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Introduction

This paper evolved from research on the question of finding criteria for the algebraicity of affine analytic sets in terms of their intersections with linear subspaces. In recent years, such criteria were obtained by various authors. The classical works here are [9], [16], [17], [21], [23], [28], [29].

The principal topic of this paper is a detailed study of intersections of germs of analytic sets with linear subspaces. As simple consequences of our main results, strong criteria for special entire analytic sets are obtained.

The organization of this paper is as follows. Chapters 1 and 2 are of preparatory nature, where we collect together some facts on Nash functions and Nash sets and derive their consequences for use in other chapters. In Chapter 3 we proceed with the study of restrictions of germs of holomorphic functions to linear subspaces.

Finally, in Chapter 4, our main results are stated and proved. We get sharp criteria for a germ of an analytic set to be a Nash germ. As the consequences of these last results we obtain, in Section 4D, some criteria for entire analytic sets to be algebraic.

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Prerequisites

As for prerequisites, the reader is expected to be familiar with:

1) *Topology*. Besides standard elementary point set topology, some basic facts on proper mappings are frequently used.
2) *Differential geometry*. Complex manifolds and properties of submersions. Note that in this paper all manifolds are assumed to be second-countable.
3) *Analytic and algebraic geometry*. Standard facts on a structure, decomposition and dimension of analytic and algebraic sets. Good references for all this material are: [10], [11], [18], [31].

Finally, we have assumed Chevalley's theorem in the following version (c.p. [18], [22], [30]).

CHEVALLEY'S THEOREM. *Suppose that N and M are finite dimensional complex vector spaces and denote by π the projection $\pi : N \times M \longrightarrow N$. If X is an irreducible algebraic subset of $N \times M$, then $\overline{\pi(X)}$ is an irreducible algebraic subset of N such that* $\dim \overline{\pi(X)} \leq \dim X$.

(Here $\overline{\pi(X)}$ is the closure of $\pi(X)$ in the standard topology of N).

CHAPTER 1

Nash functions

1A. - Definitions and basic properties

In this paper we denote by N a finite dimensional, complex vector space.

DEFINITION 1.1. Let Ω be an open subset of N. Let f be a holomorphic function on Ω. We say that f is a Nash function at $x_0 \in \Omega$ if there exist an open neighbourhood U of x_0 and a polynomial $P : N \times \mathbb{C} \longrightarrow \mathbb{C}$, $P \not\equiv 0$, such that $P(x, f(x)) = 0$ for $x \in U$. A holomorphic function defined on Ω is said to be a Nash function if it is a Nash function at every point of Ω. The family of Nash functions on Ω we denote by $\mathcal{N}(\Omega)$.

The examples of Nash functions on a fixed open subset Ω of N are the restrictions of polynomials and rational functions, holomorphic on Ω. More interesting examples will be presented in other chapters of this paper.

We start with some basic properties of Nash functions. The first one is a simple consequence of the identity principle for holomorphic functions and known facts of the algebraic geometry.

REMARK 1.2. Let D be an open connected subset of N and $x_0 \in D$. If

f is a holomorphic function defined on D, then the following statements are equivalent:

(1) f is a Nash function at x_0,

(2) $f \in \mathcal{N}(D)$,

(3) there exists a proper algebraic subset X of $N \times \mathbb{C}$ such that $f \subset X$,

(4) there exists a unique irreducible algebraic hypersurface X of $N \times \mathbb{C}$ such that $f \subset X$,

(5) there exists an irreducible polynomial $P : N \times \mathbb{C} \longrightarrow \mathbb{C}$, unique up to scalars, such that $P(x, f(x)) = 0$ for $x \in D$.

THEOREM 1.3. *Every entire Nash function is a polynomial.*

PROOF. Let $f : N \longrightarrow \mathbb{C}$ be a Nash function. The graph of f is an irreducible analytic hypersurface in the space $N \times \mathbb{C}$. By the condition (4) of Remark 1.2 there exists an irreducible algebraic hypersurface $X \subset N \times \mathbb{C}$ such that $f \subset X$. Hence $f = X$. Let $P : N \times \mathbb{C} \longrightarrow \mathbb{C}$ be an irreducible polynomial such that $X = P^{-1}(0)$, and let $\pi : N \times \mathbb{C} \longrightarrow N$ be the natural projection. Then the restriction $\pi|_X : X \longrightarrow N$ is proper, and by ([29], Lemma 1) we can assume that

$$P(x, t) = t^d + P_1(x)t^{d-1} + \ldots + P_d(x),$$

where $P_1, \ldots, P_d$ are polynomials on N.

Let us fix $x_0 \in N$ outside the set of zeros of the discriminant of P. Then $\#\{t \in \mathbb{C} : P(x_0, t) = 0\} = d$. The condition $P(x_0, t) = 0$ implies $t = f(x_0)$. Hence $d = 1$ and $f = -P_d$. This ends the proof.

THEOREM 1.4 (Lojasiewicz [15]). *Let D be an open connected subset of N, and G an open connected subset of $N \times \mathbb{C}$, $g \in \mathcal{N}(G)$, $g \not\equiv 0$. If $f : D \longrightarrow \mathbb{C}$ is a holomorphic function such that $f \subset g^{-1}(0)$, then $f \in \mathcal{N}(D)$.*

PROOF. Since $g \in \mathcal{N}(G)$, then by Remark 1.2 there exists an irreducible algebraic hypersurface $X \subset (N \times \mathbb{C}) \times \mathbb{C}$ such that $g \subset X$. Since $g \not\equiv 0$, we have $(N \times \mathbb{C}) \times \{0\} \not\subset X$. Hence

$$X^0 = \{(x, t) \in N \times \mathbb{C} : (x, t, 0) \in X\},$$

is a proper algebraic subset of $N \times \mathbb{C}$. Moreover, it is clear that $f \subset X^0$. Now, using Remark 1.2, we conclude that f is a Nash function.

DEFINITION 1.5. Let N, M be finite dimensional complex vector spaces, and Ω be an open subset of N. Let $F : \Omega \longrightarrow M$ be a holomorphic mapping. We say that F is a Nash mapping at x_0 if there exists a basis of M such that all components of F are Nash functions at x_0. A mapping F is said to be a Nash mapping if it is a Nash mapping at every point of Ω.

PROPOSITION 1.6. *Let D be an open connected subset of N, $F : D \longrightarrow M$ be a holomorphic mapping and $n = \dim N$. Then the following statements are* *equivalent:*

(1) *F is a Nash mapping,*

(2) *there exists an n-dimensional irreducible algebraic subset $X \subset N \times M$ such that $F \subset X$,*

(3) *for each basis of M, all components of F are Nash functions.*

PROOF. (1) $\Rightarrow$ (2). Let $m = \dim M$. It is obvious that there exists a basis $e_1, \dots, e_m$ of M such that $F = \sum_{i=1}^{m} e_i f_i$ and $f_i \in \mathcal{N}(D)$ for $i = 1, \dots, m$. Then we have irreducible polynomials $P_1, \dots, P_m$ on $N \times \mathbb{C}$ such that $P_i(x, f_i(x)) = 0$ for $x \in D$, $i = 1, \dots, m$. Let us fix a point $x_0 \in D$ for which all sets

$$Z_i = \{t \in \mathbb{C} : P_i(x_0, t) = 0\}, \ \ i = 1, \dots, m$$

are finite. We shall consider the algebraic set

$$Y = \{(x, y_1 e_1 + \ldots + y_m e_m) \in N \times M : P_i(x, y_i) = 0, \ \ i = 1, \dots, m\}.$$

Since the intersection

$$Y \cap (\{x_0\} \times M) = \{x_0\} \times \{y_1 e_1 + \ldots + y_m e_m \in M : y_i \in Z_i, \ \ i = 1, \dots, m\},$$

is finite, then from [18] (Prop. 3.28) we conclude that $\dim_{(x_0, F(x_0))} Y \leq n$. Moreover, F is an irreducible n-dimensional analytic subset of Y. Hence there exists an n-dimensional irreducible component X of Y such that $F \subset X$. This ends the proof.

(2) $\Rightarrow$ (3). Let $e_1, \dots, e_m$ be a basis of M. For fixed $i \in \{1, \dots, m\}$, we can find the linear form $L : M \longrightarrow \mathbb{C}$ such that $L(e_i) = 1$ and $L(e_j) = 0$ for $j \neq i$. We check at once that the i-th component of the mapping F is the composition $f = L \circ F$. Define

$$\Phi_L : N \times M \ni (x, y) \longrightarrow (x, L(y)) \in N \times \mathbb{C}.$$

Let X be an n-dimensional irreducible algebraic subset of $N \times M$ such that $F \subset X$. Now we have

$$f = \Phi_L(F) \subset \Phi_L(X) \subset \overline{\Phi_L(X)}.$$

From Chevalley's theorem, the set $\tilde{X} = \overline{\Phi_L(X)}$ is a proper algebraic subset of $N \times \mathbb{C}$. Since $f \subset \tilde{X}$, then by Remark 1.2 f is a Nash function.

(3) $\Rightarrow$ (1). Obvious.

THEOREM 1.7. *Let D and G be open subsets of N and M, respectively,* $\dim N = \dim M = n$. *Let $f : D \longrightarrow G$ be a biholomorphism. If F is a Nash mappings, so is $F^{-1} : G \longrightarrow D$.*

PROOF. There is no loss of generality in assuming that D and G are connected. It follows from Proposition 1.6 that there exists an n-dimensional irreducible algebraic subset X of $N \times M$ such that $F \subset X$. It is clear that $F^{-1} \subset \{(y,x) \in M \times N : (x,y) \in X\}$ and the proof is complete.

As a simple consequence of Theorems 1.3 and 1.7, we obtain the following corollary.

COROLLARY 1.8 (see [22]). *If $F : N \longrightarrow M$ is a polynomial biholomorphism, then $F^{-1} : M \longrightarrow N$ is a polynomial mapping too.*

1B. - Composition of Nash mappings

We now state a result we shall frequently use.

LEMMA 1.9. *Let D be an open connected subset of N and $f \in \mathcal{N}(D)$. If X is an irreducible k-dimensional algebraic subset of N, then there exists an algebraic subset Y of $N \times \mathbb{C}$ of pure dimension k such that*

$$f|_{D\cap X} \subset Y \subset X \times \mathbb{C}.$$

Moreover, if $D \cap X$ is an irreducible analytic subset of D, then we can find Y which is irreducible.

PROOF. The proof is by induction on codim X. If codim $X = 0$ then $X = N$ and by Remark 1.2 existence of Y is clear.

Now, assume that it is true for irreducible algebraic subsets of N of dimension $k+1 \leq n$; we shall prove it for k. Suppose that X is an irreducible algebraic subset of N of dimension k. We can assume that $X \cap D \neq \emptyset$. Let $\tilde{X} \subset N$ be an irreducible algebraic subset of N such that $X \subset \tilde{X}$ and dim $X = k+1$. By assumptions, there exists an algebraic subset $\tilde{Y}$ of pure dimension $k+1$ such that

$$f|_{\tilde{X}\cap D} \subset \tilde{Y} \subset \tilde{X} \times \mathbb{C}.$$

Now, to construct an algebraic set Y for X, we need only consider two cases.

(1) If $X \times \mathbb{C} \not\subset Y$, then we can define Y as the union of all irreducible k-dimensional components of $(X\times\mathbb{C})\cap\tilde{Y}$. Indeed, since $Z = (X\times\mathbb{C})\cap\tilde{Y} \subsetneq X\times\mathbb{C}$, then dim $Z \leq k$. Moreover, $f|_{X\cap D}$ is an analytic subset of $D \times \mathbb{C}$ of pure dimension k. Hence $f|_{X\cap D} \subset Y \subset X \times \mathbb{C}$.

(2) If $X\times\mathbb{C} \subset \tilde{Y}$, then dim $(X\times\mathbb{C}) =$ dim $\tilde{Y} = k+1$, and $X\times\mathbb{C}$ is an irreducible component of $\tilde{Y}$. Let $\tilde{Y}_0$ be the union of other components of $\tilde{Y}$. If we put $\varphi : \tilde{X} \cap D \ni x \longrightarrow (x, f(x)) \in N \times \mathbb{C}$, then $\varphi((\tilde{X}\setminus X) \cap D) \subset \tilde{Y}_0$. The continuity of φ and the density of $(\tilde{X}\setminus X) \cap D$ in $\tilde{X} \cap D$ imply $\varphi(\tilde{X} \cap D) \subset \tilde{Y}_0$. Hence $f|_{\tilde{X}\cap D} \subset \tilde{Y}_0$ and $X \times \mathbb{C} \not\subset Y_0$. At this point we can repeat the argument used at (1) to obtain the required set Y.

The last remark of Lemma 1.9 is a simple consequence of the irreducibility of $X \cap D$, and the proof is complete.

The aim of this section is to prove the following theorem.

THEOREM 1.10. *The composition of Nash mappings is a Nash mapping too.*

PROOF. Without loss of generality we can consider the composition of the form

$$D \xrightarrow{F} G \xrightarrow{f} \mathbb{C},$$

where D and G are open connected subsets of finite dimensional complex vector spaces N and M, respectively, F is a Nash mapping and $f \in \mathcal{N}(G)$.

Let $n = \dim N$. By Proposition 1.6, there exists a n-dimensional irreducible algebraic subset X of $N \times M$ such that $F \subset X$. Now observe that $D \times G$ is an connected subset of $N \times M$ and

$$h : D \times G \ni (x, y) \longrightarrow h(x, y) = f(y) \in \mathbb{C},$$

is a Nash function. Thus, by Lemma 1.9, we can find an algebraic subset Y of $(N \times M) \times \mathbb{C}$ of pure dimension n such that

$$h|_{X \cap (D \times G)} \subset Y \subset X \times \mathbb{C}.$$

Let us define

$$H : D \ni x \longrightarrow H(x) = (F(x), f(F(x))) \in M \times \mathbb{C}.$$

If $x \in D$ then $(x, F(x)) \in X \cap (D \times G)$ and $(x, F(x), f(F(x))) = (x, F(x), h(x, F(x))) \in h|_{X \cap (D \times G)} \subset Y$.

Hence $H \subset Y$ and, by Proposition 1.6, H is a Nash mapping. Therefore, the second component $f \circ F$ of H is a Nash function. This proves the Theorem.

COROLLARY 1.11. *If Ω is an open subset of N, then $\mathcal{N}(\Omega)$ is a subring of the ring $\mathcal{O}(\Omega)$ of holomorphic functions on Ω.*

PROOF. Let $f, g \in \mathcal{N}(\Omega)$. It is easy to see that $F = (f, g) : \Omega \longrightarrow \mathbb{C}^2$ is a Nash mapping. Define two polynomials

$$P_1 : \mathbb{C}^2 \ni (x, y) \longrightarrow x - y \in \mathbb{C},$$

$$P_2 : \mathbb{C}^2 \ni (x, y) \longrightarrow x \cdot y \in \mathbb{C}.$$

Applying Theorem 1.10 to compositions $P_1 \circ F$, $P_2 \circ F$, we get $f - g \in \mathcal{N}(\Omega)$, $f \cdot g \in \mathcal{N}(\Omega)$ and this is precisely the assertion of the Corollary.

COROLLARY 1.12. *If Ω is an open subset of $\mathbb{C}^n$, $f \in \mathcal{N}(\Omega)$, then $\frac{\partial f}{\partial x_i} \in \mathcal{N}(\Omega)$, for $i = 1, \ldots, n$.*

PROOF. We can assume that Ω is connected. Hence there exists an irreducible polynomial $P : \mathbb{C}^n \times \mathbb{C} \longrightarrow \mathbb{C}$, $P \not\equiv 0$, such that $P(x, f(x)) = 0$ for $x \in \Omega$. For a fixed $i \in \{1, \dots, n\}$, we obtain by standard calculations

$$\frac{\partial P}{\partial x_i}(x, f(x)) + \frac{\partial P}{\partial t}(x, f(x))\frac{\partial f}{\partial x_i}(x) = 0 \quad \text{for } x \in \Omega.$$

Set

$$g : \Omega \times \mathbb{C} \ni (x, t) \longrightarrow \frac{\partial P}{\partial t}(x, f(x)) \cdot t + \frac{\partial P}{\partial x_i}(x, f(x)) \in \mathbb{C}.$$

Since $\frac{\partial P}{\partial t}$, $\frac{\partial P}{\partial x_i}$ are polynomials then, by Theorem 1.10 and Corollary 1.11, $g \in \mathcal{N}(\Omega \times \mathbb{C})$. But $\frac{\partial P}{\partial t}(x, f(x))$ does not vanish identically on Ω since P is irreducible. Hence $g \not\equiv 0$ and $\frac{\partial f}{\partial x_i} \subset g^{-1}(0)$. Thus, by Theorem 1.4, we obtain the required result.

CHAPTER 2

Nash sets

2A. - Analytic sets with proper projections

In this section we recall some known theorems of the geometry that will be useful to us.

Let M be an m-dimensional complex vector space, and D an n-dimensional connected complex manifold. Let X be a purely n-dimensional analytic subset of $D \times M$ and $\pi : D \times M \ni (x, y) \longrightarrow x \in D$ the natural projection. In this section, except the last theorem, we assume that the restriction $\pi|_X : X \longrightarrow D$ is a proper mapping.

Since X has pure dimension n, the mapping $\pi|_X : X \longrightarrow D$ is a branched covering. More explicitly, the following theorem is true ([32], 2).

THEOREM 2.1. *The mapping $\pi|_X : X \longrightarrow D$ is surjective and open. There exist an integer $s = s(X)$ and a proper analytic subset $S = S(X)$ of the manifold D such that:*

(1) $\#(\pi|_X)^{-1}(x) = s$ *for* $x \in D\backslash S$,

$\#(\pi|_X)^{-1}(x) < s$ *for* $x \in S$,

(2) *for every $x \in D\backslash S$, there exists a neighbourhood $U \subset D\backslash S$ of x and holomorphic mappings $f_1, \dots, f_s : U \longrightarrow M$ such that $f_i \cap f_j = \emptyset$ for $i \neq j$ and $(U \times M) \cap X = f_i \cup \dots \cup f_s$,*

(3) *the mapping $\pi|_{X\backslash\pi^{-1}(S)} : X\backslash\pi^{-1}(S) \longrightarrow D\backslash S$ is a local biholomorphism.*

Now, let $L : M \longrightarrow \mathbb{C}$ be a linear form on M. If we set

$$\Phi_L : D \times M \ni (x,y) \longrightarrow (x, L(y)) \in D \times \mathbb{C},$$
$$X_L = \Phi_L(X),$$
$$X^L = \Phi_L^{-1}(X_L) = \Phi_L^{-1}(\Phi_L(X)),$$
$$\tilde{\pi} : D \times \mathbb{C} \ni (x,t) \longrightarrow x \in D,$$

then we obtain the following theorem.

THEOREM 2.2. *If L is a linear form on M, then*

(1) *the restriction $\Phi_L|_X : X \longrightarrow D \times \mathbb{C}$ is proper,*

(2) *the set $X_L = \Phi_L(X)$ is an analytic subset of $D \times \mathbb{C}$ of pure dimension n,*

(3) *the projection $\tilde{\pi}|_{X_L} : X_L \longrightarrow D$ is proper.*

Moreover, if $r = (m-1)s(X)+1$ and $L_1, \dots, L_r$ are linear forms on M such that every m-tuple from $L_1, \dots, L_r$ is linearly independent, then $X = \bigcap_{i=1}^{r} X^{L_i}$.

PROOF. (1) and (3) are simple topological consequences of our assumptions. By (1) and [18] (Th. 4.11), we obtain (2). The second part of the theorem follows from [19] p. 679.

Now, we consider the case $M = \mathbb{C}$.

THEOREM 2.3 (see [29], Lemma 1). *Let X be an analytic subset of $D \times \mathbb{C}$ of pure dimension n such that the projection $\pi|_X : X \longrightarrow D$ is proper. Then there exists a unique system $\sigma_1, \dots, \sigma_s$ of holomorphic functions on D such that*

$$X = \{(x,t) \in D \times \mathbb{C} : t^s + \sigma_1(x)t^{s-1} + \dots + \sigma_s(x) = 0\}.$$

In the next parts of this paper we denote by σ_X the function

$$D \times \mathbb{C} \ni (x,t) \longrightarrow \sigma_X(x,t) = t^s + \sigma_1(x)t^{s-1} + \dots + \sigma_s(x) \in \mathbb{C}.$$

Combining [29] (Lemma 1) and classical Rouche's theorem, we have

THEOREM 2.4. *If $\sigma_0, \dots, \sigma_d$, $\sigma_0 \not\equiv 0$ are holomorphic functions on the manifold D and*

$$X = \{(x,t) \in D \times \mathbb{C} : \sigma(x,t) = \sigma_0(x)t^d + \dots + \sigma_d(x) = 0\},$$

then

(1) *X is an analytic subset of $D \times \mathbb{C}$ of pure dimension n,*

(2) *the projection $\pi|_X : X \longrightarrow D$ is proper if and only if $\sigma_0(x) \neq 0$ for $x \in D$,*

(3) *if* $\pi\|_X : X \longrightarrow D$ *is proper,* $x_0 \in D\backslash S(X)$, U *and* $f_1, \dots, f_s$ *are the same as in Theorem* 2.1 *(2), then there exist positive integers* $n_1, \dots, n_s$ *such that* $n_1 + \dots + n_s = d$ *and*

$$\sigma(x,t) = \sigma_0(x) \prod_{i=1}^{s} (t - f_i(x))^{n_i}, \qquad \text{for } x \in U,\ t \in \mathbb{C}.$$

2B. - Structure of Nash sets

DEFINITION 2.5. A subset X of an open set $\Omega \subset N$ is said to be a Nash subset of Ω if for every $x_0 \in \Omega$ there exist a neighbourhood $U \subset \Omega$ of x_0 and Nash functions $f_1, \dots, f_r$ on U such that

$$X \cap U = \{x \in U : f_1(x) = f_2(x) = \dots = f_r(x) = 0\}.$$

Since every Nash set is analytic in Ω, the terminology of analytic geometry can be used here. Especially, components, irreducibility and dimension of Nash sets is defined to be the same as they are for analytic sets.

PROPOSITION 2.6. *Let* Ω *be an open subset of* N *and let* $\{X_\alpha\}_{\alpha \in A}$ *be a family of Nash subsets of* Ω. *Then*

(1) *the intersection* $\bigcap_{\alpha \in A} X_\alpha$ *is a Nash subset of* Ω,

(2) *if* $\{X_\alpha\}_{\alpha \in A}$ *is locally finite in* Ω, *then the union* $\bigcup_{\alpha \in A} X_\alpha$ *is a Nash subset of* Ω.

PROOF. Since $X_\alpha, \alpha \in A$ are analytic subsets of Ω, for every point $x_0 \in \Omega$ there exist a neighbourhood $U \subset \Omega$ of x_0 and a finite family $\{\alpha_1, \dots, \alpha_r\} \subset A$ such that

$$U \cap (\bigcup_{\alpha \in A} X_\alpha) = U \cap X_{\alpha_1} \cap \dots \cap X_{\alpha_r}.$$

Thus (1) is a simple consequence of Definition 2.5, and (2) is obvious.

LEMMA 2.7. *Let* X *be an irreducible algebraic subset of* N *and let* D *be an open connected subset of* N. *If* Y *is an irreducible analytic component of the intersection* $X \cap D$, *then* Y *is a Nash subset of* D.

PROOF. Let $k = \dim X$ and $m = \operatorname{codim} X = n - k$. It suffices to show that, for every point $z_0 \in Y$, there exists an open neighbourhood $U \subset D$ of z_0 such that $U \cap X$ is the set of common zeros of finite family of Nash functions defined on U.

Without loss of generality, we can assume that $k < n$, $z_0 = 0 \in \mathbb{C}^k \times \mathbb{C}^m$ and 0 is an isolated point of the intersection $(\{0\} \times \mathbb{C}^m) \cap Y$.

We can find two open balls B_1, B_2 in $\mathbb{C}^k$ and $\mathbb{C}^m$, respectively, with centers at the origin, such that $\overline{B}_1 \times \overline{B}_2 \subset D$ and $(B_1 \times \partial B_2) \cap Y = \emptyset$. Put $U = B_1 \times B_2$ and $\tilde{Y} = Y \cap U$. The set Y is a purely k-dimensional analytic subset of $B_1 \times \mathbb{C}^m$ such that the natural projection $\pi|_{\tilde{Y}} : \tilde{Y} \longrightarrow B_1$ is proper.

Keeping the notation of 2A with $D = B_1$ and $M = \mathbb{C}^m$, we can find linear forms $L_1, \ldots, L_r$ on $\mathbb{C}^m$, $r = (m-1)s(Y) + 1$ in the same way as in Theorem 2.2. Then $\tilde{Y} = \cap\{\tilde{Y}^{L_i} : i = 1, \ldots, r\}$. We can see that

$$Y \cap U = \{(x, y) \in U : f_1(x, y) = \ldots = f_r(x, y) = 0\},$$

where $f_i = \sigma_{\tilde{Y}_{L_i}} \circ \Phi_{L_i}$ for $i = 1, \ldots, r$.

Now it suffices to prove that $\sigma_{\tilde{Y}_{L_i}} : B_1 \times \mathbb{C} \longrightarrow \mathbb{C}$ are Nash functions for $i = 1, \ldots, r$. Let us fix $i \in \{1, \ldots, r\}$. If we set $Z = \tilde{Y}_{L_i}$, $L = L_i$, then

$$\sigma_Z(x, t) = t^s + \sigma_1(x)t^{s-1} + \ldots + \sigma_s(x),$$

where $s = s(Z)$ and $\sigma_1, \ldots, \sigma_s$ are holomorphic functions defined on the ball B_1.

We can find a point $x_0 \in B_1 \backslash S(Z)$ and its neighbourhood $G \subset B_1$ such that (2) of Theorem 2.1 holds. This means that $Z \cap (G \times \mathbb{C}) = g_1 \cup \ldots \cup g_s$, where $g_1, \ldots, g_s$ are holomorphic functions on G with disjoint graphs. Finally, by Theorem 2.4, we obtain

$$\sigma_Z(x, t) = (t - g_1(x)) \cdot \ldots \cdot (t - g_s(x)),$$

for $x \in G$ and $t \in \mathbb{C}$.

Now, observe that

$$g_j \subset Z = \tilde{Y}_L = \Phi_L(\tilde{Y}) \subset \tilde{\Phi}_L(X) \subset H = \overline{\tilde{\Phi}_L(X)},$$

where $\tilde{\Phi}_L : \mathbb{C}^k \times \mathbb{C}^m \ni (x, y) \longrightarrow (x, L(y)) \in \mathbb{C}^k \times \mathbb{C}$. By Chevalley's theorem, the set H is a proper algebraic subset of $\mathbb{C}^k \times \mathbb{C}$. Since $g_j \subset H$, for $j = 1, \ldots, s$, $g_1, \ldots, g_s$ are Nash functions on G and Theorem 1.10 implies that $\sigma_Z|_{G \times \mathbb{C}}$ is a Nash function. Since $B_1 \times \mathbb{C}$ is connected, σ_Z is a Nash function. So the proof is complete.

LEMMA 2.8. *Let Y be an irreducible analytic subset of an open connected subset D of N. If there exists an open subset $U \subset D$ such that $U \cap Y$ is a non-empty Nash subset of U, then we can find an irreducible algebraic subset X of N such that $Y \subset X$ and* $\dim X = \dim Y$.

PROOF. Let X be the smallest algebraic subset of N containing the set Y (i.e. X is Zariski's closure of Y). Then X is irreducible, $Y \subset X$ and $\dim Y \leq \dim X$.

We only need to show that dimensions of X and Y are equal.

Suppose, on the contrary, that $\dim Y < \dim X$. Since $Y \not\subset \operatorname{Sing}(X)$, $Y \backslash \operatorname{Sing}(X)$ is dense in Y. Let us fix a point $x_0 \in (Y \backslash \operatorname{Sing}(X)) \cap U$ and open connected neighbourhood $G \subset U$ of x_0 such that:

(1) the intersection $X \cap G$ is a connected submanifold of G,

(2) there exist Nash functions $f_1, \ldots, f_r$ defined on G such that

$$Y \cap G = \{x \in G : f_1(x) = \ldots = f_r(x) = 0\}.$$

Since $\dim Y < \dim X$, $Y \cap G \subsetneq X \cap G$ and there exists an integer $i \in \{1, \ldots, r\}$ such that $f_i|_{X \cap G} \neq 0$. Let $f = f_i$. By Lemma 1.9, there exists an irreducible algebraic subset $Z \subset X \times \mathbb{C}$ such that $\dim X = \dim Z, f|_{X \cap G} \subset Z$.

If $X_1 = \{x \in X : (x, 0) \in Z\}$ then $X_1 \neq X$. Indeed, supposing that $X_1 = X$, we have

$$X \times \{0\} \subset Z \quad \text{and} \quad Z = X \times \{0\},$$

because Z is irreducible. Hence $f|_{X \cap G} = 0$, which is impossible. Therefore X_1 is a proper algebraic subset of X. Moreover, if $x \in Y \cap G$, then $(x, 0) \in f|_{X \cap G} \subset Z$. thus $\emptyset \neq Y \cap G \subset X_1$ and finally, $Y \subset X_1 \subsetneq X$ contrary to the definition of the set X. This ends the proof.

THEOREM 2.9. *Let Y be an irreducible analytic subset of an open subset Ω of N. If there exists an open subset U of Ω such that $U \cap Y$ is a non-empty Nash subset of U, then Y is a Nash subset of Ω.*

PROOF. Since Y is irreducible, it is contained in a connected component of Ω. By Lemma 2.8, there exists an irreducible algebraic subset X of N of dimension $k = \dim Y$ such that $Y \subset X$.

In our situation Y is an irreducible component of the intersection $X \cap D$. Hence, by Lemma 2.7, Y is a Nash subset of Ω.

THEOREM 2.10. *Let Y be an irreducible analytic subset of an open set $\Omega \subset N$. Then Y is a Nash subset of Ω if and only if Y is an irreducible component of the intersection of Ω with a certain irreducible algebraic subset of N.*

PROOF. Let D be the connected component of Ω which contains Y and let $k = \dim Y$. If Y is a Nash subset of D, then by Lemma 2.8 there exists a k-dimensional irreducible algebraic subset X of N such that $Y \subset X$. Since dimensions of X and Y are equal, Y must be an irreducible component of the intersection $X \cap D$.

Conversely, if X is an irreducible algebraic subset of N, then in view of Lemma 2.7 every irreducible component of $X \cap D$ is a Nash subset of Ω. This proves the theorem.

THEOREM 2.11. *Let X be a Nash subset of an open subset $\Omega \subset N$ and let Y be an irreducible component of X. Then Y is a Nash subset of Ω.*

PROOF. Denote by Y' the union of the components of X different from Y. Setting $U = \Omega \setminus Y'$, we see at once that $Y \cap U = X \cap U$ is a non-empty Nash subset of U. Hence from Theorem 2.9, it follows that Y is a Nash subset of Ω, which is our claim.

We now state some results we shall frequently use

THEOREM 2.12. *An irreducible Nash subset of the space N is an irreducible algebraic subset of N.*

PROOF. Substituting $\Omega = N$ into Theorem 2.10, we get the required result.

THEOREM 2.13. *Let U be an open subset of an open set $\Omega \subset N$. Suppose that X is an analytic subset of Ω such that:*

(1) *each its irreducible component meets U,*

(2) *$U \cap X$ is a Nash subset of U.*

Then X is a Nash subset of Ω.

PROOF. Let Y be an irreducible component of X. Then $U \cap Y$ is the union of some components of $U \cap X$. Therefore Theorem 2.11 and Proposition 2.6 show that $U \cap Y$ is a non-empty Nash subset of U. Using Theorem 2.9, we can see that Y is a Nash subset of Ω. Now Proposition 2.6 completes the proof.

COROLLARY 2.14. *Let X be an analytic subset of N such that the point $0 \in N$ belongs to every irreducible component of X.*

If there exists an open neighbourhood U of 0 such that $U \cap X$ is a Nash subset of U, then X is algebraic.

PROOF. By Theorem 2.13, X is a Nash subset of N. Hence Theorems 2.11 and 2.12 imply the algebraicity of the irreducible components of X. Since the set of irreducible components of X is finite, the set X is algebraic.

THEOREM 2.15. *Let Ω be an open subset of N and let $F : \Omega \longrightarrow M$ be a holomorphic mapping. Then F is a Nash mapping if and only if F is a Nash subset of $\Omega \times M$.*

PROOF. There is no loss of generality in assuming that $M = \mathbb{C}^m$. If $F = (f_1, \ldots, f_m)$ is a Nash mapping, then by definition

$$F = \{(x, y_1, \ldots, y_m) \in \Omega \times \mathbb{C}^m : y_1 - f_1(x) = \ldots = y_m - f_m(x) = 0\},$$

is a Nash subset of $\Omega \times M$.

Conversely, we can assume that Ω is connected. In view of Lemma 2.8, there exists an n-dimensional ($n = \dim N$) algebraic subset X of $N \times M$ such that $F \subset X$. Now Proposition 1.6 completes the proof.

THEOREM 2.16. *Let Ω be an open subset of $N \times M$, and let $(x_0, y_0) \in \Omega$. Assume that*

$$G : \Omega \ni (x, y) \longrightarrow G(x, y) \in M,$$

is a Nash mapping such that $G(x_0, y_0) = 0$ and

$$\det\left(\frac{\partial G}{\partial y}(x_0, y_0)\right) \neq 0.$$

Under the above assumptions, there exist open neighbourhoods U, V of x_0 and y_0 respectively and a Nash mapping $F : U \longrightarrow V$ such that

$$F = (G^{-1}(0)) \cap (U \times V).$$

PROOF. In view of the "implicit function theorem", we obtain a holomorphic function F, for which all assertions are true. Now, Theorem 2.15 yields F is a Nash function, and the proof is complete.

2C. - Projections of Nash sets

Let M be an m-dimensional complex vector space and let Ω be an open subset of N. Let $\pi : \Omega \times M \ni (x, y) \longrightarrow x \in \Omega$ be the natural projection. Assume that X is a subset of $\Omega \times M$ such that $\pi|_X : X \longrightarrow \Omega$ is proper.

Note that, by Remmert's theorem, if X is analytic so is $\pi(X)$. By Chevalley's theorem, a similar result for algebraic set can be obtained. Moreover, the following theorem is true.

THEOREM 2.17. *Let X be a Nash subset of $\Omega \times M$ such that $\pi|_X : X \longrightarrow \Omega$ is proper. Then $\pi(X)$ is a Nash subset of Ω.*

PROOF. We first suppose that X is irreducible. Without loss of generality, we can assume that Ω is connected. Theorem 2.10 shows that there exists an irreducible algebraic subset Z of $N \times M$ such that $X \subset Z$ and $\dim Z = \dim X$. Let projection $\tilde{\pi}$ be defined by $\tilde{\pi} : N \times M \ni (x, y) \longrightarrow x \in N$. Then $\pi(X) \subset \tilde{\pi}(Z) \subset Y = \overline{\tilde{\pi}(Z)}$ and:

(1) $\pi(X)$ is an irreducible analytic subset of Ω such that $\dim \pi(X) = \dim X$,

(2) Chevalley's theorem implies Y is an irreducible algebraic subset of N.

Since $\pi(X) \subset Y$ and dimensions of X and Z are equal, $\dim \pi(X) = \dim Y$. Hence $\pi(X)$ is an irreducible component of the intersection $Y \cap \Omega$. From Lemma 2.7 we conclude that $\pi(X)$ is a Nash subset of Ω.

To prove the general case, we can express X as the union of its irreducible components X_i, $i = 1, 2 \ldots$. By Theorem 2.11, each component X_i, $i = 1, 2, \ldots$, is a Nash subset of $\Omega \times M$, so projections $\pi(X_i)$, $i = 1, 2, \ldots$, are Nash subsets of Ω. We can see that the family $\{\pi(X_i)\}_{i=1}^{\infty}$ is locally finite. Hence Proposition 2.6 shows that $\pi(X) = \bigcup_{i=1}^{\infty} \pi(X_i)$ is a Nash subset of Ω, and the proof is complete.

COROLLARY 2.18. *Let G, Ω be open subsets of the space M and N, respectively. Suppose that $F : G \longrightarrow \Omega$ is a Nash mapping.*

If X is a Nash subset of G such that the restriction $F|_X : X \longrightarrow \Omega$ is proper, then $F(X)$ is a Nash subset of Ω.

PROOF. $Y = F \cap (X \times N) = \{(x, F(x)) : x \in X\}$ is a Nash subset of $G \times N$. Of course, $\pi|_Y : Y \ni (x, y) \longrightarrow y \in \Omega$ is a proper mapping. Hence Y is closed in $M \times \Omega$, and so Y is a Nash subset of $M \times \Omega$ with a proper projection on Ω.

Therefore, by Theorem 2.17, $\pi(Y) = F(Y)$ is a Nash subset of Ω, which ends the proof.

We conclude this section with a useful theorem

THEOREM 2.19. *Let $\sigma_1, \ldots, \sigma_d$ be holomorphic functions defined on an open connected subset D of N and let*

$$X = \{(x,t) \in D \times \mathbb{C} : t^d + \sigma_1(x)t^{d-1} + \ldots + \sigma_d(x) = 0\}.$$

Then the following conditions are equivalent:

(1) *$\sigma_1, \ldots, \sigma_d$ are Nash functions,*

(2) *X is a Nash subset of $D \times \mathbb{C}$.*

PROOF. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (1). Theorem 2.4 shows that the natural projection $\pi|_X : X \longrightarrow D$ is proper. Let us fix a point $x_0 \in D \setminus S(X)$ and its connected neighbourhood U such that $(U \times \mathbb{C}) \cap X = f_1 \cup \ldots \cup f_s$, $s = s(X)$ as well as in Theorem 2.1 (2). In view of Theorem 2.4, there exist positive integers $n_1, \ldots, n_s$ such that $n_1 + \ldots + n_s = d$ and

$$t^d + \sigma_1(x)t^{d-1} + \ldots + \sigma_d(x) = \prod_{i=1}^{s}(t - f_i(x))^{n_i},$$

for $x \in U$, $t \in \mathbb{C}$. Graphs of functions $f_1, \ldots, f_s$ are irreducible Nash subsets of $U \times \mathbb{C}$. Hence Theorem 2.15 shows that $f_1, \ldots, f_s$ are Nash functions. An easy computation gives $\sigma_i|_U \in \mathcal{N}(U)$ for $i = 1, 2, \ldots, d$. Finally, since D is connected, $\sigma_1, \ldots, \sigma_d$ are Nash functions. This proves the theorem.

CHAPTER 3

Restrictions of germs of holomorphic functions to linear subspaces

3A. - Degree of a Nash function

Let D be an open connected subset of N and let $f : D \longrightarrow \mathbb{C}$ be a Nash function. Then there exists an irreducible polynomial $P : N \times \mathbb{C} \longrightarrow \mathbb{C}$, unique up to scalars, such that $P(x, f(x)) = 0$ for $x \in D$. We can write P in the form

$$P(x,t) = \sigma_0(x)t^s + \sigma_1(x)t^{s-1} + \ldots + \sigma_s(x),$$

where $s \in \mathbb{N}$, $\sigma_0, \ldots, \sigma_s$ are polynomials on N, and $\sigma_0 \not\equiv 0$. In this situation the integers:

$s =$ degree of P with respect to the variable "t",

$d =$ degree of the polynomial P,

are uniquely determined by f.

DEFINITION 3.1. Integers s, d, defined above, are called the degree and the total degree of $f \in \mathcal{N}(D)$ and they are denoted by $\deg_D f, \operatorname{Deg}_D f$, respectively.

We shall denote by:

$\mathcal{O}_x(N)$ - the ring of germs at $x \in N$ of holomorphic functions,

$\mathcal{N}_x(N)$ - the ring of germs at $x \in N$ of Nash functions,

$(f)_x$ - the germ at $x \in \Omega$ of the holomorphic function $f : \Omega \longrightarrow \mathbb{C}$.

LEMMA 3.2. *Let G and D be open connected subsets of N and suppose that $f \in \mathcal{N}(D)$ and $g \in \mathcal{N}(G)$. If there exists a point $x_0 \in D \cap G$ such that $(f)_{x_0} = (g)_{x_0}$, then $\deg_D f = \deg_G g$ and $\operatorname{Deg}_D f = \operatorname{Deg}_G g$.*

PROOF. Let $P : N \times \mathbb{C} \longrightarrow \mathbb{C}$ be an irreducible polynomial such that $P(x, f(x)) = 0$ for $x \in D$. Since $(f)_{x_0} = (g)_{x_0}$, there exists a neighbourhood $U \subset G$ of x_0 such that $P(x, g(x)) = 0$ for $x \in U$.

Obviously $\deg_D f = \deg_G g$ and $\operatorname{Deg}_D f = \operatorname{Deg}_G g$, which is our claim.

Now we are able to state the following definition.

DEFINITION 3.3. The degree and the total degree of a germ $u \in \mathcal{N}_x(N)$ is defined to be $\deg_D f$ and $\operatorname{Deg}_D f$, respectively, where f is a Nash function on an open connected neighbourhood D of x such that $(f)_x = u$. The degree and the total degree of u will be denoted by $\deg u$ and $\operatorname{Deg} u$, respectively.

Next, let Ω be an open subset of N and let $f \in \mathcal{N}(\Omega)$. For each point $x \in \Omega$ we can state

DEFINITION 3.4. The degree $\deg_x f$ and the total degree $\operatorname{Deg}_x f$, of a function f at x, is defined to be $\deg(f)_x$ and $\operatorname{Deg}(f)_x$, respectively.

Note that, in all our definitions presented in this part, we have

$$1 \leq \text{ degree } \leq \text{ total degree.}$$

Observe that, if D is an open connected subset of N and $f \in \mathcal{N}(D)$, then $\deg_D f = 1$ if and only if f is a rational function holomorphic on D.

Moreover, let us mention that, for any polynomial $P : N \longrightarrow \mathbb{C}$, we obtain

$$\operatorname{Deg}_N P = \begin{cases} 1, & \text{if } P \text{ is constant,} \\ \text{classical degree of } P, & \text{in other cases.} \end{cases}$$

Finally, let us consider the function

$$f : D \ni x \longrightarrow \frac{1}{1 + \sqrt{1 + x}} \in \mathbb{C},$$

where D is the unit disk in $\mathbb{C}$ and $\sqrt{1+x}$ is the branch of a square root of $1+x$ with value 1 at the origin.

Then $P : \mathbb{C} \times \mathbb{C} \ni (x,t) \longrightarrow xt^2 + 2t - 1 \in \mathbb{C}$ is an irreducible polynomial associated with f. Hence $\deg_D f = 2$, $\operatorname{Deg}_D f = 3$ and $\deg_x f = 2$, $\operatorname{Deg}_x f = 3$ for every $x \in D$.

3B. - Restrictions to complex lines

We now proceed to much deeper and more important results. The purpose of this section is to analyze restrictions of germs of holomorphic functions to linear subspaces.

For given $u \in \mathcal{O}_0(N)$ and a linear subspace M of N, we define $u_M \in \mathcal{O}_0(M)$ by

$$u_M = (f|_{M\cap\Omega})_0,$$

where a holomorphic function $f : \Omega \longrightarrow \mathbb{C}$, defined on an open neighbourhood $\Omega \subset N$ of 0, represents the germ u.

Let us start with a simple classical remark.

REMARK 3.5. Let $T \neq \emptyset$ be a set, $\nu \in \mathbb{N}$ and let $f_i : T \longrightarrow \mathbb{C}$, for $i = 1,\ldots,\nu$. Then $f_1,\ldots,f_\mu$ are linearly dependent in $\mathbb{C}^T$ if and only if $\det(f_i(t_j))_{i,j=1,\ldots,\nu} = 0$ for $t_j \in T$, $j = 1,2,\ldots\nu$.

Now we prove the following useful lemma.

LEMMA 3.6. *Let Ω be a complex manifold, $\Delta = \{t \in \mathbb{C} : |t| < 1\}$ and let $g : \Omega \times \mathbb{C} \to \mathbb{C}$ be a holomorphic function. Then, for $s, d \in \mathbb{N}$ such that $1 \leq s \leq d$, the set*

$$\mathcal{N}_s^d(g) = \{x \in \Omega : g(x,\cdot) \in \mathcal{N}(\Delta),\ \deg g(x,\cdot) \leq s,\ \operatorname{Deg} g(x,\cdot) \leq d\},$$

is an analytic subset of Ω.

PROOF. Let us consider the set

$$A = \{I = (p,q) \in \mathbb{N}^2 : 0 \leq p \leq s,\ 0 \leq p+q \leq d\},$$

ordered as follows

$$\begin{aligned}
I_1 &= (s, d-s),\\
I_2 &= (s, d-s-1),\\
&\vdots\\
I_{\nu-1} &= (0,1),\\
I_\nu &= (0,0), \text{ where } \nu = \tfrac{(2d-s+2)(d+1)}{2} \text{ is the number of elements of } A.
\end{aligned}$$

For each $I_j = (p,q) \in A$, we define a holomorphic function

$$f_j : \Omega \times \Delta \ni (x,t) \longrightarrow f_j(x,t) = t^q g^p(x,t) \in \mathbb{C}.$$

Let $x \in \Omega$. We can see that $x \in \mathcal{N}_s^d(g)$ if and only if the functions $f_1(x,\cdot),\dots,f_\nu(x,\cdot)$ are linearly dependent.

The function $\Lambda : \Omega \times \Delta^\nu \longrightarrow \mathbb{C}$, defined by

$$\Lambda(x,t_1,\dots,t_\nu) = \det[f_j(x,t_i)]_{i,j=1,\dots,\nu},$$

is a holomorphic function and, by Remark 3.5, $x \in \mathcal{N}_s^d(g)$ if and only if $\Lambda(x,t_1,\dots,t) = 0$ for every $(t_1,\dots,t_\nu) \in \Delta^\nu$. Hence

$$\mathcal{N}_s^d(g) = \bigcap_{(t_1,\dots,t_\nu)\in\Delta^\nu} (\Lambda(\cdot,t_1,\dots,t))^{-1}(0),$$

is an analytic subset of Ω.

THEOREM 3.7. *Let s, d be two integers such that $1 \le s \le d$. If $u \in \mathcal{O}_0(N)$, then*

$$\mathcal{N}_s^d(u) = \{a \in \mathbb{P}(N) : u_a \in \mathcal{N}_0(a), \deg u_a \le s, \operatorname{Deg} u_a \le d\},$$

is an algebraic subset of $\mathbb{P}(N)$.

PROOF. Let us fix an arbitrary norm in the space N. Then there exists a ball $B = B(0,r)$, $r > 0$, and a holomorphic function $f : B \longrightarrow \mathbb{C}$ such that $u = (f)_0$.

Let us denote by $\mathbb{P}$ the natural mapping

$$\mathbb{P} : N\backslash\{0\} \ni x \longrightarrow \mathbb{P}(x) = \mathbb{C}\cdot x \in \mathbb{P}(N).$$

For any given $x_0 \in \mathbb{P}(N)$, there exist a neighbourhood Ω of x_0 and a holomorphic mapping $s : \Omega \longrightarrow B$ such that $\mathbb{P} \circ s = \mathrm{id}_\Omega$. Put

$$g_\Omega : \Omega \times \Delta \ni (x,t) \longrightarrow f(t\cdot s(x)) \in \mathbb{C}.$$

Returning to the situation and notation of Lemma 3.6, we can see that

$$\mathcal{N}_s^d(u) \cap \Omega = \mathcal{N}_s^d(g_\Omega),$$

and so $\mathcal{N}_s^d(u)$ is an analytic subset of $\mathbb{P}(N)$. Thus, by Chow's theorem, $\mathcal{N}_s^d(u)$ is an algebraic subset of $\mathbb{P}(N)$, and the proof is complete.

COROLLARY 3.8. *Let s, d be two positive integers. If $u \in \mathcal{O}_0(N)$, then:*

(1) $\mathcal{N}^d(u) = \{a \in \mathbb{P}(N) : u_a \in \mathcal{N}_0(a),\ \operatorname{Deg} u_a \le d\}$ *is an algebraic subset of* $\mathbb{P}(N)$.

(2) $\mathcal{N}_s(u) = \{a \in \mathbb{P}(N) : u_a \in \mathcal{N}_0(a),\ \deg u_a \le s\}$ *is a countable union of algebraic subsets of* $\mathbb{P}(N)$.

(3) $\mathcal{N}(u) = \{a \in \mathbb{P}(N) : u_a \in \mathcal{N}_0(a)\}$ *is a countable union of algebraic subsets of* $\mathbb{P}(N)$.

PROOF. This is proved by observing that

(1) $\mathcal{N}^d(u) = \mathcal{N}^d_d(u)$,

(2) $\mathcal{N}_s(u) = \bigcup_{d=s}^{\infty} \mathcal{N}^d_s(u)$,

(3) $\mathcal{N}(u) = \bigcup_{d=1}^{\infty} \mathcal{N}^d(u)$.

COROLLARY 3.9. *Let* $u \in \mathcal{N}_0(N)$. *If* $\deg u = s$ *and* $\operatorname{Deg} u = d$, *then,* $\mathcal{N}^d_s(u) = \mathbb{P}(N)$.

PROOF. Let B and f be the same as in the proof of Theorem 3.7. Then there exists a polynomial

$$P : N \times \mathbb{C} \ni (x,t) \longrightarrow \sigma_0(x)t^s + \ldots + \sigma_s(x) \in \mathbb{C}$$

of degree d such that $\sigma_0 \not\equiv 0$ and $P(x, f(x)) = 0$ for $x \in B$.

Note that $X = \{a \in \mathbb{P}(N) : \sigma_0|_a = \sigma_1|_a = \ldots = \sigma_s|_a = 0\}$ is a proper algebraic subset of $\mathbb{P}(N)$ and, for $a \in X$, we have $\deg u_a \leq s$ and $\operatorname{Deg} u_a \leq d$. Thus $X \cup \mathcal{N}^d_s(u) = \mathbb{P}(N)$. By Theorem 3.7, $\mathcal{N}^d_s(u)$ is an algebraic subset of $\mathbb{P}(N)$, and so $\mathcal{N}^d_s(u) = \mathbb{P}(N)$.

The next result is central in this chapter.

THEOREM 3.10. *Let* $u \in \mathcal{O}_0(N)$ *and* $s \geq 1$. *If* $\mathcal{N}_s(u) = \mathbb{P}(N)$, *then* $u \in \mathcal{N}_0(N)$ *and* $\deg u \leq s$.

PROOF. Let us fix an arbitrary norm in N. We can find a ball $B = B(0,r)$ and a holomorphic function $f : B \longrightarrow \mathbb{C}$ such that:

(1) $u = (f)_0$

(2) $f = \sum_{i=0}^{\infty} f_i$, where f_i is a homogeneous polynomial of degree i, for $i = 0, 1, \ldots$, and the series converges uniformly in B.

Since $\mathcal{N}_s(u) = \bigcup_{d=s}^{\infty} \mathcal{N}^d_s(u) = \mathbb{P}(N)$, by Theorem 3.7, there exists an integer $d \geq s$ such that $\mathcal{N}^d_s(u) = \mathbb{P}(N)$.

Returning to the set A, defined in the proof of Lemma 3.6, for each $I_j = (p,q) \in A$, we put

$$\varphi_j : B \times \Delta \ni (x,t) \longrightarrow \varphi_j(x,t) = t^q f^p(tx) \in \mathbb{C}.$$

Our assumptions imply that $\varphi_1(x,\cdot), \ldots, \varphi_\nu(x,\cdot)$ are linearly dependent in $\mathcal{O}(\Delta)$, for $x \in B$. Observe that

$$\varphi_j(x,t) = \sum_{i=0}^{\infty} t^i Q_{j,i}(x),$$

where $Q_{j,i}$ is a homogeneous polynomial for $i = 0, 1, \ldots$.

Therefore, for every $x \in B$, there exists $(c_1, \dots, c_\nu) \in \mathbb{C}^\nu \setminus \{0\}$ such that

$$\sum_{j=1}^{\nu} c_j \left[\sum_{i=0}^{\infty} t^i Q_{j,i}(x) \right] = 0, \qquad \text{for } t \in \Delta.$$

Hence $\sum_{j=1}^{\nu} c_j Q_{j,i}(x) = 0$, for $i = 0, 1, 2, \dots$. This means that, for every $x \in B$, the system of linear equations

$$y_1 Q_{1,i}(x) + \dots + y_\nu Q_{\nu,i}(x) = 0, \;\; i = 0, 1, \dots,$$

has a non-trivial solution.

We now proceed to construct solutions which are polynomials with respect to x. For every $x \in B$, we define the sequence

$$v_i(x) = (Q_{1,i}(x), \dots, Q_{\nu,i}(x)) \in \mathbb{C}^\nu, \;\; i = 0, 1, \dots,$$

and the integer

$$r(x) = \dim \operatorname{Span} \left(\{v_i(x)\}_{i=0}^{\infty} \right).$$

By our assumptions, $r(x) < \nu$ for $x \in B$.

Let us fix $x_0 \in B$ such that $r(x_0) = \max_{x \in B} r(x)$, and integers $0 \leq i_1 < i_2 < \dots < i_m$, $m = r(x_0)$, such that the vectors $v_{i_1}(x_0), \dots, v_{i_m}(x_0)$ are linearly independent.

Let us consider the system of equations

$$(*) \qquad \begin{array}{l} y_1 Q_{1,i_1}(x) + \dots + y_\nu Q_{\nu,i_1}(x) = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_1 Q_{1,i_m}(x) + \dots + y_\nu Q_{\nu,i_m}(x) = 0. \end{array}$$

We can find integers $1 \leq j_1 < j_2 < \dots < j_m \leq \nu$ such that the determinant

$$W(x) = \begin{vmatrix} Q_{j_1,i_1}(x), \dots, Q_{j_m,i_1}(x) \\ \dots\dots\dots\dots\dots\dots \\ Q_{j_1,i_m}(x), \dots, Q_{j_m,i_m}(x) \end{vmatrix}$$

does not vanish at x_0. Then there exists an open neighbourhood $U \subset B$ of x_0 such that $W(x) \neq 0$ for $x \in U$.

Finally, choose an integer k such that $1 \leq k \leq \nu$ and $k \neq j_s$ for $s = 1, \dots, m$. By Cramer's rule, we can construct polynomials $P_{j_1}, \dots, P_{j_m}$ on N such that, for $x \in U$, the system $(*)$ has solutions $y_1(x), \dots, y_\nu(x)$, where

$$y_q(x) = \begin{cases} \dfrac{P_q(x)}{W(x)}, & q = j_1, \dots, j_m \\ 1, & q = k \\ 0, & q \neq k, \; j_1, \dots, j_m. \end{cases}$$

Setting $P_k = W$ and $P_q = 0$, for $q \neq k,\ j_1, \dots, j_m$, we have

$$\sum_{j=1}^{\nu} P_j(x) Q_{j,i_l}(x) = 0, \qquad \text{for } x \in U,\ l = 1, 2, \dots, m,$$

where $P_k \neq 0$. Obviously

$$\sum_{j=1}^{\nu} P_j(x) Q_{j,i}(x) = 0, \qquad \text{for } x \in U,\ i = 0, 1, \dots .$$

By the identity principle, these equalitites hold for every $x \in B$. This implies

$$\sum_{j=1}^{\nu} P_j(x) \varphi_j(x,t) = 0,\ x \in B,\ t \in \Delta.$$

If, for $I_j = (p,q) \in A$, we denote by $P_{p,q}$ the polynomial P_j, then

$$\sum_{p=0}^{s} \left[f^p(tx) \left(\sum_{q=0}^{d-p} t^q P_{p,q}(x) \right) \right] = 0, \qquad \text{for } (x,t) \in B \times \Delta.$$

Hence for $(x,t) \in B \times \Delta$

$$f^s(tx) \tilde{R}_s(x,t) + f^{s-1}(tx) \tilde{R}_{s-1}(x,t) + \dots + \tilde{R}_0(x,t) = 0,$$

where

$$\tilde{R}_p(x,t) = \sum_{q=0}^{d-p} t^q P_{p,q}(x).$$

We can see that there exists $t_0 \in \Delta$ such that the system of polynomials

$$R_p(x) = \tilde{R}_p(x, t_0),\ p = 0, 1, \dots, s,$$

is non-trivial. Moreover

$$R_s(x) f^s(t_0 x) + R_{s-1}(x) f^{s-1}(t_0 x) + \dots + R_0(x) = 0$$

for all $x \in B$. Hence for $x \in t_0 B$

$$R_s\left(\frac{x}{t_0}\right) f^s(x) + R_{s-1}\left(\frac{x}{t_0}\right) f^{s-1}(x) + \dots + R_0\left(\frac{x}{t_0}\right) = 0.$$

Let us define

$$\sigma_i(x) = R_{s-i}\left(\frac{x}{t_0}\right),\ i = 0, \dots, s.$$

Then, for the non-trivial system of polynomials $\sigma_0, \ldots, \sigma_s$ and for $x \in t_0 B$, we have

$$\sigma_0(x) f^s(x) + \sigma_1(x) f^{s-1}(x) + \ldots + \sigma_s(x) = 0.$$

Consequently $u = (f)_0 \in \mathcal{N}_0(N)$ and $\deg u \leq s$, which is our claim.

COROLLARY 3.11. *Suppose that $u \in \mathcal{O}_0(N)$. If, for every $a \in \mathbb{P}(N)$, u_a is the germ of a rational function, then so is u.*

PROOF. By the assumptions, $\mathcal{N}_1(u) = \mathbb{P}(N)$. Hence Theorem 3.10 implies $u \in \mathcal{N}_0(N)$ and $\deg u = 1$, which gives our claim.

COROLLARY 3.12. *If $u \in \mathcal{N}_0(N)$, then*

$$\deg u = \sup\{\deg u_a : a \in \mathbb{P}(N)\}.$$

PROOF. Indeed, by Corollary 3.9,

$$s = \sup\{\deg u_a : a \in \mathbb{P}(N)\} \leq \deg u.$$

Conversely, since $\mathcal{N}_s(u) = \mathbb{P}(N)$, Theorem 3.10 yields $\deg u \leq s$, and so $s = \deg u$, which ends the proof.

Unfortunately, in the last result, the degree cannot be changed by the total degree.

EXAMPLE 3.13. Let $\sqrt{\ } : \{x \in \mathbb{C} : |1 - x| < 1\} \longrightarrow \mathbb{C}$ be the branch of a square root such that $\sqrt{1} = 1$. We consider the function

$$f : \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 < 1\} \ni (x, y) \longrightarrow \frac{y}{1 + \sqrt{1 - x}} \in \mathbb{C},$$

and the germ $u = (f)_0$.

By a simple calculation, we obtain that $\operatorname{Deg} u_a < \operatorname{Deg} u = 3$, for every $a \in \mathbb{P}(\mathbb{C}^2)$.

THEOREM 3.14. *If $u \in \mathcal{O}_0(N)$, then the set*

$$\mathcal{N}(u) = \{a \in \mathbb{P}(N) : u_a \in \mathcal{N}_0(a)\}$$

is a countable union of algebraic subsets of $\mathbb{P}(N)$. Moreover, $\mathcal{N}(u) = \mathbb{P}(N)$ if and only if $u \in \mathcal{N}_0(N)$.

PROOF. By Corollary 3.8, we only need to show that $\mathcal{N}(u) = \mathbb{P}(N)$ implies $u \in \mathcal{N}_0(N)$. To do this, we recall that $\mathcal{N}(u) = \bigcup_{s=1}^{\infty} \mathcal{N}_s(u)$. By Corollary 3.8, there exists a positive integer s_0 such that $\mathcal{N}_{s_0}(u) = \mathbb{P}(N)$, and Theorem 3.10 completes the proof.

The next proposition shows that, in the last theorem, a sharp characterization of sets $\mathcal{N}(u)$ was given.

PROPOSITION 3.15. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of algebraic subsets of $\mathbb{P}(N)$. Then there exists a germ $u \in \mathcal{O}_0(N)$ such that $\mathcal{N}(u) = \bigcup_{i=1}^{\infty} X_i$.*

PROOF. Without loss of generality, we can assume that the sequence $\{X_i\}$ is increasing.

The construction of u will be divided into three steps.

Step 1. Let us fix an arbitrary norm $|\cdot|$ in N and a sequence $\{Q_j\}_{j=1}^{\infty}$ of non-constant homogeneous polynomials on N such that

(a) $$|Q_j(x)| \leq 1 \text{ for } |x| \leq 1, \; j = 1, 2, \ldots,$$

(b) $$X_p = \mathbb{P}\left(Q_{i_p+1}^{-1}(0) \cap Q_{i_p+2}^{-1}(0) \cap \ldots \cap Q_{i_{p+1}}^{-1}(0)\right),$$

for $p = 1, 2, \ldots,$ where $i_1 = 0, i_2, i_3, \ldots,$ is an increasing sequence of natural numbers.

Step 2. Define $P_1 = Q_1$ and

$$P_{j+1} = Q_{j+1}^{\alpha_j}, \; \alpha_j = \deg P_j + 1, \text{ for } j = 1, 2, \ldots.$$

The sequence $\{P_j\}$ has the following properties:

(a) $|P_j(x)| \leq 1$ for $|x| \leq 1, \; j = 1, 2, \ldots,$

(b) $\deg P_1 < \deg P_2 < \ldots < \deg P_j < \deg P_{j+1} < \ldots,$

(c) $a \in \bigcup_{i=1}^{\infty} X_i$ if and only if $P_j|_a = 0$ for almost all $j \in \mathbb{N}$.

Step 3. Let us define the function $f : N \longrightarrow \mathbb{C}$ by

$$f(x) = \sum_{j=1}^{\infty} \frac{P_j(x)}{(\deg P_j)!},$$

Since the series defining f is locally uniformly convergent in N, the function f is an entire holomorphic function on N. Moreover, the restriction $f|_a$ is a polynomial if and only if $a \in \bigcup_{i=1}^{\infty} X_i$. Hence, by Theorem 1.3,

$$\mathcal{N}(u) = \bigcup_{i=1}^{\infty} X_i, \text{ with } u = (f)_0,$$

which is our claim.

3C. - Grassmann manifolds

Let N be a complex vector space of dimension $n > 0$. For $x \in N\backslash\{0\}$, define $\mathbb{P}(x) = \mathbb{C}x \subset N$. If $A \subset N$, define $\mathbb{P}(A) = \{\mathbb{P}(x) : x \in A\backslash\{0\}\}$. Then $\mathbb{P}(N)$ is a connected compact complex manifold of dimension $n-1$, called the complex projective space of N. The mapping $\mathbb{P} : N\backslash\{0\} \ni x \longrightarrow \mathbb{P}(x) \in \mathbb{P}(N)$ is holomorphic and denoted by the same letter $\mathbb{P}$ for all vector spaces.

Take an integer p such that $0 \leq p \leq n-1$. The Grassmann cone, of order p to N, is defined by

$$\tilde{G}_p(N) = \{x_0 \wedge x_1 \wedge \ldots \wedge x_p : x_i \in N, \text{ for } i = 0, 1, \ldots p\} \subset \bigwedge\nolimits_{p+1} N.$$

The Grassmann manifold $G_p(N) = \mathbb{P}(\tilde{G}_p(N))$ of order p to N is a compact connected submanifold of $\mathbb{P}(\bigwedge_{p+1} N)$ of dimension $(n-p-1)(p+1)$. If $p = 0$, then $G_0(N) = \mathbb{P}(N)$.

Note that Chow's theorem implies that $G_p(N)$ is an algebraic subset of $\mathbb{P}(\bigwedge_{p+1} N)$. Then algebraic subsets of $G_p(N)$ are well defined.

Each $\xi = \mathbb{P}(x_0 \wedge \ldots \wedge x_p) \in G_p(N)$ may be regarded as:

(1) a point of $G_p(N)$,

(2) a complex line $\mathbb{C}(x_0 \wedge \ldots \wedge x_p) \subset \tilde{G}_p(N)$,

(3) a $(p+1)$-dimensional subspace $\mathrm{Span}\{x_0, \ldots, x_p\}$ of N,

(4) a p-plane $\mathbb{P}(\mathrm{Span}\{x_0, \ldots, x_p\})$ in $\mathbb{P}(N)$.

In this paper, all identifications of ξ will be denoted by the same symbol ξ (cp. [27]).

Let p and q be integers with $0 \leq q \leq p < n$. Then the "short flag"

$$F_{p,q}(N) = \{(\xi, \eta) \in G_p(N) \times G_q(N) : \xi \supset \eta\},$$

is a connected compact complex submanifold of $G_p(N) \times G_q(N)$ of dimension $(n-p-1)(p+1) + (p-q)(q+1)$ (see [27], §1).

If $q = 0$, write $F_p(N) = F_{p,0}(N)$. Then

$$F_p(N) = \{(\xi, a) \in G_p(N) \times \mathbb{P}(N) : a \in \xi\}.$$

Let

$$p_1 : F_{p,q}(N) \ni (\xi, \eta) \longrightarrow \xi \in G_p(N), \text{ and}$$
$$p_2 : F_{p,q}(N) \ni (\xi, \eta) \longrightarrow \eta \in G_q(N)$$

be the natural projections. Then p_1 and p_2 are holomorphic proper surjective submersions. Moreover, if $\xi \in G_p(N)$ and $\eta \in G_q(N)$, then $p_1^{-1}(\xi)$ and $p_2^{-1}(\eta)$ are connected compact complex manifolds of dimensions $(p-q)(q+1)$ and $(p-q)(n-p-1)$, respectively.

Now, observe that for $\eta \in G_q(N)$, we have the natural identification between $p_2^{-1}(\eta)$ and

$$G_p(N,\eta) = \{\xi \in G_p(N) : \eta \subset \xi\}.$$

Then $G_p(N,\eta)$ is a connected compact complex submanifold of $G_p(N)$ of dimension $(p-q)(n-p-1)$ (see [27], §1 for details).

We now state a useful lemma

LEMMA 3.16. *Let D and G be connected complex manifolds and let $f : D \longrightarrow G$ be a surjective holomorphic submersion with connected fibers. If X is an analytic subset of D, then*

$$Y = \{y \in G : f^{-1}(y) \subset X\},$$

is an analytic subset of G.

PROOF. If $n = \dim D$, $m = \dim G$, then $m < n$. Let us define

$$\pi : \Delta^n \ni (t_1, \dots, t_n) \longrightarrow (t_1, \dots, t_m) \in \Delta^m, \text{ where } \Delta = \{t \in \mathbb{C} : |t| < 1\}.$$

For any fixed $y_0 \in G$ and $x_0 \in f^{-1}(y_0)$, there exist neighbourhoods U of x_0, V of y_0 and biholomorphic mappings

$$h : U \longrightarrow \Delta^n, \; g : V \longrightarrow \Delta^m,$$

such that the diagram

$$\begin{array}{ccc} D \supset U & \xrightarrow{f|_U} & V \subset G \\ \downarrow h & & \downarrow g \\ \Delta^n & \xrightarrow{\pi} & \Delta^m \end{array}$$

commutes ([4], th. 10.3.1).

We only need to show that $Y \cap V$ is an analytic subset of V. Let us first compute that

$$Y \cap V = \{y \in V : (f|_U)^{-1}(y) \subset U \cap X\}.$$

($\supset$). If $y \in V$ and $(f|_U)^{-1}(y) \subset U \cap X$, then $f^{-1}(y) \cap U \subset f^{-1}(y) \cap X$. Since $f^{-1}(y)$ is connected, $X \cap f^{-1}(y) = f^{-1}(y)$. Hence $f^{-1}(y) \subset X$, which implies $y \in Y \cap V$.

($\subset$). Obvious.

Now, let $\tilde{X} = h(U \cap X)$ and $\tilde{Y} = g(V \cap Y)$. By the preliminary step, we get

$$\tilde{Y} = \{(t_1, \dots, t_m) \in \Delta^m : \pi^{-1}(t_1, \dots, t_m) \subset \tilde{X}\}.$$

Moreover, for every $(t_{m+1}, \dots, t_n) \in \Delta^{n-m}$,

$$\tilde{X}_{(t_{m+1},\dots,t_n)} = \left[\left(\Delta^m \times \{(t_{m+1}, \dots, t_n)\}\right) \cap \tilde{X}\right]$$

is an analytic subset of Δ^m. Then

$$\tilde{Y} = \cap \left\{\tilde{X}_{(t_{m+1},\dots,t_n)} : (t_{m+1}, \dots, t_n) \in \Delta^{n-m}\right\},$$

in an analytic subset of Δ^m. Since $Y \cap V = g^{-1}(\tilde{V})$, $Y \cap V$ is an analytic subset of V.

The same reasoning holds for each $y_0 \in G$, which completes the proof.

On $G_p(N)$ we can consider:

(1) the classical topology of a complex manifold,

(2) Zariski's topology in which closed subsets are algebraic subsets of $G_p(N)$.

In the sequel, we shall construct a special topology on the Grassmann manifold weaker then Zariski's one, useful to study restrictions of germs of holomorphic functions to linear subspaces.

For an integer p, with $0 \le p < n$ and $A \subset G_p(N)$, we denote by $\overline{A}^Z$ the Zariski's closure of A. Moreover define

$$\cup A = \{a \in \mathbb{P}(N) : \text{ there exists } \xi \in A \text{ such that } a \in \xi\}.$$

Let us consider the diagram

$$\begin{array}{ccc} \mathbb{P}(N) & \xleftarrow[p_2]{} & F_p(N) \\ & & \downarrow p_1 \\ & & G_p(N). \end{array}$$

We can see at once that $\cup A = p_2(p_1^{-1}(A))$. Then $\cup A$ is an algebraic subset of $\mathbb{P}(N)$, provided A i algebraic.

For $B \subset \mathbb{P}(N)$, we set

$$g_p(B) = \{\xi \in G_p(N) : \xi \subset B\}.$$

We can write $g_p(B) = \{\xi \in G_p(N) : p_1^{-1}(\xi) \subset p_2^{-1}(B)\}$. Hence Lemma 3.16 and Chow's theorem imply the algebraicity of $g_p(B)$, provided B is algebraic.

Finally, for any given $A \subset G_p(N)$, we define

$$d(A) = g_p(\cup(\overline{A}^Z)).$$

The last remarks show that $d(A)$ is an algebraic subset of $G_p(N)$.

LEMMA 3.17. *For d all properties of a closure are satisfied. Precisely:*

$$d(\emptyset) = \emptyset, \tag{1}$$

$$A \subset d(A), \tag{2}$$

$$d(A \cup B) = d(A) \cup d(B), \tag{3}$$

$$d(d(A)) = d(A), \tag{4}$$

where $A, B \subset G_p(N)$.

PROOF. (1) and (2) are obvious.

(3). It is clear that $d(A \cup B) \supset d(A) \cup d(B)$. If $\xi \in d(A \cup B)$, then $\xi \subset \cup\overline{A \cup B}^Z$. Hence $\xi \subset (\cup\overline{A}^Z) \cup (\cup\overline{B}^Z)$. Since ξ is irreducible and $\cup\overline{A}^Z$, $\cup\overline{B}^Z$ are algebraic subsets of $\mathbb{P}(N)$, $\xi \subset \cup\overline{A}^Z$ or $\xi \subset \cup\overline{B}^Z$. Thus $\xi \in d(A) \cup d(B)$, and so (3) is proved.

(4). Of course $d(d(A)) \supset d(A)$. If $\xi \in d(d(A))$, then $\xi \subset \cup d(A)$. Since $\cup d(A) \subset \cup\overline{A}^Z$, $\xi \subset \cup\overline{A}^Z$ and $\xi \in d(A)$. This implies $d(d(A)) \subset d(A)$, and the lemma follows.

Now we are able to define a topology on $G_p(N)$ in which $d(A)$ is the closure of A.

DEFINITION 3.18. A topology on $G_p(N)$, given by the closure d, we call it the weak algebraic topology.

In our topology, a subset A of $G_p(N)$ is closed if and only if $A = d(A)$. Note that each closed subset of $G_p(N)$ must be algebraic.

LEMMA 3.19. *If $A \subset G_p(N)$, then the following statements are equivalent:*

(1) *A is closed in the weak algebraic topology,*

(2) *there exists an algebraic subset X of $\mathbb{P}(N)$ such that $A = g_p(X)$.*

PROOF. (1) $\Rightarrow$ (2). If we take $X = \cup A$, then $A = d(A) = g_p(X)$.

(2) $\Rightarrow$ (1). If $A = g_p(X)$, then $d(A) = g_p(\cup g_p(X))$. Since $g_p(\cup g_p(X)) \subset g_p(X) = A$, $d(A) \subset A$. This implies $d(A) = A$, which ends the proof.

If $0 < p < n - 1$, then the weak algebraic topology and Zariski's one are different. The set $A = G_p(N, a)$, where $a \in \mathbb{P}(N)$, is closed in Zariski's topology byt $d(A) = G_p(N) \neq A$.

THEOREM 3.20. *Let* Y *be closed in the weak algebraic topology, with* $\emptyset \neq Y \subsetneqq G_p(N)$, *then* codim $Y \geq p+1$.

The proof is based on the following lemma

LEMMA 3.21. *If* X *is an algebraic subset of* $\mathbb{P}(N)$ *of dimension* k *and* $0 \leq p \leq k$, *then* codim $g_p(X) \geq (p+1)(n-k-1)$.

PROOF. There is no loss of generality in assuming that X is irreducible. For a given $\xi_0 \in Y = g_p(X)$, there exists a linear subspace $\tilde{\eta}$ of N such that, for $\eta = \mathbb{P}(\tilde{\eta})$, the following conditions hold:

(1) $\eta \supset \xi_0$,

(2) dim $\eta = n+p-k-1$,

(3) dim $(\eta \cap X) = p$.

Then the intersection $Y \cap G_p(\tilde{\eta})$ is finite and $\xi_0 \in Y \cap G_p(\tilde{\eta})$. Hence $\operatorname{codim}_{\xi_0} Y \geq \dim G_p(\tilde{\eta}) = (p+1)(n-k-1)$. This inequality holds for each $\xi_0 \in Y$, and the lemma follows.

PROOF OF THEOREM 3.20. By Lemma 3.19, there exists a proper algebraic subset X of $\mathbb{P}(N)$ such that $Y = g_p(X)$. Hence Lemma 3.21 implies codim $Y \geq (p+1)(n - \dim X - 1) \geq p+1$.

COROLLARY 3.22. *In the case* $G_0(N) = \mathbb{P}(N)$, *the weak algebraic topology and Zariski's one are equivalent. A set* $Y \subsetneqq G_{n-2}(N)$ *is closed in the weak algebraic topology if and only if* Y *is finite.*

PROOF. The first part of the corollary is a simple consequence of Definition 3.18. The second one follows from Theorem 3.20.

3D. - Restrictions to linear subspaces

Let $u \in \mathcal{O}_0(N)$ and let p be an integer with $1 \leq p \leq n = \dim N$. Define

$$\mathcal{N}(u,p) = \{\xi \in G_{p-1}(N) : u_\xi \in \mathcal{N}_0(\xi)\}.$$

In this section, we want to apply the results obtained in 3B and 3C to the study of a structure of the set $\mathcal{N}(u,p)$.

THEOREM 3.23. *Let* p *be an integer with* $1 \leq p \leq n$. *For* $A \subset G_{p-1}(N)$, *the following conditions are equivalent:*

(1) *A is F_σ in the weak algebraic topology,*

(2) *there exists* $u \in \mathcal{O}_0(N)$ *such that* $A = \mathcal{N}(u,p)$.

Moreover, $u \in \mathcal{O}_0(N)$ *is the germ of a Nash function if and only if* $\mathcal{N}(u,p) = G_{p-1}(N)$.

PROOF. The second part of the theorem follows clearly from Theorems 1.10 and 3.14.

To prove the equivalence of (1) and (2), observe that $\mathcal{N}(u,p) = g_{p-1}(\mathcal{N}(u))$. Indeed, by Theorem 3.14, $\xi \in \mathcal{N}(u,p)$ if and only if $\xi \subset \mathcal{N}(u)$.

(1) $\Rightarrow$ (2). Let $A = \bigcup_{i=1}^{\infty} A_i$, where A_i is closed in the weak algebraic topology, for $i = 1,2,\ldots$. Setting $B_i = \cup A_i$, $i = 1,2,\ldots$, we have that $B = \cup A = \bigcup_{i=1}^{\infty} B_i$ is F_σ in Zariski's topology on $\mathbb{P}(N)$. According to Proposition 3.15, there exists $u \in \mathcal{O}_0(N)$ such that $B = \mathcal{N}(u)$. Hence $\mathcal{N}(u,p) = g_{p-1}(B)$.

The irreducibility of projective $(p-1)$-planes implies

$$g_{p-1}(B) = \bigcup_{i=1}^{\infty} g_{p-1}(B_i) = \bigcup_{i=1}^{\infty} d(A_i) = \bigcup_{i=1}^{\infty} A_i = A,$$

which gives $\mathcal{N}(u,p) = A$.

(2) $\Rightarrow$ (1). Let $u \in \mathcal{O}_0(N)$. By Theorem 3.14, $\mathcal{N}(u) = \bigcup_{i=1}^{\infty} X_i$, where X_i is an algebraic subset of $\mathbb{P}(N)$, $i = 1,2,\ldots$. By the preliminary step, we show that

$$\mathcal{N}(u,p) = g_{p-1}(\mathcal{N}(u)) = \bigcup_{i=1}^{\infty} g_{p-1}(X_i).$$

Lemma 3.19 yields, $A_i = g_{p-1}(X_i)$, $i = 1,2,\ldots$, are closed in the weak algebraic topology. Hence $A = \mathcal{N}(u,p) = \bigcup_{i=1}^{\infty} A_i$ is F_σ in this topolgy. This proves the theorem.

COROLLARY 3.24. *For $A \subset G_{n-2}(N)$, $n \geq 2$, the following conditions are equivalent:*

(1) *A is countable or $A = G_{n-2}(N)$,*

(2) *there exists $u \in \mathcal{O}_0(N)$ such that $A = \mathcal{N}(u, n-1)$.*

PROOF. This is a simple consequence of Theorem 3.23 and Corollary 3.22.

COROLLARY 3.25. *If $u \in \mathcal{O}_0(N) \setminus \mathcal{N}_0(N)$ with $p < n$, then $\mathcal{N}(u,p) = \bigcup_{i=1}^{\infty} Y_i$, where Y_i is an algebraic subset of $G_{p-1}(N)$ such that* codim $Y_i \geq p$, *for $i = 1,2,\ldots$.*

PROOF. By Theorem 3.23, $\mathcal{N}(u,p) = \bigcup_{i=1}^{\infty} Y_i$, where Y_i is a proper algebraic subset of $G_{p-1}(N)$ which is closed in the weak algebraic topology, for $i = 1,2,\ldots$. If $Y_i \neq \emptyset$, then Theorem 3.20 yields codim $Y_i \geq p$, $i = 1,2,\ldots$, and the proof is complete.

CHAPTER 4

Intersections of analytic sets with linear subspaces

4A. - Intersections with special linear subspaces

Let N and M be two finite dimensional complex vector spaces, with $\dim N = n$, and let B be an open convex neighbourhood of $0 \in N$. Suppose that X is an analytic subset of $B \times M$ of pure dimension n such that the natural projection $\pi|_X : X \ni (x, y) \longrightarrow x \in B$ is proper.

Under the above assumptions, the following lemma holds

LEMMA 4.1. *If p is an integer with $1 \leq p \leq n$, then the set*

$$A = \{\xi \in G_{p-1}(N) : (\xi \times M) \cap X \text{ is a Nash subset } B \times M\},$$

is F_σ in the weak algebraic topolgy on $G_{p-1}(N)$. Moreover $A = G_{p-1}(N)$ if and only if X is a Nash subset of $B \times M$.

PROOF. The proof falls naturally into two parts.

Part 1. Assume that $M = \mathbb{C}$. By Theorem 2.3, there exist $\sigma_1, \ldots, \sigma_s \in \mathcal{O}(B)$, with $s = s(X)$, such that

$$X = \{(x, t) \in B \times \mathbb{C} : t^s + \sigma_1(x)t^{s-1} + \ldots + \sigma_s(x) = 0\}.$$

Theorem 2.19 shows that $X \cap (\xi \times \mathbb{C})$ is a Nash subset of $B \times \mathbb{C}$ if and only if

$$\sigma_1|_{\xi \cap B}, \ldots, \sigma_s|_{\xi \cap B} \in \mathcal{N}(B \cap \xi).$$

Since $B \cap \xi$ is an open connected subset of ξ, Remark 1.2 implies $\sigma_i|_{\xi \cap B} \in \mathcal{N}(B \cap \xi)$ if and only if $((\sigma_i)_0)_\xi \in \mathcal{N}_0(\xi)$, for $i = 1, 2, \ldots, s$. Hence

$$A = \bigcap_{i=1}^{s} \mathcal{N}((\sigma_i)_0, p),$$

From Theorem 3.23, the sets $\mathcal{N}((\sigma_i)_0, p)$, $i = 1, \ldots, s$, are F_σ in the weak algebraic topology on $G_{p-1}(N)$, then so is A.

If $A = G_{p-1}(N)$, then $\mathcal{N}((\sigma_i)_0, p) = G_{p-1}(N)$, for $i = 1, \ldots, s$. Theorem 3.23 yields $(\sigma_i)_0 \in \mathcal{N}_0(N)$, for $i = 1, \ldots, s$. Hence $\sigma_i \in \mathcal{N}(B)$, $i = 1, \ldots, s$, and so X is a Nash subset of $B \times \mathbb{C}$. Thus Part 1 is proved.

Part 2. We can now turn to the general case, where M is a complex vector space of dimension m.

Let us choose linear forms $L_1, \ldots, L_r$ on the space M, with

$r(m-1)s(X)+1$ (as in Theorem 2.2). Keeping the notation of 2A, we set $X_i = X_{L_i}$, for $i = 1,2,\ldots,r$. By Theorem 2.2 and Corollary 2.18, $X \cap (\xi \times M)$ is a Nash subset of $B \times M$ if and only if $X_i \cap (\xi \times \mathbb{C})$ is a Nash subset of $B \times \mathbb{C}$, for $i = 1,\ldots,r$. The sets

$$A_i = \{\xi \in G_{p-1}(N) : (\xi \times \mathbb{C}) \cap X_i \text{ is a Nash subset of } B \times \mathbb{C}\},$$

for $i = 1,\ldots,r$, by Part 1, are F_σ in the weak algebraic topology, and so is $A = \bigcap_{i=1}^{r} A_i$.

Moreover, if $A = G_{p-1}(N)$, then $A_i = G_{p-1}(N)$, for $i = 1,\ldots,r$. By Part 1, X_i is a Nash subset of $B \times \mathbb{C}$, for $i = 1,\ldots,r$. Theorem 2.2 shows that

$$X = \bigcap_{i=1}^{r} X^{L_i} = \bigcap_{i=1}^{r} \Phi_{L_i}^{-1}(X_i),$$

ans so X is a Nash subset of $B \times M$, which completes the proof.

We now turn to study the intersection of germs of analytic sets with linear subspaces. The germ of $Y \subset N$, at $x \in N$, will be denoted by $(Y)_x$. We introduce the following natural definition.

DEFINITION 4.2. The germ V at x is called a Nash germ if there exist an open neighbourhood U of x and a Nash subset Y of U such that $V = (Y)_x$.

For simplicity, we shall assume that $x = 0$ and we shall write $\{0\}$ instead of $(\{0\})_0$.

Fixing a linear subspace ξ of N and a germ V at 0, we set V_ξ to be the germ defined by $V_\xi = V \cap (\xi)_0$.

Now, let us look at an identification. Suppose that p, q are integers, with $0 \le q < p < n$, and let $\eta \in G_q(N)$. We can find a linear subspace η' of N such that

(1) $\eta \cap \eta' = 0$ and

(2) $\eta + \eta' = N$.

Putting $s = p - q - 1$, we have the natural mapping

$$\chi : G_p(N,\eta) \ni \xi \longrightarrow \xi \cap \eta' \in G_s(\eta').$$

It is easy to verify that χ is a biholomorphic mapping. Moreover, by Chow's theorem, χ is a homeomorphism in the Zariski topologies. Since $G_p(N,\eta) \subset G_p(N)$, then the induced weak algebraic topology on $G_p(N,\eta)$ is well defined and we can state the following proposition.

PROPOSITION 4.3. *The mapping χ is a homeomorphism in the weak algebraic topologies.*

PROOF. We only need to show that

$$\chi(d(A) \cap G_p(N,\eta)) = d(\chi(A)),$$

for every $A \subset G_p(N,\eta)$. To see this, let us fix $A \subset G_p(N,\eta)$ and denote $B = \overline{A}^Z$. Proceeding step by step, we can write

$$\begin{aligned}\chi(d(A) \cap G_p(N,\eta)) &= \{\xi \cap \eta' : \xi \in G_p(N), \eta \subset \xi \subset \cup B\}\\ &= \{\mu \in G_s(\eta') : \mu + \eta \subset \cup B\}.\end{aligned}$$

Similarly,

$$\begin{aligned}d(\chi(A)) &= \{\mu \in G_s(\eta') : \mu \subset \cup \overline{\chi(A)}^Z\}\\ &= \{\mu \in G_s(\eta') : \mu \subset \cup \chi(B)\}\\ &= \{\mu \in G_s(\eta') : \mu \subset B\}.\end{aligned}$$

Since $B \subset G_p(N,\eta)$, the conditions:

(1) $\mu + \eta \subset \cup B$,

(2) $\mu \subset \cup B$,

are equivalent for every $\mu \in G_s(\eta')$.

Hence the required equality holds, and the proof is complete.

Now, let V be a germ of an analytic set of pure dimension k at $0 \in N$, with $1 < k < n$. Suppose that $\eta \in G_{n-k-1}(N)$ and p is an integer such that $n - k < p < n$. We want to analyze the structure of the set

$$\mathcal{N}(V,\eta,p) = \{\xi \in G_{p-1}(N,\eta) : V_\xi \text{ is a Nash germ}\}.$$

THEOREM 4.4. *Under the above assumptions, if* $V \cap (\eta)_0 = \{0\}$, *then* $\mathcal{N}(V,\eta,p)$ *is* F_σ *in the weak algebraic topology on* $G_p(N,\eta)$. *Moreover,* $\mathcal{N}(V,\eta,p) = G_{p-1}(N,\eta)$ *if and only if* V *is a Nash germ.*

PROOF. Let us choose $\eta' \in G_{k-1}(N)$ such that $\eta' \cap \eta = \{0\}$, $\eta' + \eta = N$, and let $\pi : N = \eta' + \eta \ni x + y \longrightarrow x \in \eta'$ be the natural projection. Then there exist an open convex neighbourhood B of $0 \in \eta'$ and an analytic subset X of $B + \eta$ of pure dimension k such that

(1) $(X)_0 = V, X \cap \eta = \{0\}$ and

(2) $\pi|_X : X \longrightarrow B$ is a proper mapping.

We can see, by Theorem 2.13, that $X \cap (\mu + \eta)$ is a Nash subset of $B + \eta$ if and only if $V \cap (\mu + \eta)_0$ is a Nash germ, provided $\mu \in G_{p+k-n-1}(\eta')$.

By Lemma 4.1, we know that the set

$$A = \{\mu \in G_{p+k-n-1}(\eta') : (\mu + \eta) \cap X \text{ is a Nash subset of } B + \eta\}$$

is F_σ in the weak algebraic topology. Moreover, $A = G_{p+k-n-1}(\eta')$ if and only if X is a Nash set. Our identification

$$\chi^{-1} : G_{p+k-n-1}(\eta') \ni \mu \longrightarrow \mu + \eta \in G_{p-1}(N,\eta)$$

yields $\mathcal{N}(V,\eta,p)=\chi^{-1}(A)$. Thus Proposition 4.3 completes the proof.

COROLLARY 4.5 *If V is no Nash germ and if $(\eta)_0\cap V=\{0\}$, then $\mathcal{N}(V,\eta,p)=\bigcup_{i=1}^{\infty}Y_i$, where Y_i is an algebraic subset of $G_{p-1}(N,\eta)$ with* codim $Y_i\geq p+k-n$, *for $i=1,2,\ldots$.*

PROOF. By Theorem 4.4, $\mathcal{N}(V,\eta,p)=\bigcup_{i=1}^{\infty}Y_i$, where Y_i is a proper subset of $G_{p-1}(N,\eta)$ which is closed in the weak algebraic topology for $i=1,2,\ldots$, Hence Proposition 4.3 yields $\chi(Y_i)\subsetneq G_{p+k-n-1}(\eta')$ is closed in the weak algebraic topology. It follows from Theorem 3.20 that codim $\chi(Y_i)\geq p+k-n$. Hence, codim $Y_i\geq p+k-n$, for $i=1,2,\ldots$, which is our claim.

Note that dim $G_{p-1}(N,\eta)=(n-p)(p+k-n)$. Thus, in the above corollary, we can write dim $Y_i\leq(n-p-1)(p+k-n)$. Therefore, in the case $p=n-1$, we obtain the following

COROLLARY 4.6. *If V is no Nash germ and if $(\eta)_0\cap V=\{0\}$, then the set $\mathcal{N}(V,\eta,n-1)$ is countable.*

Finally observe that, without the assumption $(\eta)_0\cap V=\{0\}$, the results 4.4, 4.5, 4.6 are false.

EXAMPLE 4.7. Put: $N=\mathbb{C}^3$, $V=(\{(x,y,z)\in\mathbb{C}^3: y=\sin x\})_0$ $\eta=\{(x,y,z)\in\mathbb{C}^3: x=y=0\}$ and $p=2$.

It is easy to verify that $\mathcal{N}(V,\eta,2)=G_1(\mathbb{C}^3,\eta)$ but V is no Nash germ. Obviously, we have $(\eta)_0\cap V=(\eta)_0\neq\{0\}$.

4B. - Tangent cones

In this section, we review some facts on tangent cones. We shall restrict our attention to tangent cones at the origin of a complex vector space N of dimension n.

DEFINITION 4.8. Suppose that $X\subset N$ and $x\in N$, then we write $x\in\mathrm{Tan}(X)$ if there exist sequences $\{x_\nu\}\subset X$, $\{\lambda_\nu\}\subset\mathbb{C}$ such that the following two conditions are satisfied:

(1) $x_\nu\longrightarrow 0(\nu\longrightarrow\infty)$,

(2) $\lambda_\nu x_\nu\longrightarrow x(\nu\longrightarrow\infty)$.

The set Tan(X) is called the tangent cone of X at 0.

We can see that Tan(X) is determined by the germ $(X)_0$, so we are able to state the next definition.

DEFINITION 4.9. Let $V=(X)_0$, then the tangent cone of V, denoted by Tan(V), is defined to be equal to Tan(X).

THEOREM 4.10. *If V is an analytic germ at $0\in N$ of pure dimension k, then* Tan(V) *is a purely k-dimensional homogeneous algebraic subset of N.*

PROOF (see [31], p. 214).

COROLLARY 4.11. *Let V be an analytic germ of pure dimension k at $0 \in N$ and let $\xi \in G_{p-1}(N)$ with $p \geq n-k$. If* $\dim(\xi \cap \operatorname{Tan}(V)) = p+k-n$, *then* $\dim((\xi)_0 \cap V) = p+k-n$.

PROOF. Let us write $(\xi)_0 \cap V = W_1 \cup \ldots \cup W_s$, where W_i, $i = 1, \ldots, s$, are irreducible components of $(\xi)_0 \cap V$. Obviously $\dim W_i \geq p+k-n$, for $i = 1, \ldots, s$. Moreover, it is clear that $\operatorname{Tan}((\xi)_0 \cap V) \subset \operatorname{Tan}(V) \cap \xi$. Thus, by Theorem 4.10, $\dim W_i \leq p+k-n$, $i = 1, \ldots, s$. This completes the proof.

Now, let V be an analytic germ at $0 \in N$ of pure dimension k, with $0 < k < n$. For an integer $p, n-k \leq p < n$, denote

$$T_p(V) = \{\xi \in G_{p-1}(N) : \dim(\xi \cap \operatorname{Tan}(V)) > p+k-n\}.$$

It is easy to see that $\xi \in T_p(V)$ if and only if the intersection $\xi \cap \operatorname{Tan}(V)$ is not proper. Our purpose is to characterize a structure of $T_p(V)$. We start with the following lemma.

LEMMA 4.12. *Let X be a homogeneous algebraic subset of N of pure dimension $k (0 < k < n)$. If p, s are two integers such that*

(1) $0 < p < n$, *and*

(2) $\max(1, p+k-n) \leq s \leq \min(p, k)$,

then the set

$$A_s = \{\xi \in G_{p-1}(N) : \dim(\xi \cap X) \geq s\},$$

is an algebraic subset of $G_{p-1}(N)$ with codim $A_s \geq s(n+s-p-k)$.

PROOF. Let us look at the diagram introduced in 3C

$$\begin{array}{ccc} \mathbb{P}(N) & \xleftarrow{p_2} & F_{p-1}(N) \\ & & \downarrow p_1 \\ & & G_{p-1}(N). \end{array}$$

if $Y = \mathbb{P}(X)$, then we can write

$$A_s = \cap\{p_1(p_2^{-1}(\mu \cap Y)) : \mu \in G_{n-s}(N)\}.$$

Hence all A_s are algebraic subsets of $G_{p-1}(N)$.
Now, let $A_{s_0} \neq \emptyset$. Define

$$q = \max\{s : s_0 \leq s \leq \min(p,k), \dim A_s = \dim A_{s_0}\}.$$

We first prove that there exists $\xi_0 \in A_q$ such that $\dim(\xi_0 \cap X) = q$ and $\dim_{\xi_0} A_q = \dim A_q$. Indeed, if $q = \min(p,k)$, then $\dim(\xi \cap X) = q$ for every

$\xi \in A_q$. Hence, for every $\xi_0 \in A_q$ such that $\dim_{\xi_0} A_q = \dim A_q$, our conditions are satisfied.

In the case $q < \min(p, k)$, we have $\dim A_{q+1} < \dim A_q$. Thus there exists $\xi_0 \in A_q$ such that $\dim_{\xi_0} A_q = \dim A_q$ and $\xi_0 \notin A_{q+1}$. Since $\xi_0 \in A_q \setminus A_{q+1}$, $\dim(\xi_0 \cap X) = q$.

We are now in a position to estimate $\operatorname{codim}_{\xi_0} A_q = \operatorname{codim} A_q$. Let us consider two cases.

Case 1. If $q = p$, then $A_q = g_{p-1}(Y)$. By Lemma 3.21 $\operatorname{codim} A_q \geq p(n-k)$. Thus $\operatorname{codim} A_{s_0} = \operatorname{codim}_{\xi_0} A_q \geq p(n-k) \geq s_0(n + s_0 - p - k)$.

Case 2. If $q < p$, then we can find two subspaces $\eta \in G_{p-q-1}(N)$. $\eta' \in G_{n+p-q-1}(N)$ such that:

(a) $\eta \subset \xi_0$,

(b) $\eta \cap X = \{0\}$,

(c) $\eta + \eta' = N$.

Denote by π the natural projection $\pi : N = \eta' + \eta \ni x + y \longrightarrow x \in \eta'$. Since $\eta \cap X = \{0\}$ and X is a homogeneous algebraic subset of N, the restriction $\pi|_X : X \longrightarrow \eta'$ is proper. Hence $Z = \pi(X)$ is a homogeneous algebraic subset of η' of pure dimension $\dim Z = k < n + q - p = \dim \eta'$.

Let us consider $G_{p-1}(N, \eta) = \{\mu + \eta : \mu \in G_{q-1}(\eta')\}$. It is easy to verify that

$$\xi_0 \in G_{p-1}(N, \eta) \cap A_q = \{\mu + \eta : \mu \in G_{q-1}(\eta'), \mu \subset Z\}.$$

By Lemma 3.21, we obtain

$$\operatorname{codim}_{G_{q-1}(\eta')}\{\mu \in G_{q-1}(\eta') : \mu \subset Z\} \geq q(n + q - p - k).$$

Now, the identification χ defined in 4A implies

$$\operatorname{codim}_{G_{p-1}(N,\eta)}(G_{p-1}(N, \eta) \cap A_q) \geq q(n + q - p - k).$$

Thus $\operatorname{codim}_{\xi_0} A_q \geq q(n + q - p - k) \geq s_0(n + s_0 - p - k)$. Since $\operatorname{codim}_{\xi_0} A_q = \operatorname{codim} A_{s_0}$, $\operatorname{codim} A_{s_0} \geq s_0(n + s_0 - p - k)$, which completes the proof.

THEOREM 4.13. *Let V be an analytic germ at $0 \in N$ of pure dimension k, $0 < k < n$. If p is an integer, with $n - k \leq p \leq n$, then the set*

$$T_p(V) = \{\xi \in G_{p-1}(N) : \dim(\xi \cap \operatorname{Tan}(V)) > p + k - n\},$$

is an algebraic subset of $G_{p-1}(N)$ such that $\operatorname{codim} T_p(V) \geq p + k - n + 1$.

PROOF. Substituting $X = \operatorname{Tan}(V)$ and $s = p + k - n + 1$ into the assumptions of Lemma 4.12, we see that $T_p(V) = A_s$. Therefore the lemma gives the required result.

4C. - Intersections with all linear subspaces

Let V be an analytic germ at $0 \in N$ of pure dimension k, $1 < k < n$. For a fixed integer p, with $n - k \leq p < n$, we can define two subsets of $G_{p-1}(N)$

$$T_p(V) = \{\xi \in G_{p-1}(N) : \dim(\xi \cap \operatorname{Tan}(V)) > p + k - n\},$$

and

$$U_p(V) = \{\xi \in G_{p-1}(N) : \dim(\xi \cap \operatorname{Tan}(V)) = p + k - n\}.$$

By Theorem 4.13, we conclude that $T_p(V)$ is a proper algebraic subset of $G_{p-1}(N)$. Hence $U_p(V) = G_{p-1}(N)\backslash T_p(V)$ is an open connected subset of $G_{p-1}(N)$. We shall apply these two sets to characterize the set

$$\mathcal{N}(V,p) = \{\xi \in G_{p-1}(N) : V_\xi \text{ is a Nash germ}\}.$$

We can now formulate our main result.

THEOREM 4.14. *Let V be an analytic germ at $0 \in N$ of pure dimension k, $1 < k < n$. Suppose that p is an integer, with $n - k < p < n$. Then:*

(1) *V is a Nash germ if and only if $\mathcal{N}(V,P) = G_{p-1}(N)$*

(2) *if V is no Nash germ, then $\mathcal{N}(V,p) \cap U_p(V) = \bigcup_{i=1}^{\infty} Y_i$, where Y_i is an analytic subset of $U_p(V)$, with* codim $Y_i \geq p + k - n$, *for $i = 1, 2, \ldots$.*

Note that this theorem yields information about $\mathcal{N}(V,p)$ outside $T_p(V)$. But, by Theorem 4.13, codim $T_p(V) \geq p + k - n + 1$.

The proof will be divided into a sequence of lemmata.

LEMMA 4.15. *Let N, M be two complex vector spaces of dimension n and m, respectively. Suppose that D is a q-dimensional connected complex manifold. Let G be an open connected subset of $N \times D$ containing $\{0\} \times D$, and let Y be an analytic subset of $G \times M$ of pure dimension $n + q$ such that the projection $\pi|_Y : Y \ni (x, y, z) \longrightarrow (x, z) \in G$ is proper. Put $Y^z = \{(x, y) \in N \times M : (x, z, y) \in Y\}$ for $z \in D$.*

If $Y^z \cap (\{0\} \times M) = \{0\}$ for every $z \in D$, then the set

$$E = \{(a, z) \in \mathbb{P}(N) \times D : (Y^z \cap (a \times M))_0 \textit{ is a Nash germ}\}$$

is a countable union of analytic subset of $\mathbb{P}(N) \times D$.

PROOF. We can assume that the sets $G^z = \{x \in N : (x, z) \in G\}$, $z \in G$, are convex neighbourhoods of $0 \in N$. Indeed, if we choose a certain norm $|\cdot|$ on the space N and we set

$$\varrho : D \ni Z \longrightarrow \varrho(z) = \sup\{r > 0 : \overline{K}(0, r) \times \{z\} \subset G\} \in (0, +\infty]$$

and

$$\tilde{G} = \{(x, z) \in N \times D : |x| < \varrho(z)\},$$

then $\tilde{G} \subset G$ has the required property, and we can replace G by $\tilde{G}$.

Under the above assumption, we can rewrite E as

$$E = \{(a, z) \in \mathbb{P}(N) \times D : Y^z \cap (a \times M) \text{ is a Nash subset of } G^z \times M\}.$$

Now, we apply the methods of 2A to prove that E is a countable union of analytic subsets of $\mathbb{P}(N) \times D$.

Let $L_1, \ldots, L_r : M \longrightarrow \mathbb{C}$ be the same linear forms as in Theorem 2.2 and let $Y_i = \Phi_{L_i}(Y) \subset G \times \mathbb{C}$, for $i = 1, \ldots, r$. By Lemma 2.3, for every $i \in \{1, \ldots, r\}$, there exist functions $\sigma_1^i, \ldots, \sigma_{s_i}^i$ holomorphic on D, $s_i = s(Y_i)$, such that

$$Y_i = \{((x, z), t) \in G \times \mathbb{C} : t^{s_i} + \sigma_1^i(x, z)t^{s_i - 1} + \ldots + \sigma_{s_i}^i(x, z) = 0\}.$$

Define

$$E_j^i = \{(a, z) \in \mathbb{P}(N) \times D : \sigma_j^i(\cdot, z), a \cap G^z \in \mathcal{N}(a \cap G^z)\},$$

for $i = 1, \ldots, r$ and $1 \leq j \leq s_i$. By Corollary 2.18 and Theorem 2.19, we obtain

$$E = \bigcap\{E_j^i : i = 1, \ldots, r, \ \ 1 \leq j \leq s_i\}.$$

Now, we only need to show that every E_j^i is a countable union of analytic subsets of $\mathbb{P}(N) \times D$. To do this, fix $i \in \{1, \ldots, r\}$ and $j \in \{1, \ldots, s_i\}$. We can consider the functions $\sigma = \sigma_j^i \in \mathcal{O}(G)$ and

$$f : (G \setminus (\{0\} \times D)) \times \Delta \ni ((x, z), t) \longrightarrow \sigma(xt, z) \in \mathbb{C}.$$

Therefore, by Lemma 3.6, the set

$$W = \left\{(x, z) \in G \setminus (\{0\} \times D) | f_{(x,z)} : \Delta \ni t \longrightarrow f((x, z), t) \in \mathbb{C} \atop \text{is a Nash function} \right\}$$

is a countable union of analytic subsets W_i, $i = 1, 2, \ldots$, of $G \setminus (\{0\} \times D)$.

It is easy to see that

$$W = \{(x, z) \in G \setminus (\{0\} \times D) : \sigma(\cdot, z)|_{(\mathbb{C}x) \cap G^z} \in \mathcal{N}((\mathbb{C}x) \cap G^z)\}.$$

Observe that the mapping

$$F : G \setminus (\{0\} \times D) \ni (x, z) \longrightarrow (\mathbb{P}(x), z) \in \mathbb{P}(N) \times D,$$

satisfies the assumptions of Lemma 3.16, and so

$$\begin{aligned} E_j^i &= \{(a, z) \in \mathbb{P}(N) \times D : F^{-1}(a, z) \subset W\} \\ &= \bigcup_{l=1}^{\infty} \{(a, z) \in \mathbb{P}(N) \times D : F^{-1}(a, z) \subset W_l\}, \end{aligned}$$

is a countable union of analytic subsets of $\mathbb{P}(N) \times D$. This completes the proof.

LEMMA 4.16. *Let N, M and D be as above. Suppose that Ω is an open subset of $N \times D \times M$ and that X is an analytic subset of Ω of pure dimension $n+q$, with $\{0\} \times D \times \{0\} \subset X$. If $D \times \{0\}$ is a connected component of*

$$X_0 = \{(z, y) \in D \times M : (0, z, y) \in X\},$$

and if we put

$$X^z = \{(x, y) \in N \times M : (x, z, y) \in X\} \text{ for } z \in D,$$

then the set

$$E = \{(a, z) \in \mathbb{P}(N) \times D : (X^z \cap (a \times M))_0 \text{ is a Nash germ}\},$$

is a countable union of analytic subsets of $\mathbb{P}(N) \times D$.

PROOF. Put $\Omega_0 = \{(z, y) \in D \times M : (0, z, y) \in \Omega\}$, and note that $D \times \{0\} \subset \Omega_0$ is a connected component of X_0, analytic in Ω_0. The assumptions imply $\tilde{X}_0 = X_0 \backslash (D \times \{0\})$ is an analytic subset of Ω_0. Setting

$$F_1 = D \times \{0\}, \text{ and}$$
$$F_2 = \tilde{X}_0 \cup ((D \times M) \backslash \Omega_0),$$

we obtain two closed disjoint subsets of the product $D \times M$.

Then there exists an open neighbourhood U_1 of F_1 such that $\overline{U}_1 \cap F_2 = \emptyset$. This means that $\overline{U}_1 \subset \Omega_0$ and $U_1 \cap X_0 = D \times \{0\}$. Choose a bounded neighbourhood B of 0 in M and define $U = (D \times B) \cap U_1$. Thus

(1) $\overline{U} \subset \Omega_0$, and

(2) $\overline{U} \cap X_0 = D \times \{0\}$.

Now, $Z = X \cap (N \times U)$ is an analytic subset of $(N \times U) \cap \Omega$ of pure dimension $n+q$ such that $\{0\} \times D \times \{0\} \subset Z$ and $Z \subset N \times D \times B$. We observe that $(\overline{Z} \backslash Z) \cap (\{0\} \times D \times M) = \emptyset$. Indeed, we have

$$\begin{aligned}
&\overline{Z} \cap (\{0\} \times D \times M) \subset \overline{X} \cap (N \times \overline{U}) \cap (\{0\} \times D \times M) \\
&= \overline{X} \cap (\{0\} \times \overline{U}) = (\{0\} \times X_0) \cap (\{0\} \times \overline{U}) \\
&= \{0\} \times D \times \{0\}.
\end{aligned}$$

Therefore

$$\overline{Z} \cap (\{0\} \times D \times M) = \{0\} \times D \times \{0\} \subset Z \cap (\{0\} \times D \times M)$$

implies the required property of Z.

Let us denote by π the natural projection $\pi : N \times D \times M \ni (x, z, y) \longrightarrow (x, z) \in N \times D$.

Since $\overline{Z} \subset N \times D \times \overline{B}$, the restriction $\pi|_{\overline{Z}} : \overline{Z} \longrightarrow N \times D$ is a proper mapping. Hence $\pi(\overline{Z}\backslash Z)$ is a closed subset of $N \times D$ with $\pi(\overline{Z}\backslash Z) \cap (\{0\} \times D) = \emptyset$.

Let G be a connected component of $(N \times D)\backslash\pi(\overline{Z}\backslash Z)$ containing $\{0\} \times D$. Putting $Y = Z \cap (G \times M)$, we see that $\pi|_Y : Y \longrightarrow G$ is a proper mapping and

$$(X^z \cap (a \times M))_0 = (Y^z \cap (a \times M))_0, \quad \text{for } z \in D, \ a \in \mathbb{P}(N).$$

Applying Lemma 4.15 to G and Y, we get the required result.

LEMMA 4.17. *Let V be an analytic germ at $0 \in N$ of pure dimension k, $1 < k < n$. If $p = n - k + 1$, then the set $\mathcal{N}(V, p) \cap U_p(V)$ is a countable union of analytic subsets of $U_p(V)$. Moreover, $\mathcal{N}(V, p) \supset U_p(V)$ if and only if V is a Nash germ.*

PROOF. It is easy to verify that

$$S(V) = \{(\eta, y) \in U_{n-k}(V) \times N : y \in \eta\}$$

is a closed complex submanifold of $U_{n-k}(V) \times N$ of dimension $n - k + \dim G_{n-k-1}(N)$. Put

$$\tilde{\varphi} : N \times U_{n-k}(V) \times N \ni (x, \eta, y) \longrightarrow x + y \in N,$$

$$\varphi = \tilde{\varphi}|_{N \times S(V)} : N \times S(V) \ni (x, (\eta, y)) \longrightarrow x + y \in N.$$

Let B be an open neighbourhood of $0 \in N$ and let Y be an analytic subset of B of pure dimension k such that $(Y)_0 = V$. If $\Omega = \tilde{\varphi}^{-1}(B)$, then $X = \varphi^{-1}(Y) = \tilde{\varphi}^{-1}(Y) \cap (N \times S(V))$ is an analytic subset of Ω. Since φ is a submersion, X is an analytic set of pure dimension $n + \dim G_{n-k-1}(N) = n + \dim U_{n-k}(V)$.

Now, we have an open subset Ω of $N \times U_{n-k}(V) \times N$, and an analytic set $X \subset \Omega$ of pure dimension $n + \dim U_{n-k}(V)$. In this situation, we want to apply Lemma 4.16. To do it, we only need to show that:

(1) $\{0\} \times U_{n-k}(V) \times \{0\} \subset X$, and

(2) $U_{n-k}(V) \times \{0\}$ is a connected component of

$$X_0 = \{(\eta, y) \in U_{n-k}(V) \times N : (0, \eta, y) \in X\}.$$

The first required condition is obvious. To prove the second one, observe that

$$X_0 = \{(\eta, y) \in U_{n-k}(V) \times N : y \in Y \cap \eta\}$$

is an analytic subset of $\Omega_0 = U_{n-k}(V) \times B$. Suppose, on the contrary, that $U_{n-k}(V) \times \{0\}$ is no connected component of X_0. Then there exist sequences $\{\eta_\nu\} \subset U_{n-k}(V)$, $\{y_\nu\} \subset N\backslash\{0\}$ and $\eta_0 \in U_{n-k}(V)$ such that:

(1) $\eta_\nu \longrightarrow \eta_0$,

(2) $y_\nu \in \eta_\nu \cap Y$, for $\nu = 1, 2, \ldots$,

(3) $y_\nu \longrightarrow 0$.

Choose a certain norm $|\cdot|$ in N. There is no loss of generality in assuming that

$$\frac{y_\nu}{|y_\nu|} \longrightarrow y_0, |y_0| = 1. \tag{4}$$

Since $\eta_\nu \longrightarrow \eta_0$ and $\frac{y_\nu}{|y_\nu|} \longrightarrow y_0$, $0 \neq y_0 \in \eta_0$, setting $\lambda_\nu = \frac{1}{|y_\nu|}$, we have

$$\lambda_\nu y_\nu \longrightarrow y_0, \; y_\nu \in Y \text{ and } y_\nu \longrightarrow 0.$$

By Definition 4.8, $y_0 \in \eta_0 \cap \mathrm{Tan}(V)$. Hence $\mathrm{Tan}(V) \cap \eta_0 \neq \{0\}$, contrary to $\eta_0 \in U_{n-k}(V)$. Thus all assumptions of Lemma 4.16 are satisfied, and so the set

$$E = \{(a, \eta) \in \mathbb{P}(N) \times U_{n-k}(V) : (X^\eta \cap (a \times N))_0 \text{ is a Nash germ}\}$$

is a countable union of analytic subsets of $\mathbb{P}(N) \times U_{n-k}(V)$.

Now, let $C = \mathbb{P}(\mathrm{Tan}(V))$. Then C is an algebraic subset of $\mathbb{P}(N)$ and $\tilde{E} = E \cap (C \times U_{n-k}(V))$ is a countable union of analytic subsets of $C \times U_{n-k}(V)$.

Moreover, under our definitions,

$$\tilde{E} = \{(a, \eta) \in C \times U_{n-k}(V) : a + \eta \in \mathcal{N}(V, p)\}.$$

To characterize $\mathcal{N}(V, p) \cap U_p(V)$, $p = n - k + 1$, we apply the flag manifolds (Section 3C).

Observe that

$$F = \{(\xi, \eta) \in F_{p-1,p-2}(N) : \xi \in U_p(V), \eta \in U_{p-1}(V)\}$$

is an open subset of the manifold $F_{p-1,p-2}(N)$. Moreover, the mapping $p_1 : F \ni (\xi, \eta) \longrightarrow \xi \in U_p(V)$ is a surjective submersion. Define

$$Z = \{(\xi, \eta) \in F : \xi \in \mathcal{N}(V, p) \cap U_p(V)\}, \text{ and}$$
$$\psi : C \times U_{p-1}(V) \ni (a, \eta) \longrightarrow (a + \eta, \eta) \in U_p(V) \times U_{p-1}(V).$$

We shall compute that $Z = \psi(E)$. Indeed, if $(a, \eta) \in \tilde{E}$, then $a + \eta \in \mathcal{N}(V, p)$ and $a + \eta \in U_p(V)$, and so $\psi(a, \eta) \in Z$. Conversely, if $\xi \in \mathcal{N}(V, p) \cap U_p(V)$ and $\eta \in U_{p-1}(V)$, $\eta \subset \xi$, then, for every $a \in C \cap \xi$, $a + \eta = \xi$. Hence $\psi(a, \eta) = (\xi, \eta)$ and $(a, \eta) \in \tilde{E}$.

Since ψ is a proper holomorphic mapping, the set Z is a countable union of analytic subsets of F. The mapping $p_1 : F \longrightarrow U_p(V)$ satisfies assumptions of Lemma 3.16, and so

$$\mathcal{N}(V, p) \cap U_p(V) = \{\xi \in U_p(V) : p_1^{-1}(\xi) \subset Z\}$$

is a countable union of analytic subsets of $U_p(V)$. This proves the first part of the lemma.

To prove the second one, assume that $\mathcal{N}(V,p) \supset U_p(V)$. Choosing $\eta \in U_{p-1}(V)$, we have $G_{p-1}(N,\eta) \subset U_p(V)$. Moreover, $(\eta)_0 \cap V = \{0\}$. Since

$$\mathcal{N}(V,\eta,p) = \mathcal{N}(V,p) \cap G_{p-1}(N,\eta) = G_{p-1}(N,\eta),$$

Theorem 4.4 implies V is a Nash germ.

Thus the proof is complete.

COROLLARY 4.18. *Let V be an analytic germ at $0 \in N$ of pure dimension k, $1 < k < n$. If $p = n - k + 1$ and if* int $\mathcal{N}(V,p) \neq \emptyset$, *then V is a Nash germ.*

PROOF. Since int $\mathcal{N}(V,p) \neq \emptyset$, we have $\operatorname{int}(\mathcal{N}(V,p) \cap U_p(V)) \neq \emptyset$. Suppose, on the contrary, that V is no Nash germ. By Lemma 4.17, $\operatorname{int}(\mathcal{N}(V,p) \cap U_p(V)) = \emptyset$. This contradicts our assumptions.

LEMMA 4.19. *Let V be an analytic germ at $0 \in N$ of pure dimension k, $1 < k < n$. If p is an integer with $n-k < p < n$, then the set $\mathcal{N}(V,p) \cap U_p(V)$ is a countable union of analytic subsets of $U_p(V)$.*

PROOF. Let us consider the set

$$F = \{(\xi,\eta) \in F_{p-1,n-k}(N) : \xi \in U_p(V), \eta \in U_{n-k+1}(V)\},$$

and the mappings

$$\begin{array}{ccc} U_{n-k+1}(V) & \xleftarrow[p_2|_F]{} & F \\ & & \downarrow p_1|_F \\ & & U_p(V). \end{array}$$

The set F is an open subset of $F_{p-1,n-k}(N)$ and $p_1|_F$ is a surjective submersion with connected fibres. Put $Y = \mathcal{N}(V, n-k+1) \cap U_{n-k+1}(V)$. Lemma 3.16 implies that the set

$$Z = \{\xi \in U_p(V) : (p_1|_F)^{-1}(\xi) \subset (p_2|_F)^{-1}(Y)\}$$

is a countable union of analytic subsets of $U_p(V)$.

Observe that $\mathcal{N}(V,p) \cap U_p(V) = Z$. We can assume here that $p > n - k + 1$. In fact, if $\xi \in \mathcal{N}(V,p) \cap U_p(V)$ and if $(\xi,\eta) \in (p_1|_F)^{-1}(\xi)$, then $\eta \subset \xi$ and $\eta \in U_{n-k+1}(V)$. Since $\eta \subset \xi, \eta \in \mathcal{N}(V, n-k+1)$ and $(\xi,\eta) \in (p_2|_F)^{-1}(Y)$, we have $\mathcal{N}(V,p) \cap U_p(V) \subset Z$.

Conversely, let us fix $\xi \in Z$. By definition of Z, if $\eta \in U_{n-k+1}(V)$ and if $\eta \subset \xi$, then $\eta \in Y$. Thus $\eta \in \mathcal{N}(V, n-k+1)$. Note that $\tilde{V} = V \cap (\xi)_0$ is an analytic germ at $0 \in \tilde{N} = \xi$ of pure dimension $\tilde{k} = k + p - n$, $1 < \tilde{k} < \tilde{n} = \dim \tilde{N} = p$. Putting $\tilde{p} = n - k + 1$, observe that

$$\mathcal{N}(\tilde{V},p) \supset \{\eta \in U_{n-k+1}(V) : \eta \subset \xi\} = G_{n-k}(\xi) \setminus T_{n-k+1}(V).$$

By Corollary 4.18, $\tilde{V}$ is a Nash germ. Hence $\xi \in \mathcal{N}(V,p) \cap U_p(V)$. This gives the required inclusion $Z \subset \mathcal{N}(V,p) \cap U_p(V)$, and completes the proof.

PROOF OF THEOREM 4.14. We first observe that $\mathcal{N}(V,p) = G_{p-1}(N)$, provided V is a Nash germ. Conversely, suppose that the equality holds, and choose certain $\eta \in U_{n-k}(V)$. We have $(\eta)_0 \cap V = \{0\}$ and $G_{p-1}(N,\eta) \subset U_p(V) \subset \mathcal{N}(V,p)$. Thus, by Theorem 4.4, V is a Nash germ.

Now, we prove part (2) of the theorem. By Lemma 4.19, we can write

$$\mathcal{N}(V,p) \cap U_p(V) = \bigcup_{i=1}^{\infty} Y_i,$$

where Y_i is an analytic subset of $U_p(V)$, for $i = 1,2,\ldots$. Let us fix $i_0 \in \mathbb{N}$ and $\xi_{i_0} \in Y_{i_0}$. Since $\xi_0 \in U_p(V)$, there exists $\eta \in U_{n-k}(V)$ such that $\eta \subset \xi_0$. It is obvious that

$$\xi_0 \in Y_{i_0} \cap G_{p-1}(N,\eta) \subset \mathcal{N}(V,\eta,p),$$

and so, by Corollary 4.5, $\operatorname{codim}_{\xi_0}(Y_{i_0} \cap G_{p-1}(N,\eta)) \geq p + k - n$. Thus $\operatorname{codim}_{\xi_0} Y_{i_0} \geq p + k - n$. Similarly, $\operatorname{codim}_{\xi} Y_i \geq p + k - n$, for every positive integers i and $\xi \in Y_i$. Hence $\operatorname{codim} Y_i \geq p+k-n$, for $i = 1,2,\ldots$. This ends the proof.

Let us mention that we have to distinguish the set $T_p(V)$ of "singular subspaces" also in the case where all subspaces intersect the germ V properly.

EXAMPLE 4.20. Let V be the germ of the set

$$X = \{(x,y,z) \in \mathbb{C}^3 : z = x(y - \sin x)\}$$

at $0 \in \mathbb{C}^3$. Then all 2-dimensional subspaces intersect V properly. But

$$\mathcal{N}(V,2) = G_1(C^3, \{0\} \times \mathbb{C} \times \{0\}) \setminus \{\mathbb{P}(\mathbb{C} \times \mathbb{C} \times \{0\})\}$$

is no countable union of analytic subsets of $G_1(\mathbb{C}^3)$.

Next we want to present some corollaries of Theorem 4.14.

COROLLARY 4.21. *If V is no Nash germ and if $T_p(V) = \emptyset$, then $\mathcal{N}(V,p) = \bigcup_{i=1}^{\infty} Y_i$, where Y_i is an algebraic subset of $G_{p-1}(N)$ with* $\operatorname{codim} Y_i \geq p+k-n$, *for $i = 1,2,\ldots$.*

PROOF. Since $U_p(V) = G_{p-1}(N)$, Theorem 4.14 and Chow's theorem give the required result.

COROLLARY 4.22. *If an analytic germ V of pure dimension $n-1$ is no Nash germ, then there exist $\xi_1,\ldots,\xi_r \in G_{n-2}(N)$ and a sequence $\{Y_i\}$ of algebraic subsets of $G_{n-2}(N)$ with* $\dim Y_i \leq 1$, $i = 1,2,\ldots$, *such that*

$$\mathcal{N}(V,n-1) = \left(\bigcup_{i=1}^{\infty} Y_i\right) \setminus \{\xi_1,\ldots,\xi_r\}.$$

PROOF. By Theorem 4.13, the set $T_{n-1}(V)$ is finite. Setting $T_{n-1}(V) = \{\xi_1,\dots,\xi_s\}$, we have $U_{n-1}(V) = G_{n-2}(N)\setminus\{\xi_1,\dots,\xi_s\}$. Now, Theorem 4.14 shows that $U_{n-1}(V) \cap \mathcal{N}(V,n-1) = \bigcup_{i=1}^{\infty} X_i$, where X_i is an irreducible analytic subset of $U_{n-1}(V)$ with $\dim X_i \leq 1$, for $i = 1,2,\dots$. Suppose that $\mathcal{N}(V,n-1)\cap T_{n-1}(V) = \{\xi_{r+1},\dots,\xi_s\}$. Then

$$\begin{aligned}\mathcal{N}(V,n-1) &= \left[\left(\bigcup_{i=1}^{\infty}\overline{X}_i\right)\setminus\{\xi_1,\dots,\xi_s\}\right]\cup\{\xi_{r+1},\dots,\xi_s\}\\ &= \left(\bigcup_{i=1}^{\infty}\overline{X}_i\right)\cup\{\xi_1,\dots,\xi_r\}.\end{aligned}$$

If we put $Y_i = \overline{X}_i$, $i = 1,2,\dots$, then the Remmert-Stein theorem (see [10], p. 170) completes the proof.

We now state a useful characterization of the sets $\mathcal{N}(V,p)$.

THEOREM 4.23. *If V is no Nash germ, then there exists a sequence $\{X_i\}$ of complex submanifolds of $G_{p-1}(N)$ with* codim $X_i \geq p+k-n$, *for* $i = 1,2,\dots$, *such that* $\mathcal{N}(V,p) \subset \bigcup_{i=1}^{\infty} X_i$.

PROOF. We can write $\mathcal{N}(V,p) \subset \bigcup_{i=0}^{\infty} Y_i$, where $Y_0 = T_p(V)$ and Y_i is an analytic subset of $U_p(V)$ with codim $Y_i \geq p+k-n$, for $i = 1,2,\dots$.

Since codim $Y_0 \geq p+k-n+1$, we see that every Y_i can be presented as a countable union of submanifolds of the required dimensions. This ends the proof.

We shall now construct an example showing that the estimation of dimension of $\mathcal{N}(V,p)$, presented in previous theorems, is optimal.

EXAMPLE 4.24. Let k,p and n be integers such that:

(1) $1 < k < n$, and

(2) $n-k < p < n$.

Take $N = \mathbb{C}^n = \mathbb{C}\times\mathbb{C}^{n-k}\times\mathbb{C}^{k-1}$ and

$$X = \{(x,y,z)\in\mathbb{C}\times\mathbb{C}^{n-k}\times\mathbb{C}^{k-1} : y_i = x\,\sin^i x,\ i = 1,\dots,n-k\}.$$

It is easy to see that X is a connected complex submanifold of N. Moreover $V = (X)_0$ is no Nash germ.

Let $\eta = \{0\}\times\{0\}\times\mathbb{C}^{k-1}$ and let

$$Y = \{\xi \in G_{p-1}(N) : \text{ there exists } \varsigma \in G_{n-2}(N,\eta) \text{ such that } \xi \subset \varsigma\}.$$

We can prove that Y is an irreducible algebraic subset of $G_{p-1}(N)$ with codim $Y = p+k-n$.

Next, observe that, for every $\varsigma \in G_{n-2}(N,\eta)$, we have $(\varsigma)_0 \cap V = (\eta)_0$. Therefore, for every $\xi \in Y$, $(\xi)_0 \cap V = (\eta \cap \xi)_0$ is a Nash germ. Hence we conclude that $Y \subset \mathcal{N}(V,p)$ and codim $Y = p+k-n$.

4D. - Criteria for analytic sets to be algebraic

In this section, we present certain criteria for analytic sets to be algebraic in terms of its intersections with linear subspaces. A basic problem is to relate the algebraicity of an analytic subset $X \subset N$ to that of their intersections $X \cap \xi$ with linear subspaces ξ of fixed dimension.

We shall restrict out attention to a simple case where:

(a) X is an irreducible analytic subset of N of dimension k, with $0 \in X$,

(b) $\xi \in G_{p-1}(N)$ and $p+k > n$.

Our assumptions show that every component of intersections $X \cap \xi$ has dimension at least $r = p+k-n > 0$. Let us define

$$A(X,p) = \{\xi \in G_{p-1}(N) : X \cap \xi \text{ is algebraic}\}.$$

In the remainder of this section, we assume X to be transcendental. Recall the following known facts:

(1) int $A(X,p) = \emptyset$, ([21], 1973)

(2) the Lebesgue measure of $A(X,p)$ is equal to 0, ([9], 1978)

(3) if $p = n-1$, then $A(X,p)$ is a locally pluripolar subset of $G_{p-1}(N)$, ([17], 1981).

For a recent account of the theory, we refer also [16], [19], [20], [23], [26], [28].

It is easy to see that, by contraposition of the presented results, we obtain criteria for analytic sets to be algebraic. For example:

$$\text{if int } A(X,p) \neq \emptyset, \text{ then } X \text{ is algebraic.}$$

We now state two of the consequences of the results proved in 4C.

THEOREM 4.25. *In the above situation, we have*

$$A(X,p) \subset \bigcup_{i=1}^{\infty} Y_i,$$

where Y_i is a complex submanifold of $G_{p-1}(N)$ such that codim $Y_i \geq r$, *for $i = 1,2,\ldots$.*

PROOF. By Corollary 2.14, $V = (X)_0$ is no Nash germ. Since $A(X,p) \subset \mathcal{N}(V,p)$, Theorem 4.23 completes the proof.

Similarly, by Theorem 4.22, we obtain the following result.

THEOREM 4.26. *If $p = k = n - 1$, then*

$$A(X,p) \subset \bigcup_{i=1}^{\infty} Y_i,$$

where Y_i is an algebraic curve in $G_{n-2}(N)$, for $i = 1, 2, \ldots$.

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