boundary integrals of (1.8) and (2.1) agree in y_0.

3. - Interpretation of (1.2) in terms of differential forms.

Let $\Omega \subseteq \mathbb{R}^n$ be bounded, with C^1-boundary $\partial\Omega$. Let $u \in C(\partial\Omega; \mathbb{R}^n)$. We would like to rewrite the formula for $T(u)$ given by (1.3) in the language of differential forms. This is possible, but in general the outward unit normal to $\partial\Omega$ will enter the resulting expression explicity, making the kernel non-universal.

Suppose, from now on, that $\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}, \rho \in C^1(\mathbb{R}^n), |\nabla\rho| = 1$ on $\partial\Omega = \{x \in \mathbb{R}^n : \rho(x) = 0\}$ (so that $\nabla\rho$ is the outward unit normal to $\partial\Omega$).

We can decompose a $u \in C(\partial\Omega; \mathbb{R}^n)$ in its tangential and normal part at each $y \in \partial\Omega$:

$$u(y) = v(y) + \phi(y)(\nabla\rho)(y), \tag{3.1}$$

with $\phi(y) \in \mathbb{R}$, and $v(y) \perp \nabla\rho(y)$; v and ϕ are continuous.

If we extend u in a C^1 way to a neighbourhood of $\partial\Omega$, and if we let $i : \partial\Omega \to \mathbb{R}^n$ be the inclusion, then $i^* u = v$ (u, v considered as one-forms), and v is a one-form on the C^1-manifold $\partial\Omega$.

Define the $(n-2)$-forms $\Theta_k(y, x)$ for $y \neq x$ by

$$\Theta_k(y, x) := \sum_{j : j \neq k} (-1)^{j+k} \text{ sgn } (j-k) \frac{\partial E_x}{\partial y_j} dy_1 \wedge ...[k]...[j]... \wedge dy_n, \tag{3.2}$$

and define the double form $\Theta(y, x)$ on $\mathbb{R}^n \times \mathbb{R}^n \backslash \{(y, y) : y \in \mathbb{R}^n\}$

$$\Theta(y, x) := \sum_{k=1}^{n} \Theta_k(y, x) dx_k;$$

$\Theta(y, x)$ is an $(n-2)$-form in y and a one-form in x.

THEOREM 3.1. *All notation as above. Then, as a one-form in x on $\mathbb{R}^n \backslash \partial\Omega, T(u)$ can be written as*

$$T(u)(x) = \int_{\partial\Omega} v(y) \wedge \Theta(y, x) - d_x[\int_{\partial\Omega} \phi(y) E_x(y)(\nabla\rho \lfloor \omega)(y)]. \tag{3.3}$$

($u = v + \phi\nabla\rho$ as in (3.1)).

PROOF. $T(u) = T(v) + T(\phi\nabla\rho)$. First we investigate $T(v)$. The condition $v \perp \nabla\rho$ in (3.1) can be expressed as

$$v \lfloor \omega = \sum_{j=1}^{n} (-1)^{j-1} v_j dy_1 \wedge ...[j]... \wedge dy_n = 0 \text{ on } \partial\Omega,$$

$v = (v_1, ..., v_n)$. (This can easily be checked pointwise on $\partial\Omega$ by introducing suitable linear coordinates of $\mathbb{R}^n$ at each point of $\partial\Omega$; cf. the end of section

2.) Hence,

$$(-1)^{k-1} v_k dy_1 \wedge ...[k]... \wedge dy_n = -\sum_{j:j\neq k} (-1)^{j-1} v_j dy_1 \wedge ...[j]... \wedge dy_n.$$

Substitute this in the defining expression for $T(v)_k(x)$, which then becomes,

$$T(v)_k(x) = \int_{\partial\Omega} \left\{ -(-1)^{k-1} \left(\sum_{l:l\neq k} v_l \frac{\partial E_x}{\partial y_l} \right) dy_1 \wedge ...[k]... \wedge dy_n \right.$$

$$\left. + \sum_{j:j\neq k} (-1)^{j-1} v_k \frac{\partial E_x}{\partial y_j} dy_1 \wedge ...[j]... \wedge dy_n \right\}$$

$$= \int_{\partial\Omega} (v_1 dy_1 + ... + v_n dy_n) \wedge \Theta_k(y, x).$$

We now turn to $T(\phi \nabla \rho)$. Since $d\rho = 0$ on $\partial\Omega$,

$$\frac{\partial \rho}{\partial y_l} dy_l = -\frac{\partial \rho}{\partial y_1} dy_1 - ...[l]... - \frac{\partial \rho}{\partial y_n} dy_n \text{ on } \partial\Omega.$$

Take the wedge product with $dy_1 \wedge ...[l]...[k]... \wedge dy_n, l \neq k$. Then

$$(-1)^{k-1} \frac{\partial \rho}{\partial y_l} dy_1 \wedge ...[k]... \wedge dy_n = (-1)^{l-1} \frac{\partial \rho}{\partial y_k} dy_1 \wedge ...[l]... \wedge dy_n,$$

and hence,

$$T(\phi \nabla \rho)_k(x) = \int_{\partial\Omega} \sum_{j=1}^{n} (-1)^{j-1} \phi \frac{\partial \rho}{\partial y_j} \frac{\partial E_x}{\partial y_k} dy_1 \wedge ...[j]... \wedge dy_n$$

$$= -\int_{\partial\Omega} \phi \frac{\partial E_x}{\partial x_k} (\nabla \rho \lfloor \omega).$$

4. - Characterization of the tangential parts of boundary values of Cauchy-Riemann vector fields.

Let $\Omega = \{x \in \mathbb{R}^n : \rho(x) < 0\}$ be bounded, with $\rho \in C^2(\mathbb{R}^n)$, $|\nabla \rho| = 1$ on $\partial\Omega$, and let $i : \partial\Omega \to \mathbb{R}^n$ denote the inclusion map. Suppose that $u \in C^1(\Omega; \mathbb{R}^n)$ satisfies the generalized Cauchy-Riemann equations on Ω. Then obviously $d(i^* u) = 0$.

We will say that $u \in C^1(\partial\Omega;\mathbb{R}^n)$ satisfies the *induced Cauchy-Riemann equations* if $dv = 0$, with $u = v + \phi\nabla\rho$ as in (3.1).

THEOREM 4.1. *If $u \in C^1(\partial\Omega;\mathbb{R}^n)$ satisfies the induced Cauchy-Riemann equations, then $T(u)$ satisfies the generalized Cauchy-Riemann equations on $\mathbb{R}^n\backslash\partial\Omega$.*

PROOF. To begin with, for any vector field u on $\partial\Omega$, $T(u)$ satisfies div $T(u) = 0$.

Write $w = T(u)$ and evaluate $\frac{\partial w_k}{\partial x_k}$ by differentiating under the integral sign; use $\frac{\partial E_x}{\partial x_k}(y) = -\frac{\partial E_x}{\partial y_k}(y)$. Then for $x \notin \partial\Omega$,

$$-\frac{\partial w_k}{\partial x_k}(x) = \int_{\partial\Omega} (-1)^{k-1}\left(u_k\frac{\partial^2 E_x}{\partial y_k^2} - \sum_{l:l\neq k} u_l \frac{\partial^2 E_x}{\partial y_l \partial y_k}\right) dy_1 \wedge ...[k]... \wedge dy_n$$
$$+ \sum_{j:j\neq k}(-1)^{j-1}\left(u_j\frac{\partial^2 E_x}{\partial y_k^2} + u_k\frac{\partial^2 E_x}{\partial y_k \partial y_j}\right) dy_1 \wedge ...[j]... \wedge dy_n.$$

The coefficient of $dy_1 \wedge ...[k].... \wedge dy_n$ in the resulting integral formula for $-$ div w is equal to

$$(-1)^{k-1}\left(u_k\frac{\partial^2 E_x}{\partial y_k^2} - \sum_{l:l\neq k} u_l\frac{\partial^2 E_x}{\partial y_l\partial y_k}\right) +$$
$$+ \sum_{m:m\neq k}(-1)^{k-1}\left(u_k\frac{\partial^2 E_x}{\partial y_m^2} + u_m\frac{\partial^2 E_x}{\partial y_m \partial y_k}\right) = (-1)^{k-1}u_k\Delta E_x = 0 \text{ on } \partial\Omega,$$

since $\Delta E_x(y) = 0$ on $\mathbb{R}^n\backslash\{x\}$.

To prove that $dT(u) = 0$ we use theorem 3.1. We will need the following result.

LEMMA 4.2. *There exists a double form $A(y,x)$ on $\mathbb{R}^n \times \mathbb{R}^n\backslash\{(y,y) : y \in \mathbb{R}^n\}$ which is an $(n-3)$-form in y and a 2-form in x, such that*

$$d_x\Theta(y,x) = d_yA(y,x).$$

PROOF. By (3.2),

$$d_x\Theta(y,x) = \sum_k d_x\Theta_k(y,k) \wedge dx_k$$
$$= (-1)^{n-1}\sum_{k.l}\Big(\sum_{j:j\neq k}(-1)^{j+k} \text{ sgn } (j-k)\frac{\partial^2 E_x}{\partial y_l \partial y_j} dy_1 \wedge$$
$$\wedge ...[k]...[j]... \wedge dy_n\Big) \wedge dx_l \wedge dx_k.$$

Combining the terms for a fixed pair of indices l and k, with $l < k$, say, the coefficient of $dx_l \wedge dx_k$ becomes, neglecting the sign $(-1)^{n-1}$,

$$(4.1)\qquad \begin{cases} \sum\limits_{j:j\neq k,l} (-1)^{j+k} \operatorname{sgn}(j-k) \dfrac{\partial^2 E_x}{\partial y_l \partial y_j} dy_1 \wedge ...[k]...[j]... \wedge dy_n \\ - \sum\limits_{j:j\neq k.l} (-1)^{j+l} \operatorname{sgn}(j-l) \dfrac{\partial^2 E_x}{\partial y_k \partial y_j} dy_1 \wedge ...[l]...[j]... \wedge dy_n \\ +(-1)^{l+k} \operatorname{sgn}(l-k) \left(\dfrac{\partial^2 E_x}{\partial y_l^2} + \dfrac{\partial^2 E_x}{\partial y_k^2} \right) dy_1 \wedge ...[k]...[l]... \wedge dy_n. \end{cases}$$

Since $\Delta E_x = 0$ on $\mathbb{R}^n \backslash \{x\}$ the last term of (4.1) becomes

$$(-1)^{l+k} \operatorname{sgn}(k-l) \left(\sum_{j:j\neq l.k} \frac{\partial^2 E_x}{\partial y_j^2} \right) dy_1 \wedge ...[k]...[l]... \wedge dy_n.$$

We claim that (4.1) is equal to $d_y A_{l.k}$, where

$$A_{l.k} := \sum_{j:j\neq l.k} (-1)^{j-1}(-1)^{k-1}(-1)^{l-1} \operatorname{sgn}(j-l) \operatorname{sgn}(j-k) \operatorname{sgn}(k-l) \\ \frac{\partial E_x}{\partial y_j} dy_1 \wedge ...[l]...[k]...[j]... \wedge dy_n.$$

This is left to the reader to verify. Now put $A(y,x) = \sum_{l<k} A_{l,k} dx_l \wedge dx_k$.

We now finish the proof of 4.1. Write $u = v + \phi \nabla \rho$ as in (3.1). Then by (3.3) and Stokes' theorem, if $x \notin \partial\Omega$,

$$d_x T(u)(x) = \int_{\partial\Omega} v(y) \wedge d_x \Theta(y,x) = \int_{\partial\Omega} v(y) \wedge d_y A(y,x)$$
$$= \int_{\partial\Omega} dv(y) \wedge A(y,x) = 0,$$

since u satisfies the induced Cauchy-Riemann equations.

The following proposition corresponds to a special case of the Plemelj-Sokhotskii formulas from one-variable complex analysis.

PROPOSITION 4.3. *Let $\Omega \subseteq \mathbb{R}^n$ be bounded with C^1-boundary $\partial\Omega$. Put $\Omega^+ := \Omega$, $\Omega^- := \mathbb{R}^n \backslash \overline{\Omega}$. Let $u \in C^1(\partial\Omega; \mathbb{R}^n)$ and define u^+, u^- on Ω^+, Ω^-, respectively, by*

$$(4.2)\qquad u^{\pm} := T(u)|\Omega^{\pm}.$$

Then $u^{\pm} \in C(\overline{\Omega^{\pm}})$ and $u^+(y) - u^-(y) = u(y)$ on $\partial\Omega$.

For the proof one can imitate the proof of lemma 1.4 in Aizenberg and Yuzhakov [1].

Formula (4.2) also shows some resemblance to a formula from potential theory (which can be attributed to Green, cf. Burkhardt and Meyer [2], pp. 470, 471), which asserts that the jump across $\partial\Omega$ of the normal derivative of the single layer potential of a $\phi \in C(\partial\Omega)$ is equal to ϕ. (Recall that Cauchy-Riemann vector fields are, essentially, gradients of harmonic functions.) There are differences, however: the normal derivative $\partial E_x/\partial n_x$ is, for $(x, y) \in \partial\Omega\times\partial\Omega$, less singular than the kernel of $T(u)$.

REMARK 4.4. If $\partial\Omega$ is of class $C^{k,\alpha}(k \in \mathbb{N}, 0 < \alpha \leq 1)$ and if $u \in C^k(\partial\Omega)$, then $u^{\pm} \in C^k(\overline{\Omega^{\pm}})$. This follows from inspection of the proof of proposition 4.3, using theorem 2 in chapter II, §20 of Günter [3].

We now arrive at the characterization promised in the title of this section.

THEOREM 4.5. *Let $\Omega \subseteq \mathbb{R}^n$, $n \geq 3$, be bounded, open (not necessarily connected) such that each component of $\mathbb{R}^n\backslash\overline{\Omega}$ is simply connected. Suppose that $\partial\Omega$ is of class $C^{k+1,\alpha}$ for some $k \geq 1, 0 < \alpha \leq 1$ and that $v \in C^{k+1}(\partial\Omega;\mathbb{R}^n)$ is a tangential vector field on $\partial\Omega$ which satisfies the induced Cauchy-Riemann equations $dv = 0$. Then v is the tangential part of the restriction to $\partial\Omega$ of a $C^{k,\beta}$ Cauchy-Riemann vector field on $\overline{\Omega}$, $\beta < \alpha$ arbitrary.*

In the special case where Ω is a ball this result is due to Korányi and Vági [4].

PROOF. Let $v \in C^{k+1}(\partial\Omega;\mathbb{R}^n), v$ tangential, $dv = 0$. It is sufficient to show that there exists a $\phi \in C^{k,\beta}(\partial\Omega;\mathbb{R}^n)$ such that for $u := v + \phi\nabla\rho$ on $\partial\Omega, T(u) = 0$ on $\Omega^- = \mathbb{R}^n\backslash\overline{\Omega}$. For then u^+ satisfies the generalized Cauchy-Riemann equations on Ω, by theorem 4.1 and $u^+ = v+\phi\nabla\rho$ on $\partial\Omega$ by proposition 4.3. That $u^+ \in C^{k,\beta}(\overline{\Omega})$ follows from theorem 3.1, combined with remark 4.4 applied to v^+ and [3], theorem 4 of §19, chapter II applied to $T(\phi\nabla\rho)$.

Consider $v^- = T(v)|\Omega^-$. Since each component of Ω^- is simply connected there exists a harmonic function V on Ω^- such that $v^- = \nabla V$ on Ω^-. We can choose V such that $|V(x)| = O(|x|^{2-n})$, $|x| \to \infty$, since $|v^-(x)| = O(|x|^{1-n})$, $|x| \to \infty$. In particular, $V(x)$ is harmonic at ∞.

It would now be sufficient to show that on each component of Ω^- V differs at most a constant from a single layer potential

$$P(\phi) = \int_{\partial\Omega} E_x(y)\phi(y)(\nabla\rho\lfloor\omega)(y). \tag{4.3}$$

For then by formula (3.3), $T(v + \phi\nabla\rho) = \nabla(V - P(\phi)) = 0$ on Ω^-.

Let $\partial V/\partial n$ denote the outward normal derivative of V on $\partial\Omega^- = \partial\Omega$. Since $v \in C^{k+1}(\partial\Omega)$, certainly $\partial V/\partial n \in C^{k,\beta}(\partial\Omega)$ for $0 < \beta < \alpha$; cf. remark 4.4.

The solution to the exterior Neumann problem with boundary value of the normal derivative equal $\partial V/\partial n$ can be written as a single layer potential $P(\phi)$ for some $\phi \in C^{k,\beta}(\partial\Omega), \beta < \alpha$ arbitrary, cf. Günter [3], in particular the lemma in §18 of chapter III. By uniqueness of the solution to the exterior Neumann problem we are done.

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