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Index estimates and critical points of functionals not satisfying Palais-Smale

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Index Estimates and Critical Points of Functionals Not Satisfying Palais-Smale

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1. - Introduction

We begin by recalling the definition of a Hamiltonian system with n degrees of freedom. Define a $2n \times 2n$ matrix J by

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1}$$

If $H : \mathbb{R}^{2n} \to \mathbb{R}$ is a smooth function, the Hamiltonian system associated with H is the system of ODE's:

$$\dot{x} = JH'(x). \tag{2}$$

We are interested in finding periodic solutions of this equation, that is, in solving the boundary-value problem

$$\begin{cases} \dot{x} = JH'(x) \\ x(0) = x(T) \end{cases} \tag{3}$$

for prescribed T. There is a well-known existence result:

THEOREM 1. *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be strictly convex and C^1. Assume that*

$$H(x) > H(0) = 0 \qquad \forall x \neq 0 \tag{4}$$

$$H(x)\|x\|^{-2} \to 0 \qquad \textit{when } x \to 0 \tag{5}$$

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and that, for some constants $\alpha > 2$ and $R > 0$ we have:

(6) $$\|x\| \geq R \Longrightarrow H(\lambda x) \geq \lambda^\alpha H(x) \qquad \forall \lambda \geq 1.$$

Then, for every $T > 0$, the boundary-value problem (3) has a solution with minimal period T. ∎

Under more general assumptions, Rabinowitz proved in [R1] the existence of a non-trivial solution (the solution $x(t) \equiv 0$ being considered trivial) and in [R2] the existence of an unbounded sequence of solutions. In his statement, condition (6) is replaced by the following one:

(7) $$(H'(x), x) \geq \alpha H(x)$$

which turns out to be equivalent to (6) in the convex case.

The existence of a solution with minimal period T is due to Ekeland and Hofer [EH]. Earlier, Ambrosetti and Mancini [AM], and then Girardi and Matzeu ([GM1][GM2]) had proved the existence of such a solution under more restrictive conditions on H.

In all these papers, condition (6) or (7) is crucial to prove that the Palais-Smale condition holds in the associated variational problem. In fact, this condition first appears in the seminal paper of Ambrosetti and Rabinowitz, where it is introduced specifically for that purpose. On the other hand, it is clearly a technical condition, which is hard to justify on physical grounds.

The purpose of this paper is to replace condition (6) by the condition that the Hessian $H''(x)$ goes to infinity (in the sense that its smallest eigenvalue goes to $+\infty$) as $\|x\| \to \infty$. In fact, we shall even be able to treat the case when H is defined only on a convex open subset of $\mathbb{R}^{2n}$, and goes to $+\infty$ on the boundary. We shall prove the following.

THEOREM 2. *Let Ω be a convex open subset of $\mathbb{R}^{2n}$ containing the origin. Let $H \in C^2(\Omega, \mathbb{R})$ be such that*

(8) $$H(x) \geq H(0) = 0 \qquad \forall x \in \Omega$$

(9) $$H''(x) \quad \textit{is positive definite} \qquad \forall x \neq 0$$

(10) $$H''(0) = 0$$

(11) $$H''(x)^{-1} \to 0 \qquad \textit{as} \qquad \|x\| \to \infty \qquad \textit{or} \qquad \|x\| \to \partial\Omega$$

Then, for every T, the boundary value problem (3) has a solution with minimal period T. ∎

By assumption (11), we mean that for every $a > 0$, there exist $\rho > 0$ and $\eta > 0$ such that

(12) $$\|x\| > \rho \quad \text{or} \quad d(x, \partial\Omega) \leq \eta \Longrightarrow H''(x) \leq aI$$

This implies that $H(x)\|x\|^{-2} \to +\infty$ when $\|x\| \to \infty$ or $x \to \partial\Omega$. More precisely, for every $a' > 0$ there exist $\rho' > 0$ and $\eta' > 0$ such that

$$\|x\| > \rho' \quad \text{or} \quad d(x, \partial\Omega) \leq \eta' \Longrightarrow H(x)\|x\|^{-2} \geq a'. \tag{13}$$

Assumptions (6) and (11) are not directly comparable. Theorem 1 only requires H to be strictly convex and C^1, in which case (6) amounts to (7), which does not imply anything on the behaviour of the Hessian $H''(x)$, if it exists. Theorem 2 requires H to be C^2 and strictly convex, but even then (11) does not imply (6) (take $H(x) = x^2 \log\|x\|$ for $\|x\|$ large).

The most natural assumption, subsuming (6) and (11), would be

$$H(x)\|x\|^{-2} \to +\infty \quad \text{when} \quad \|x\| \to \infty \quad \text{or} \quad x \to \partial\Omega \tag{14}$$

or, failing that,

$$\|H'(x)\|\,\|x\|^{-1} \to +\infty \quad \text{when} \quad \|x\| \to \infty \quad \text{or} \quad x \to \partial\Omega \tag{15}$$

The authors do not know wether assumptions (14) or (15) are enough for existence. As it is, assumptions (11) enables us to extend theorem 1 to the case when H is defined on a proper subset Ω of $\mathbb{R}^{2n}$ – and this is the first result of its kind (for first order system; for second order ones see the discussion in section 3).

In the following section, we prove theorem 1. In the last section, we extend the results to second-order systems.

2. - Hamiltonian systems

We begin by recalling the proof of a weaker version of theorem 1, which will be enough for our purposes.

PROPOSITION 3. *Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be strictly convex and C^1. Assume that:*

$$H(x) > H(0) = 0 \qquad \forall x \neq 0 \tag{1}$$

$$H(x)\|x\|^{-2} \to 0 \qquad \textit{when} \qquad x \to 0 \tag{2}$$

$$H(\lambda x) \geq \lambda^\alpha H(x) \qquad \forall \lambda \geq 1, \quad \|x\| \geq R \tag{3}$$

$$H(x) \leq \frac{k^\alpha}{\alpha}\|x\|^\alpha \qquad \forall x. \tag{4}$$

for suitable constants $\alpha > 2$, $R > 0$ and $k > 0$. Then, for every $T > 0$, the Hamiltonian system

$$\dot{x} = JH'(x) \tag{5}$$

has a periodic solution with minimal period T. ∎

The proof is divided in two parts:

PART 1: EXISTENCE: (Ekeland [E1])

We introduce Clarke's dual action functional

$$\psi(u) = \int_0^T \left[\frac{1}{2}(Ju, \Pi u) + H^*(-Ju)\right] dt \tag{6}$$

defined on the space

$$L_0^\beta = \left\{u \in L^\beta(0, T; \mathbb{R}^{2n}) | \int_0^T u(t)\, dt = 0\right\} \tag{7}$$

Here H^* is the Fenchel conjugate (or Legendre transform) of the convex function H. Since H is strictly convex, H^* is C^1 and condition (3) yields by duality:

$$H^*(\lambda y) \leq \lambda^\beta H^*(y) \qquad \forall \lambda \geq 1, \quad ||y|| \geq R' \tag{8}$$

with $\alpha^{-1} + \beta^{-1} = 1$. The operator $\Pi : L_0^\beta \to L_0^\alpha$ is defined by

$$\frac{d}{dt}(\Pi u) = u \qquad \int_0^T (\Pi u) dt = 0 \tag{9}$$

and it is compact. The resulting functional ψ is C^1 and satisfies condition (C) of Palais-Smale.

By Clarke's dual action principle (see [C1], [C2] and [CE]), if $\bar{u}$ is a critical point of ψ on L_0^β, then there exists some solution $\bar{x}$ of problem (5) such that $\frac{d\bar{x}}{dt} = \bar{x}$. In order words, $\bar{x} = \Pi \bar{u} + \xi$ for some constant $\xi \in \mathbb{R}^{2n}$.

The problem is then to find a critical point $\bar{u} \neq 0$ for ψ. This is done by noting that:

(a) ψ has a local minimum at the origin:

$$\exists \rho > 0 : \qquad \inf\{\psi(u) |\ ||u|| = \rho\} > \psi(0) = 0 \tag{10}$$

(b) ψ is negative at some point far away:

$$\exists u_1 : \qquad ||u_1|| > \rho \text{ and } \psi(u_1) < \psi(0) = 0.$$

By a celebrated theorem of Ambrosetti and Rabinowitz (see [AR]), this, together with condition (C), is enough to ensure the existence of some critical point $\bar{u} \neq 0$.

PART 2: MINIMAL PERIOD: (Ekeland and Hofer [EH])

Since $H''(x)$ is positive definite for $x \neq 0$, we have the formula:

$$(H^*)''(y)H''(x) = I \qquad \text{for } y = H'(x) \tag{11}$$

whereby H^* is C^2 on $\mathbb{R}^{2n}\backslash\{0\}$. We know that $\bar{u} = \frac{d\bar{x}}{dt}$, where $\bar{x}$ solves equation (5). It follows that $\bar{u}$ must be C^1, and $\bar{u}(t) \neq 0$ for all t.

We may therefore associate with $\bar{u}$ a well-defined quadratic form q_T on L_0^2, given by:

$$q_T(v,v) = \int_0^T [(Jv, \Pi v) + (A(t)Jv, Jv)]\, dt \tag{12}$$

with

$$A(t) = H''(\bar{x}(t))^{-1} = (H^*)''\left(-J\frac{d\bar{x}}{dt}(t)\right). \tag{13}$$

We then have a splitting

$$L_0^2 = E_+ \oplus E_0 \oplus E_- \tag{14}$$

into positive, null and negative subspace for q_T, where E_0 and E_- are finite-dimensional. We define the *index* $i_T(\bar{x})$of the periodic solution $\bar{x}$ to be:

$$i_T(\bar{x}) = \dim E_- \tag{15}$$

Hofer [H] has shown that applying the Ambrosetti-Rabinowitz theorem to a C^2 functional with Freedholm derivative yields a critical point with Morse index ≤ 1. This result does not apply directly to the present situation, because the functional ψ is not C^2, but can be adapted, so that

$$i_T(\bar{x}) \leq 1. \tag{16}$$

From then on, using a geometrical interpretation of the index devised by Ekeland [E2], one can show that $\bar{x}$ must have minimal period T. ■

We now proceed to reduce the general case to the particular situation of proposition 1. The idea, is that a bound on the index of the critical point (here $i_T(\bar{x}) \leq 1$) will yield an L^∞ estimate on the solution $\bar{x}$.

PROOF OF THEOREM 2: Choose $h_0 > 1$ such that:

$$(x \in \Omega \text{ and } H(x) \geq h_0) \Longrightarrow H''(x) > \frac{2\pi}{T} I \tag{17}$$

Set:

$$\tilde{\Omega} := \{x \in \Omega | H(x) < h_0\} \tag{18}$$

$\tilde{\Omega}$ is an open bounded convex set. Construct a function $\tilde{H} \in C^2(\mathbb{R}^{2n}, \mathbb{R})$ such that:

$$\tilde{H}(x) = H(x), \qquad \forall x \in \tilde{\Omega} \tag{19}$$

$$\tilde{H}''(x) > \frac{2\pi}{T} I, \qquad \forall x \notin \tilde{\Omega} \tag{20}$$

$$\exists r > 0,\ \exists c > 0: \qquad \|x\| > r \Longrightarrow \tilde{H}(x) = c\|x\|^4 \tag{21}$$

One can, for instance, proceed as follows. Choose a number $h_1 > h_0$ such that:

$$H(x) \geq h_1 \Longrightarrow H''(x) \geq \left(\frac{2\pi}{T} + 1\right) I \tag{22}$$

Now consider the Fenchel conjugate H^* of H. It is convex and finite everywhere. Choose some large number $A > 0$ and write:

$$H_A^*(y) := \begin{cases} H^*(y) & \text{if } \|y\| \leq A \\ +\infty & \text{otherwise} \end{cases} \tag{23}$$

Take the Fenchel conjugate again, thereby defining $H_A := (H_A^*)^*$. It is convex, finite everywhere, and grows linearly at infinity. If A has been chosen large enough, we will have:

$$H(x) \leq 2h_1 \Longrightarrow H(x) = H_A(x) \tag{24}$$

Now define:

$$H_A^\delta := \max\{H_A(x),\ \delta\|x\|^2\} \tag{25}$$

For $\delta > 0$ small enough, we have

$$H(x) \leq 2h_1 \Longrightarrow H(x) = H_A(x) = H_A^\delta(x) \tag{26}$$

Finally, choose an increasing C^2 convex function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\varphi(h) = h \qquad \text{for } 0 \leq h \leq h_1 \tag{27}$$

$$\varphi(h) = \gamma h^2 \qquad \text{for } h \geq 2h_1 \tag{28}$$

where $\gamma > 0$ is a large constant. Now set:

$$\hat{H}(x) := \varphi(H_A^\delta(x)) \tag{29}$$

$\hat{H}$ is a convex function, which coincides with H on the set where $H(x) \leq h_1$, and with $\gamma\delta^2\|x\|^4$ when $\|x\|$ is large. If γ is large enough, we will have $\hat{H}''(x) \geq (\frac{2\pi}{T}I)$ at every point x where $\hat{H}$ is C^2 and $H(x) \geq h_1$. Smoothing $\hat{H}$ down, we get a C^2 function $\tilde{H}$ satisfying (19) to (21).

Apply proposition 1 to $\tilde{H}$. We get a solution $\bar{x}$ of the problem

$$\dot{x} = J\tilde{H}'(x) \tag{30}$$

$$x(0) = x(T) \tag{31}$$

such that $\bar{x}$ has minimal period T. We claim thath:

$$\exists t_0 : \qquad \tilde{H}''(\bar{x}(t_0)) \leq \frac{2\pi}{T}I \tag{32}$$

Indeed, by formula (16), we have $i_T(\bar{x}) \leq 1$, that is, the quadratic form q_T of formula (12) has negative eigenspace of dimension 0 or 1. If (32) did not hold, we would have $\tilde{H}''(\bar{x}(t)) > \frac{2\pi}{T}I$ for all t, and hence:

$$A(t) < \frac{T}{2\pi}I \tag{33}$$

On the other hand, the operator $-J\Pi$ is known to have an n dimensional negative eigenspace F_-, corresponding to the eigenvalue $-\frac{T}{2\pi}$. Substituting into q_T, we get:

$$\forall v \in F_-, \qquad q_T(v,v) < -\frac{T}{2\pi}\|v\|^2 + \frac{T}{2\pi}\|v\|^2 < 0 \tag{34}$$

So we have found an n-dimensional subspace on which the restriction of q_T is negative definite, contradicting $i_T(\bar{x}) \leq 1$. Formula (32) is proved.

Condition (20) then implies that $\tilde{H}(\bar{x}(t_0)) < h_0$. Since $\tilde{H}$ is an integral of the motion:

$$\tilde{H}(\bar{x}(t)) < h_0 \qquad \forall t \tag{35}$$

Condition (27) then implies that $\tilde{H}$ coincides with H on a neighbourhood of the trajectory $\bar{x}$. So $\tilde{H}'(\bar{x}(t)) = H'(\bar{x}(t))$, and $\bar{x}$ in fact solves the equation

$$\dot{x} = JH'(x) \tag{36}$$

■

3. - Second-order systems

The preceding results do not apply directly to second-order systems:

$$\begin{cases} \ddot{q} + V'(q) = 0 \\ q(0) = q(T) \\ \dot{q}(0) = \dot{q}(T) \end{cases} \tag{1}$$

because the corresponding Hamiltonian:

$$H(p,q) = \frac{1}{2}p^2 + V(q) \tag{2}$$

cannot satisfy any superquadraticity assumption. A special statement is needed, with its own proof.

THEOREM 4. *Let Ω be a convex open subset of $\mathbb{R}^{2n}$ containing the origin. Let $V \in C^2(\Omega;\mathbb{R})$ be such that*

$$V(q) \geq V(0) = 0 \qquad \forall q \in \Omega \tag{3}$$

$$V''(q) \text{ is positive definite} \qquad \forall q \neq 0 \tag{4}$$

$$V''(0) = 0 \tag{5}$$

$$V''(q)^{-1} \to 0 \qquad \text{when} \qquad \|q\| \to \infty \quad \text{or} \quad q \to \partial\Omega \tag{6}$$

Then, for every $T > 0$, problem (1) has a solution with minimal period T. ■

When Ω is a bounded subset of $\mathbb{R}^n$, we are dealing with a potential well. Benci [B] was the first to prove this kind of result; he does not assume Ω to be convex, but requires that

$$\frac{1}{V(x)}V'(x) \cdot D'(x) \to +\infty \tag{7}$$

when $x \to \partial\Omega$. Here $D(x) = -\operatorname{dist}(x, \partial\Omega)$. This assumption is here replaced by (6). Minimality of the period was first proved by Ambrosetti and Coti Zelati [ACZ] in the convex case, using the dual approach.

To prove theorem 4, we have to work in the Lagrangian formalism. The standard action functional, defined on $W^{1,\alpha}(\mathbb{R}/T\mathbb{Z};\mathbb{R}^{2n})$, ($\alpha > 2$ will be chosen later) is:

$$\varphi(q) = \int_0^T \left[\frac{1}{2}\dot{q}^2 + V(q)\right] dt \tag{8}$$

whereas the corresponding dual action is

$$\varphi(u) = \int_0^T \left[-\frac{1}{2}(\Pi u)^2 + V^*(u) \right] dt \tag{9}$$

for $u \in L_0^\beta$, $\alpha^{-1} + \beta^{-1} = 1$. As before, Πu denotes the antiderivative of u with mean value zero. If $\bar{u}$ is a critical point of ψ on $\mathbb{R}^{2n}$, then there exists a solution $\bar{q}$ of problem (1) such that $\frac{d^2}{dt^2}\bar{q} = \bar{u}$.

The proof now proceeds in two steps:

STEP 1: Assume in addition that there exist $\alpha > 2$ and $r > 0$ such that

$$\limsup_{\|q\| \to \infty} V(q)\|q\|^{-\alpha} < \infty \tag{10}$$

$$\|q\| \geq r \Longrightarrow V(\lambda q) \geq \lambda^\alpha V(q) \qquad \forall \lambda \geq 1 \tag{11}$$

We then argue as in the preceding section by applying the Ambrosetti-Rabinowitz mountain-pass theorem, and showing that the critical point has index 0 or 1. So theorem 3 obtains under the additional assumptions (10) and (11).

STEP 2: In the general case, that is, when V no longer satisfies (10) and (11), we proceed as follows. With every $k \in \mathbb{N}$, we associate the set:

$$\Omega_k := \{q \in \Omega | V(q) < k\} \tag{12}$$

Ω_k is an open bounded set of $\mathbb{R}^n$. Prooceding as in the proof of theorem 1, we construct a function $V_k \in C^2(\mathbb{R}^n, \mathbb{R})$ such that:

$$V_k \leq V_{k+1} \leq V \qquad \forall k \in \mathbb{N} \tag{13}$$

$$V_k(q) = V(q) \qquad \forall q \in \Omega_k \tag{14}$$

$$\forall \lambda > 0, \ \exists M : \qquad \forall k \geq M, \ q \notin \Omega_M \Longrightarrow V_k''(q) \geq \lambda I \tag{15}$$

$$\exists r > 0, \ \exists c_k > 0 : \ \|q\| > r \Longrightarrow V_k(q) = c_k \|q\|^4 \tag{16}$$

V_k satisfies the additional conditions (10) and (11). By step 1, the problem

$$\begin{cases} \ddot{q} + V_k'(q) = 0 \\ q(t+T) = q(T) \end{cases} \tag{17}$$

has a solution q_k, with minimal period T and index ≤ 1. The latter is the index of the quadratic form $\psi_k''(u_k)$ on L_0^2, with $u_k := \frac{d^2}{dt^2}q_k$ and

$$(18)\qquad \begin{aligned}(\psi_k''(u_k)v, v) : &= \int_0^T [-(\Pi v)^2 + ((V_k^*)''(u_k)v, v)]\,dt \\ &= \int_0^T [-(\Pi v)^2 + (V_k''(q_k)^{-1}v, v)]\,dt\end{aligned}$$

The critical point u_k for ψ_k found by the Ambrosetti-Rabinowitz theorem (see section 2). The corresponding critical value γ_k is given by:

$$(19)\qquad \gamma_k := \inf_{c\in\Gamma} \max_{0\leq s\leq 1} \psi_k(c(s))$$

where Γ is the set of all continuous paths $c : [0,1] \to L_0^{\frac{4}{3}}$ such that $c(0) = 0$ and $\psi_k(c(1)) < 0$. Since $\psi_k(0) = 0$, we have $\gamma_k > 0$, and since $V_k \leq V_{k+1}$, we have $V_k^* \geq V_{k+1}^*$. It follows that the sequence γ_k is bounded:

$$(20)\qquad 0 \leq \gamma_{k+1} \leq \gamma_k$$

From duality theory, we have:

$$(21)\qquad \gamma_k = \int_0^T \left[\frac{1}{2}\dot{q}^2 - V_k(q_k)\right] dt$$

We also introduce the constants:

$$(22)\qquad h_k := \frac{1}{2}\dot{q}_k(t)^2 + V_k(q_k(t))$$

If the $V_k \circ q_k$, $k \in \mathbb{N}$, are uniformly bounded, say $V_k(q_k(t)) \leq b$, the problem is over. Indeed, it then follows from (14) and (12) that $q_k(t) \in \Omega_k$ as soon as $k \geq b$, so that q_k is in fact a solution of problem (1), with minimal period T.

So all we have to do is to show that the sequence $||V_k \circ q_k||_\infty$ is bounded. Assume otherwise. Then we may assume that

$$(23)\qquad ||V_k \circ q_k|| \to \infty$$

so that $h_k \to \infty$ by formula (22).

We will now relate the h_k to the γ_k. We have, by adding (21) and (22):

$$(24)\qquad \gamma_k + Th_k = \int_0^T \dot{q}_k^2\,dt$$

Take any $M \in \mathbb{N}$, and set

$$(25)\qquad m_k = \max\{V_k(q_k(t)) | 0 \leq t \leq T\} \to \infty$$

We take k so large that $m_k > M$. Denote by μ the Lebesgue measure on $[0, T]$ and set:

$$A_k := \mu\{t | V_k(q_k(t)) \geq M\} \tag{26}$$

From equations (22) and (24) it follows that:

$$\gamma_k + T h_k \geq \int_{V_k(q_k)<M} \dot{q}_k^2 \, dt \geq 2(h_k - M)(T - \mu(A_k)) \tag{27}$$

Rewrite this inequality as follows

$$\gamma_k \geq h_k(T - 2\mu(A_k)) - (2M(T - \mu(A_k)) \tag{28}$$

Since γ_k is bounded and $h_k \to +\infty$, we must have $T - 2\mu(A_k) \leq 0$ for all but a finite number of $k \in \mathbb{N}$. Thus we have proved that:

$$\forall M \ \exists K \quad : \quad \forall k \geq K, \quad \mu(A_k) \geq \frac{T}{2} \tag{29}$$

By condition (15), there is an M so large that:

$$(k \geq M \text{ and } V_k(q) > M) \Longrightarrow V_k''(q) \geq 2\left(\frac{4\pi}{T}\right)^2 \tag{30}$$

Pick the corresponding K from formula (29). With K fixed in this way, pick a vector $\xi \in \mathbb{R}^n$, and set, for $t \geq 0$:

$$\mu_K(t) := 2\pi\mu(A_K \cap [0, t])/\mu(A_K) \tag{31}$$

$$v_K(t) := \begin{cases} \xi \ \sin \ \mu_K(t) & \text{if } t \in A_K \\ 0 & \text{otherwise} \end{cases} \tag{32}$$

The function v_K has been built to have mean value zero, so $v_K \in L_0^2(0, T)$. Since A_K is closed, $[0, T]/A_K$ is a countable union of subintervals $(t_n, t_n + \delta_n)$, $n \in \mathbb{N}$, so that:

$$\int_0^t v_K(s) \, ds = \begin{cases} \frac{\mu(A_K)}{2\pi}\xi - \frac{\mu(A_K)}{2\pi}\xi \ \cos \ \mu_K(t) & \text{if } t \in A_K \\ \frac{\mu(A_K)}{2\pi}\xi - \frac{\mu(A_K)}{2\pi}\xi \ \cos \ \mu_K(t_n) & \text{if } t_n < t < t_n + \delta_n \end{cases} \tag{33}$$

Hence:

$$(\Pi v_K)(t) = \int_0^t v_K(s) \, ds - \frac{1}{T}\int_0^T \left(\int_0^t v_K(s) \, ds\right) \, dt$$

with

$$\int_0^T \left(\int_0^t v_K(s)\,ds \right) dt = \frac{\mu(A_K)}{2\pi} \xi T - \frac{\mu(A_K)}{2\pi} \xi \sum_{n=0}^{\infty} \delta_n \cos \mu_K(t_n) \tag{35}$$

We have:

$$\int_0^T (\Pi v_K)^2\,dt = \int_0^T \left(\int_0^t v_K(s)\,ds \right)^2 dt - \frac{1}{T} \left(\int_0^T \left(\int_0^t v_K(s)\,ds \right) dt \right)^2 \tag{36}$$

Substituting (33) and (35) into the right-hand side yields (after simplifying):

$$\int_0^T (\Pi v_K)^2\,dt =$$

$$= \frac{\mu(A_K)^2}{4\pi^2} \|\xi\|^2 \left[\frac{T}{2} + \sum_{n=0}^{\infty} \delta_n \cos^2 u_K(t_n) - \frac{1}{T} \left(\sum_{n=0}^{\infty} \delta_n \cos u_K(t_n) \right)^2 \right]$$

$$\geq \frac{\mu(A_K)^2}{4\pi^2} \|\xi\|^2 \left[\frac{T}{2} + \left(1 - \frac{1}{T} \sum_{n=0}^{\infty} \delta_n \right) \left(\sum_{n=0}^{\infty} \delta_n \cos^2 u_K(t_n) \right) \right] \tag{37}$$

$$= \frac{\mu(A_K)^2}{4\pi^2} \|\xi\|^2 \left[\frac{T}{2} + \frac{\mu(A_K)}{T} \left(\sum_{n=0}^{\infty} \delta_n \cos^2 u_K(t_n) \right) \right]$$

$$= \frac{\mu(A_K)^2}{4\pi^2} \|\xi\|^2 \frac{T}{2}$$

Finally, we use (29) to get a uniform estimate:

$$\int_0^T (\Pi v_K)^2\,dt \geq \left(\frac{T}{4\pi} \right)^2 \mu(A_K) \|\xi\|^2 \tag{38}$$

We now substitute this into the quadratic form $\psi''_K(u_K)$ given by formula (18). Recall also that v_K vanishes outside A_K and that estimate (30) holds on A_K:

$$(\psi''_K(u_K)v_K, v_K) \leq - \left(\frac{T}{4\pi} \right)^2 \mu(A_K) \|\xi\|^2 + \int_0^T (V''_K(q_K)^{-1} v_K, v_K)\,dt$$

$$\leq \mu(A_K) \|\xi\|^2 \left[- \left(\frac{T}{4\pi} \right)^2 + \frac{1}{2} \left(\frac{T}{4\pi} \right)^2 \right] \tag{39}$$

$$= -\frac{1}{2} \left(\frac{T}{4\pi} \right)^2 \mu(A_K) \|\xi\|^2$$

Since ξ varies in $\mathbb{R}^n$, we have found an n-dimensional space of functions v_K where the restriction of $\psi''_K(u_K)$ is negative definite. This means that the index of q_K, the solution of problem (19) associated with u_K, is at least n. But we know that this index is ≤ 1. If $n > 1$, we have a contradiction, so that assumption (25) cannot hold, and the theorem is proved. The case $n = 1$ is of course trivial, and can be handled directly. ∎

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