

ANNALI DELLA  
SCUOLA NORMALE SUPERIORE DI PISA  
*Classe di Scienze*

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*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4<sup>e</sup> série, tome 17,  
n° 4 (1990), p. 569-581*

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# Index Estimates and Critical Points of Functionals Not Satisfying Palais-Smale

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## 1. - Introduction

We begin by recalling the definition of a Hamiltonian system with  $n$  degrees of freedom. Define a  $2n \times 2n$  matrix  $J$  by

$$(1) \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

If  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  is a smooth function, the Hamiltonian system associated with  $H$  is the system of ODE's:

$$(2) \quad \dot{x} = JH'(x).$$

We are interested in finding periodic solutions of this equation, that is, in solving the boundary-value problem

$$(3) \quad \begin{cases} \dot{x} = JH'(x) \\ x(0) = x(T) \end{cases}$$

for prescribed  $T$ . There is a well-known existence result:

THEOREM 1. *Let  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  be strictly convex and  $C^1$ . Assume that*

$$(4) \quad H(x) > H(0) = 0 \quad \forall x \neq 0$$

$$(5) \quad H(x) \|x\|^{-2} \rightarrow 0 \quad \text{when } x \rightarrow 0$$

Pervenuto alla Redazione il 24 Ottobre 1989 e in forma definitiva il 28 Giugno 1990.

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Supported by US Army under contract # DAJA.

and that, for some constants  $\alpha > 2$  and  $R > 0$  we have:

$$(6) \quad \|x\| \geq R \implies H(\lambda x) \geq \lambda^\alpha H(x) \quad \forall \lambda \geq 1.$$

Then, for every  $T > 0$ , the boundary-value problem (3) has a solution with minimal period  $T$ . ■

Under more general assumptions, Rabinowitz proved in [R1] the existence of a non-trivial solution (the solution  $x(t) \equiv 0$  being considered trivial) and in [R2] the existence of an unbounded sequence of solutions. In his statement, condition (6) is replaced by the following one:

$$(7) \quad (H'(x), x) \geq \alpha H(x)$$

which turns out to be equivalent to (6) in the convex case.

The existence of a solution with minimal period  $T$  is due to Ekeland and Hofer [EH]. Earlier, Ambrosetti and Mancini [AM], and then Girardi and Matzeu ([GM1][GM2]) had proved the existence of such a solution under more restrictive conditions on  $H$ .

In all these papers, condition (6) or (7) is crucial to prove that the Palais-Smale condition holds in the associated variational problem. In fact, this condition first appears in the seminal paper of Ambrosetti and Rabinowitz, where it is introduced specifically for that purpose. On the other hand, it is clearly a technical condition, which is hard to justify on physical grounds.

The purpose of this paper is to replace condition (6) by the condition that the Hessian  $H''(x)$  goes to infinity (in the sense that its smallest eigenvalue goes to  $+\infty$ ) as  $\|x\| \rightarrow \infty$ . In fact, we shall even be able to treat the case when  $H$  is defined only on a convex open subset of  $\mathbb{R}^{2n}$ , and goes to  $+\infty$  on the boundary. We shall prove the following.

**THEOREM 2.** *Let  $\Omega$  be a convex open subset of  $\mathbb{R}^{2n}$  containing the origin. Let  $H \in C^2(\Omega, \mathbb{R})$  be such that*

$$(8) \quad H(x) \geq H(0) = 0 \quad \forall x \in \Omega$$

$$(9) \quad H''(x) \quad \text{is positive definite} \quad \forall x \neq 0$$

$$(10) \quad H''(0) = 0$$

$$(11) \quad H''(x)^{-1} \rightarrow 0 \quad \text{as} \quad \|x\| \rightarrow \infty \quad \text{or} \quad \|x\| \rightarrow \partial\Omega$$

Then, for every  $T$ , the boundary value problem (3) has a solution with minimal period  $T$ . ■

By assumption (11), we mean that for every  $a > 0$ , there exist  $\rho > 0$  and  $\eta > 0$  such that

$$(12) \quad \|x\| > \rho \quad \text{or} \quad d(x, \partial\Omega) \leq \eta \implies H''(x) \leq aI$$

This implies that  $H(x)\|x\|^{-2} \rightarrow +\infty$  when  $\|x\| \rightarrow \infty$  or  $x \rightarrow \partial\Omega$ . More precisely, for every  $a' > 0$  there exist  $\rho' > 0$  and  $\eta' > 0$  such that

$$(13) \quad \|x\| > \rho' \quad \text{or} \quad d(x, \partial\Omega) \leq \eta' \implies H(x)\|x\|^{-2} \geq a'.$$

Assumptions (6) and (11) are not directly comparable. Theorem 1 only requires  $H$  to be strictly convex and  $C^1$ , in which case (6) amounts to (7), which does not imply anything on the behaviour of the Hessian  $H''(x)$ , if it exists. Theorem 2 requires  $H$  to be  $C^2$  and strictly convex, but even then (11) does not imply (6) (take  $H(x) = x^2 \log \|x\|$  for  $\|x\|$  large).

The most natural assumption, subsuming (6) and (11), would be

$$(14) \quad H(x)\|x\|^{-2} \rightarrow +\infty \quad \text{when} \quad \|x\| \rightarrow \infty \quad \text{or} \quad x \rightarrow \partial\Omega$$

or, failing that,

$$(15) \quad \|H'(x)\| \|x\|^{-1} \rightarrow +\infty \quad \text{when} \quad \|x\| \rightarrow \infty \quad \text{or} \quad x \rightarrow \partial\Omega$$

The authors do not know whether assumptions (14) or (15) are enough for existence. As it is, assumption (11) enables us to extend theorem 1 to the case when  $H$  is defined on a proper subset  $\Omega$  of  $\mathbb{R}^{2n}$  – and this is the first result of its kind (for first order system; for second order ones see the discussion in section 3).

In the following section, we prove theorem 1. In the last section, we extend the results to second-order systems.

## 2. - Hamiltonian systems

We begin by recalling the proof of a weaker version of theorem 1, which will be enough for our purposes.

PROPOSITION 3. *Let  $H : \mathbb{R}^{2n} \rightarrow \mathbb{R}$  be strictly convex and  $C^1$ . Assume that:*

- (1)  $H(x) > H(0) = 0 \quad \forall x \neq 0$
- (2)  $H(x)\|x\|^{-2} \rightarrow 0 \quad \text{when} \quad x \rightarrow 0$
- (3)  $H(\lambda x) \geq \lambda^\alpha H(x) \quad \forall \lambda \geq 1, \quad \|x\| \geq R$
- (4)  $H(x) \leq \frac{k^\alpha}{\alpha} \|x\|^\alpha \quad \forall x.$

for suitable constants  $\alpha > 2$ ,  $R > 0$  and  $k > 0$ . Then, for every  $T > 0$ , the Hamiltonian system

$$(5) \quad \dot{x} = JH'(x)$$

has a periodic solution with minimal period  $T$ . ■

The proof is divided in two parts:

PART 1: EXISTENCE: (Ekeland [E1])

We introduce Clarke's dual action functional

$$(6) \quad \psi(u) = \int_0^T \left[ \frac{1}{2}(Ju, \Pi u) + H^*(-Ju) \right] dt$$

defined on the space

$$(7) \quad L_0^\beta = \left\{ u \in L^\beta(0, T; \mathbb{R}^{2n}) \mid \int_0^T u(t) dt = 0 \right\}$$

Here  $H^*$  is the Fenchel conjugate (or Legendre transform) of the convex function  $H$ . Since  $H$  is strictly convex,  $H^*$  is  $C^1$  and condition (3) yields by duality:

$$(8) \quad H^*(\lambda y) \leq \lambda^\beta H^*(y) \quad \forall \lambda \geq 1, \quad \|y\| \geq R'$$

with  $\alpha^{-1} + \beta^{-1} = 1$ . The operator  $\Pi : L_0^\beta \rightarrow L_0^\alpha$  is defined by

$$(9) \quad \frac{d}{dt}(\Pi u) = u \quad \int_0^T (\Pi u) dt = 0$$

and it is compact. The resulting functional  $\psi$  is  $C^1$  and satisfies condition (C) of Palais-Smale.

By Clarke's dual action principle (see [C1], [C2] and [CE]), if  $\bar{u}$  is a critical point of  $\psi$  on  $L_0^\beta$ , then there exists some solution  $\bar{x}$  of problem (5) such that  $\frac{d\bar{x}}{dt} = \bar{x}$ . In other words,  $\bar{x} = \Pi\bar{u} + \xi$  for some constant  $\xi \in \mathbb{R}^{2n}$ .

The problem is then to find a critical point  $\bar{u} \neq 0$  for  $\psi$ . This is done by noting that:

(a)  $\psi$  has a local minimum at the origin:

$$(10) \quad \exists \rho > 0 : \quad \inf\{\psi(u) \mid \|u\| = \rho\} > \psi(0) = 0$$

(b)  $\psi$  is negative at some point far away:

$$\exists u_1 : \quad \|u_1\| > \rho \text{ and } \psi(u_1) < \psi(0) = 0.$$

By a celebrated theorem of Ambrosetti and Rabinowitz (see [AR]), this, together with condition (C), is enough to ensure the existence of some critical point  $\bar{u} \neq 0$ .

PART 2: MINIMAL PERIOD: (Ekeland and Hofer [EH])

Since  $H''(x)$  is positive definite for  $x \neq 0$ , we have the formula:

$$(11) \quad (H^*)''(y)H''(x) = I \quad \text{for } y = H'(x)$$

whereby  $H^*$  is  $C^2$  on  $\mathbb{R}^{2n} \setminus \{0\}$ . We know that  $\bar{u} = \frac{d\bar{x}}{dt}$ , where  $\bar{x}$  solves equation (5). It follows that  $\bar{u}$  must be  $C^1$ , and  $\bar{u}(t) \neq 0$  for all  $t$ .

We may therefore associate with  $\bar{u}$  a well-defined quadratic form  $q_T$  on  $L_0^2$ , given by:

$$(12) \quad q_T(v, v) = \int_0^T [(Jv, \Pi v) + (A(t)Jv, Jv)] dt$$

with

$$(13) \quad A(t) = H''(\bar{x}(t))^{-1} = (H^*)'' \left( -J \frac{d\bar{x}}{dt}(t) \right).$$

We then have a splitting

$$(14) \quad L_0^2 = E_+ \oplus E_0 \oplus E_-$$

into positive, null and negative subspace for  $q_T$ , where  $E_0$  and  $E_-$  are finite-dimensional. We define the *index*  $i_T(\bar{x})$  of the periodic solution  $\bar{x}$  to be:

$$(15) \quad i_T(\bar{x}) = \dim E_-$$

Hofer [H] has shown that applying the Ambrosetti-Rabinowitz theorem to a  $C^2$  functional with Freedholm derivative yields a critical point with Morse index  $\leq 1$ . This result does not apply directly to the present situation, because the functional  $\psi$  is not  $C^2$ , but can be adapted, so that

$$(16) \quad i_T(\bar{x}) \leq 1.$$

From then on, using a geometrical interpretation of the index devised by Ekeland [E2], one can show that  $\bar{x}$  must have minimal period  $T$ . ■

We now proceed to reduce the general case to the particular situation of proposition 1. The idea, is that a bound on the index of the critical point (here  $i_T(\bar{x}) \leq 1$ ) will yield an  $L^\infty$  estimate on the solution  $\bar{x}$ .

PROOF OF THEOREM 2: Choose  $h_0 > 1$  such that:

$$(17) \quad (x \in \Omega \text{ and } H(x) \geq h_0) \implies H''(x) > \frac{2\pi}{T} I$$

Set:

$$(18) \quad \tilde{\Omega} := \{x \in \Omega \mid H(x) < h_0\}$$

$\tilde{\Omega}$  is an open bounded convex set. Construct a function  $\tilde{H} \in C^2(\mathbb{R}^{2n}, \mathbb{R})$  such that:

$$(19) \quad \tilde{H}(x) = H(x), \quad \forall x \in \tilde{\Omega}$$

$$(20) \quad \tilde{H}''(x) > \frac{2\pi}{T}I, \quad \forall x \notin \tilde{\Omega}$$

$$(21) \quad \exists r > 0, \exists c > 0 : \quad \|x\| > r \implies \tilde{H}(x) = c\|x\|^4$$

One can, for instance, proceed as follows. Choose a number  $h_1 > h_0$  such that:

$$(22) \quad H(x) \geq h_1 \implies H''(x) \geq \left(\frac{2\pi}{T} + 1\right)I$$

Now consider the Fenchel conjugate  $H^*$  of  $H$ . It is convex and finite everywhere. Choose some large number  $A > 0$  and write:

$$(23) \quad H_A^*(y) := \begin{cases} H^*(y) & \text{if } \|y\| \leq A \\ +\infty & \text{otherwise} \end{cases}$$

Take the Fenchel conjugate again, thereby defining  $H_A := (H_A^*)^*$ . It is convex, finite everywhere, and grows linearly at infinity. If  $A$  has been chosen large enough, we will have:

$$(24) \quad H(x) \leq 2h_1 \implies H(x) = H_A(x)$$

Now define:

$$(25) \quad H_A^\delta := \max\{H_A(x), \delta\|x\|^2\}$$

For  $\delta > 0$  small enough, we have

$$(26) \quad H(x) \leq 2h_1 \implies H(x) = H_A(x) = H_A^\delta(x)$$

Finally, choose an increasing  $C^2$  convex function  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  such that

$$(27) \quad \varphi(h) = h \quad \text{for } 0 \leq h \leq h_1$$

$$(28) \quad \varphi(h) = \gamma h^2 \quad \text{for } h \geq 2h_1$$

where  $\gamma > 0$  is a large constant. Now set:

$$(29) \quad \hat{H}(x) := \varphi(H_A^\delta(x))$$

$\hat{H}$  is a convex function, which coincides with  $H$  on the set where  $H(x) \leq h_1$ , and with  $\gamma\delta^2\|x\|^4$  when  $\|x\|$  is large. If  $\gamma$  is large enough, we will have  $\hat{H}''(x) \geq (\frac{2\pi}{T}I)$  at every point  $x$  where  $\hat{H}$  is  $C^2$  and  $H(x) \geq h_1$ . Smoothing  $\hat{H}$  down, we get a  $C^2$  function  $\tilde{H}$  satisfying (19) to (21).

Apply proposition 1 to  $\tilde{H}$ . We get a solution  $\bar{x}$  of the problem

$$(30) \quad \dot{x} = J\tilde{H}'(x)$$

$$(31) \quad x(0) = x(T)$$

such that  $\bar{x}$  has minimal period  $T$ . We claim that:

$$(32) \quad \exists t_0 : \quad \tilde{H}''(\bar{x}(t_0)) \leq \frac{2\pi}{T}I$$

Indeed, by formula (16), we have  $i_T(\bar{x}) \leq 1$ , that is, the quadratic form  $q_T$  of formula (12) has negative eigenspace of dimension 0 or 1. If (32) did not hold, we would have  $\tilde{H}''(\bar{x}(t)) > \frac{2\pi}{T}I$  for all  $t$ , and hence:

$$(33) \quad A(t) < \frac{T}{2\pi}I$$

On the other hand, the operator  $-J\Pi$  is known to have an  $n$  dimensional negative eigenspace  $F_-$ , corresponding to the eigenvalue  $-\frac{T}{2\pi}$ . Substituting into  $q_T$ , we get:

$$(34) \quad \forall v \in F_-, \quad q_T(v, v) < -\frac{T}{2\pi}\|v\|^2 + \frac{T}{2\pi}\|v\|^2 < 0$$

So we have found an  $n$ -dimensional subspace on which the restriction of  $q_T$  is negative definite, contradicting  $i_T(\bar{x}) \leq 1$ . Formula (32) is proved.

Condition (20) then implies that  $\tilde{H}(\bar{x}(t_0)) < h_0$ . Since  $\tilde{H}$  is an integral of the motion:

$$(35) \quad \tilde{H}(\bar{x}(t)) < h_0 \quad \forall t$$

Condition (27) then implies that  $\tilde{H}$  coincides with  $H$  on a neighbourhood of the trajectory  $\bar{x}$ . So  $\tilde{H}'(\bar{x}(t)) = H'(\bar{x}(t))$ , and  $\bar{x}$  in fact solves the equation

$$(36) \quad \dot{x} = JH'(x)$$

■



### 3. - Second-order systems

The preceding results do not apply directly to second-order systems:

$$(1) \quad \begin{cases} \ddot{q} + V'(q) = 0 \\ q(0) = q(T) \\ \dot{q}(0) = \dot{q}(T) \end{cases}$$

because the corresponding Hamiltonian:

$$(2) \quad H(p, q) = \frac{1}{2}p^2 + V(q)$$

cannot satisfy any superquadraticity assumption. A special statement is needed, with its own proof.

**THEOREM 4.** *Let  $\Omega$  be a convex open subset of  $\mathbb{R}^{2n}$  containing the origin. Let  $V \in C^2(\Omega; \mathbb{R})$  be such that*

$$(3) \quad V(q) \geq V(0) = 0 \quad \forall q \in \Omega$$

$$(4) \quad V''(q) \text{ is positive definite} \quad \forall q \neq 0$$

$$(5) \quad V''(0) = 0$$

$$(6) \quad V''(q)^{-1} \rightarrow 0 \quad \text{when} \quad \|q\| \rightarrow \infty \quad \text{or} \quad q \rightarrow \partial\Omega$$

*Then, for every  $T > 0$ , problem (1) has a solution with minimal period  $T$ . ■*

When  $\Omega$  is a bounded subset of  $\mathbb{R}^n$ , we are dealing with a potential well. Benci [B] was the first to prove this kind of result; he does not assume  $\Omega$  to be convex, but requires that

$$(7) \quad \frac{1}{V(x)} V'(x) \cdot D'(x) \rightarrow +\infty$$

when  $x \rightarrow \partial\Omega$ . Here  $D(x) = -\text{dist}(x, \partial\Omega)$ . This assumption is here replaced by (6). Minimality of the period was first proved by Ambrosetti and Coti Zelati [ACZ] in the convex case, using the dual approach.

To prove theorem 4, we have to work in the Lagrangian formalism. The standard action functional, defined on  $W^{1,\alpha}(\mathbb{R}/T\mathbb{Z}; \mathbb{R}^{2n})$ , ( $\alpha > 2$  will be chosen later) is:

$$(8) \quad \varphi(q) = \int_0^T \left[ \frac{1}{2} \dot{q}^2 + V(q) \right] dt$$

whereas the corresponding dual action is

$$(9) \quad \varphi(u) = \int_0^T \left[ -\frac{1}{2}(\Pi u)^2 + V^*(u) \right] dt$$

for  $u \in L_0^\beta$ ,  $\alpha^{-1} + \beta^{-1} = 1$ . As before,  $\Pi u$  denotes the antiderivative of  $u$  with mean value zero. If  $\bar{u}$  is a critical point of  $\psi$  on  $\mathbb{R}^{2n}$ , then there exists a solution  $\bar{q}$  of problem (1) such that  $\frac{d^2}{dt^2}\bar{q} = \bar{u}$ .

The proof now proceeds in two steps:

STEP 1: Assume in addition that there exist  $\alpha > 2$  and  $r > 0$  such that

$$(10) \quad \limsup_{\|q\| \rightarrow \infty} V(q)\|q\|^{-\alpha} < \infty$$

$$(11) \quad \|q\| \geq r \implies V(\lambda q) \geq \lambda^\alpha V(q) \quad \forall \lambda \geq 1$$

We then argue as in the preceding section by applying the Ambrosetti-Rabinowitz mountain-pass theorem, and showing that the critical point has index 0 or 1. So theorem 3 obtains under the additional assumptions (10) and (11).

STEP 2: In the general case, that is, when  $V$  no longer satisfies (10) and (11), we proceed as follows. With every  $k \in \mathbb{N}$ , we associate the set:

$$(12) \quad \Omega_k := \{q \in \Omega \mid V(q) < k\}$$

$\Omega_k$  is an open bounded set of  $\mathbb{R}^n$ . Proceeding as in the proof of theorem 1, we construct a function  $V_k \in C^2(\mathbb{R}^n, \mathbb{R})$  such that:

$$(13) \quad V_k \leq V_{k+1} \leq V \quad \forall k \in \mathbb{N}$$

$$(14) \quad V_k(q) = V(q) \quad \forall q \in \Omega_k$$

$$(15) \quad \forall \lambda > 0, \exists M : \quad \forall k \geq M, q \notin \Omega_M \implies V_k''(q) \geq \lambda I$$

$$(16) \quad \exists r > 0, \exists c_k > 0 : \quad \|q\| > r \implies V_k(q) = c_k \|q\|^4$$

$V_k$  satisfies the additional conditions (10) and (11). By step 1, the problem

$$(17) \quad \begin{cases} \ddot{q} + V_k'(q) = 0 \\ q(t+T) = q(t) \end{cases}$$

has a solution  $q_k$ , with minimal period  $T$  and index  $\leq 1$ . The latter is the index

of the quadratic form  $\psi_k''(u_k)$  on  $L_0^2$ , with  $u_k := \frac{d^2}{dt^2} q_k$  and

$$(18) \quad \begin{aligned} (\psi_k''(u_k)v, v) &:= \int_0^T [-(\Pi v)^2 + ((V_k^*)''(u_k)v, v)] dt \\ &= \int_0^T [-(\Pi v)^2 + (V_k''(q_k)^{-1}v, v)] dt \end{aligned}$$

The critical point  $u_k$  for  $\psi_k$  found by the Ambrosetti-Rabinowitz theorem (see section 2). The corresponding critical value  $\gamma_k$  is given by:

$$(19) \quad \gamma_k := \inf_{c \in \Gamma} \max_{0 \leq s \leq 1} \psi_k(c(s))$$

where  $\Gamma$  is the set of all continuous paths  $c : [0, 1] \rightarrow L_0^{\frac{3}{2}}$  such that  $c(0) = 0$  and  $\psi_k(c(1)) < 0$ . Since  $\psi_k(0) = 0$ , we have  $\gamma_k > 0$ , and since  $V_k \leq V_{k+1}$ , we have  $V_k^* \geq V_{k+1}^*$ . It follows that the sequence  $\gamma_k$  is bounded:

$$(20) \quad 0 \leq \gamma_{k+1} \leq \gamma_k$$

From duality theory, we have:

$$(21) \quad \gamma_k = \int_0^T \left[ \frac{1}{2} \dot{q}^2 - V_k(q_k) \right] dt$$

We also introduce the constants:

$$(22) \quad h_k := \frac{1}{2} \dot{q}_k(t)^2 + V_k(q_k(t))$$

If the  $V_k \circ q_k$ ,  $k \in \mathbb{N}$ , are uniformly bounded, say  $V_k(q_k(t)) \leq b$ , the problem is over. Indeed, it then follows from (14) and (12) that  $q_k(t) \in \Omega_k$  as soon as  $k \geq b$ , so that  $q_k$  is in fact a solution of problem (1), with minimal period  $T$ .

So all we have to do is to show that the sequence  $\|V_k \circ q_k\|_\infty$  is bounded. Assume otherwise. Then we may assume that

$$(23) \quad \|V_k \circ q_k\| \rightarrow \infty$$

so that  $h_k \rightarrow \infty$  by formula (22).

We will now relate the  $h_k$  to the  $\gamma_k$ . We have, by adding (21) and (22):

$$(24) \quad \gamma_k + T h_k = \int_0^T \dot{q}_k^2 dt$$

Take any  $M \in \mathbb{N}$ , and set

$$(25) \quad m_k = \max\{V_k(q_k(t)) | 0 \leq t \leq T\} \rightarrow \infty$$

We take  $k$  so large that  $m_k > M$ . Denote by  $\mu$  the Lebesgue measure on  $[0, T]$  and set:

$$(26) \quad A_k := \mu\{t | V_k(q_k(t)) \geq M\}$$

From equations (22) and (24) it follows that:

$$(27) \quad \gamma_k + Th_k \geq \int_{V_k(q_k) < M} \dot{q}_k^2 dt \geq 2(h_k - M)(T - \mu(A_k))$$

Rewrite this inequality as follows

$$(28) \quad \gamma_k \geq h_k(T - 2\mu(A_k)) - (2M(T - \mu(A_k)))$$

Since  $\gamma_k$  is bounded and  $h_k \rightarrow +\infty$ , we must have  $T - 2\mu(A_k) \leq 0$  for all but a finite number of  $k \in \mathbb{N}$ . Thus we have proved that:

$$(29) \quad \forall M \exists K : \forall k \geq K, \mu(A_k) \geq \frac{T}{2}$$

By condition (15), there is an  $M$  so large that:

$$(30) \quad (k \geq M \text{ and } V_k(q) > M) \implies V_k''(q) \geq 2 \left(\frac{4\pi}{T}\right)^2$$

Pick the corresponding  $K$  from formula (29). With  $K$  fixed in this way, pick a vector  $\xi \in \mathbb{R}^n$ , and set, for  $t \geq 0$ :

$$(31) \quad \mu_K(t) := 2\pi\mu(A_K \cap [0, t]) / \mu(A_K)$$

$$(32) \quad v_K(t) := \begin{cases} \xi \sin \mu_K(t) & \text{if } t \in A_K \\ 0 & \text{otherwise} \end{cases}$$

The function  $v_K$  has been built to have mean value zero, so  $v_K \in L_0^2(0, T)$ . Since  $A_K$  is closed,  $[0, T]/A_K$  is a countable union of subintervals  $(t_n, t_n + \delta_n)$ ,  $n \in \mathbb{N}$ , so that:

$$(33) \quad \int_0^t v_K(s) ds = \begin{cases} \frac{\mu(A_K)}{2\pi} \xi - \frac{\mu(A_K)}{2\pi} \xi \cos \mu_K(t) & \text{if } t \in A_K \\ \frac{\mu(A_K)}{2\pi} \xi - \frac{\mu(A_K)}{2\pi} \xi \cos \mu_K(t_n) & \text{if } t_n < t < t_n + \delta_n \end{cases}$$

Hence:

$$(\Pi v_K)(t) = \int_0^t v_K(s) ds - \frac{1}{T} \int_0^T \left( \int_0^t v_K(s) ds \right) dt$$

with

$$(35) \quad \int_0^T \left( \int_0^t v_K(s) ds \right) dt = \frac{\mu(A_K)}{2\pi} \xi T - \frac{\mu(A_K)}{2\pi} \xi \sum_{n=0}^{\infty} \delta_n \cos \mu_K(t_n)$$

We have:

$$(36) \quad \int_0^T (\Pi v_K)^2 dt = \int_0^T \left( \int_0^t v_K(s) ds \right)^2 dt - \frac{1}{T} \left( \int_0^T \left( \int_0^t v_K(s) ds \right) dt \right)^2$$

Substituting (33) and (35) into the right-hand side yields (after simplifying):

$$(37) \quad \begin{aligned} \int_0^T (\Pi v_K)^2 dt &= \\ &= \frac{\mu(A_K)^2}{4\pi^2} \|\xi\|^2 \left[ \frac{T}{2} + \sum_{n=0}^{\infty} \delta_n \cos^2 u_K(t_n) - \frac{1}{T} \left( \sum_{n=0}^{\infty} \delta_n \cos u_K(t_n) \right)^2 \right] \\ &\geq \frac{\mu(A_K)^2}{4\pi^2} \|\xi\|^2 \left[ \frac{T}{2} + \left( 1 - \frac{1}{T} \sum_{n=0}^{\infty} \delta_n \right) \left( \sum_{n=0}^{\infty} \delta_n \cos^2 u_K(t_n) \right) \right] \\ &= \frac{\mu(A_K)^2}{4\pi^2} \|\xi\|^2 \left[ \frac{T}{2} + \frac{\mu(A_K)}{T} \left( \sum_{n=0}^{\infty} \delta_n \cos^2 u_K(t_n) \right) \right] \\ &= \frac{\mu(A_K)^2}{4\pi^2} \|\xi\|^2 \frac{T}{2} \end{aligned}$$

Finally, we use (29) to get a uniform estimate:

$$(38) \quad \int_0^T (\Pi v_K)^2 dt \geq \left( \frac{T}{4\pi} \right)^2 \mu(A_K) \|\xi\|^2$$

We now substitute this into the quadratic form  $\psi_K''(u_K)$  given by formula (18). Recall also that  $v_K$  vanishes outside  $A_K$  and that estimate (30) holds on  $A_K$ :

$$(39) \quad \begin{aligned} (\psi_K''(u_K)v_K, v_K) &\leq - \left( \frac{T}{4\pi} \right)^2 \mu(A_K) \|\xi\|^2 + \int_0^T (V_K''(q_K))^{-1} v_K, v_K dt \\ &\leq \mu(A_K) \|\xi\|^2 \left[ - \left( \frac{T}{4\pi} \right)^2 + \frac{1}{2} \left( \frac{T}{4\pi} \right)^2 \right] \\ &= - \frac{1}{2} \left( \frac{T}{4\pi} \right)^2 \mu(A_K) \|\xi\|^2 \end{aligned}$$

Since  $\xi$  varies in  $\mathbb{R}^n$ , we have found an  $n$ -dimensional space of functions  $v_K$  where the restriction of  $\psi_K''(u_K)$  is negative definite. This means that the index of  $q_K$ , the solution of problem (19) associated with  $u_K$ , is at least  $n$ . But we know that this index is  $\leq 1$ . If  $n > 1$ , we have a contradiction, so that assumption (25) cannot hold, and the theorem is proved. The case  $n = 1$  is of course trivial, and can be handled directly. ■

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