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The Isoperimetric Inequality for a Minimal Surface with Radially Connected Boundary

JAIGYOUNG CHOE

As an extension of the classical isoperimetric inequality for domains in Euclidean space, it is conjectured that for any k-dimensional compact minimal submanifold M of $\mathbb{R}^n$,

$$\text{Volume}(M)^{k-1} \leq k^{-k}\omega_k^{-1}\ \text{Volume}(\partial M)^k, \tag{1}$$

where ω_k is the volume of the k-dimensional unit ball.

The first partial proof of the conjecture was obtained by Carleman [C] in 1921, who used complex function theory to prove the isoperimetric inequality for simply connected minimal surfaces in $\mathbb{R}^n$. Then, using the Weierstrass representation formula for minimal surfaces in $\mathbb{R}^3$, Osserman and Schiffer [OS] showed in 1975 that the isoperimetric inequality holds for doubly connected minimal surfaces in $\mathbb{R}^3$. Their method was generalized by Feinberg [F] to doubly connected minimal surfaces in $\mathbb{R}^n$ for all n.

Recently Li, Schoen, and Yau [LSY] proved the isoperimetric inequality for minimal surfaces in $\mathbb{R}^n$ with weakly connected boundaries. The boundary $\partial\Sigma$ of a surface Σ in $\mathbb{R}^n$ is said to be *weakly connected* if there exists a rectangular coordinate system $\{x_i\}_{1\leq i\leq n}$ of $\mathbb{R}^n$ none of whose coordinate hyperplanes $\{x_i = \text{const}\}$ ever separate $\partial\Sigma$. The Poincaré inequality plays a crucial role in their paper. They also showed that for any minimal surface Σ in $\mathbb{R}^3$ with two boundary components (not necessarily doubly connected), $\partial\Sigma$ is weakly connected and hence Σ satisfies the isoperimetric inequality.

In this paper we give another sufficient condition for minimal surfaces in $\mathbb{R}^n$ to satisfy the isoperimetric inequality. Let $C \subset \mathbb{R}^n$ be a union of curves and p a point in $\mathbb{R}^n$. We say that C is *radially connected* from p if $I = \{r : r = \text{dist}(p, q), q \in C\}$ is a connected interval, that is, no spheres centered at p separate C. Obviously, every connected set in $\mathbb{R}^n$ is radially connected from any $p \in \mathbb{R}^n$. We prove that the isoperimetric inequality holds for every minimal surface Σ in $\mathbb{R}^n$ whose boundary $\partial\Sigma$ is radially connected from a point in Σ.

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Note that, like weak connectedness of [LSY], radial connectedness of $\partial\Sigma$ places no restrictions on the topology of the surface Σ. It immediately follows that every minimal surface Σ in $\mathbb{R}^n (n \geq 3)$ with two boundary components satisfies the isoperimetric inequality since $\partial\Sigma$ is radially connected from a point of Σ which is the same distance away from each component of $\partial\Sigma$.

Most papers about the isoperimetric inequality for minimal surfaces have used the inequality

$$2 \text{ Area}(\Sigma) = \int_{\partial\Sigma} r \frac{\partial r}{\partial \nu} \leq \int_{\partial\Sigma} r,$$

where r is the Euclidean distance function from a fixed point $p \in \mathbb{R}^n$ and ν is the outward unit normal vector to $\partial\Sigma$ on Σ. In this paper, however, we use a sharper inequality

$$(2) \qquad 2 \text{ Area}(\Sigma) = \int_{\partial\Sigma} r \frac{\partial r}{\partial \nu} \leq 2 \text{ Area}(p \mathbin{⋇} \partial\Sigma),$$

where $p \mathbin{⋇} \partial\Sigma$ is the cone from p over $\partial\Sigma$. Therefore the isoperimetric inequality for the minimal surface Σ will follow if we prove the isoperimetric inequality for the cone $p \mathbin{⋇} \partial\Sigma$. Since every cone can be developed onto a plane domain, preserving its area and the length of its boundary, the original isoperimetric inequality on the minimal surface reduces to the classical isoperimetric inequality on the plane, which is known to be true.

In this process of developing the cone $p \mathbin{⋇} \partial\Sigma$ onto a plane domain $D \ni p$, we should verify that the plane curve $C \subset \partial D$ developed from $\partial\Sigma$ is connected and that C winds around p by at least 360°. Connectedness of C follows from radial connectedness of $\partial\Sigma$ by way of cutting and inserting arguments. To verify the second fact, we have to show that if $p \in \Sigma$, then

$$(3) \qquad \text{Length}((p \mathbin{⋇} \partial\Sigma)^\infty \cap \partial B^n(p, 1)) \geq 2\pi,$$

where $(p \mathbin{⋇} \partial\Sigma)^\infty$ is the infinite cone obtained by indefinitely extending $p \mathbin{⋇} \partial\Sigma$ across $\partial\Sigma$, and $B^n(p, 1)$ is the n-dimensional unit ball with center at p. We prove (3) by showing that $\log r$ is a subharmonic function on the minimal surface Σ. (3) was obtained also by Gromov [G].

Although we show that higher dimensional minimal submanifolds M^k, $k \geq 3$, have the properties corresponding to (2) and (3), our argument does not carry over to the proof of higher dimensional isoperimetric inequality (1) with $k \geq 3$. This is partly because in general one can develop the cone $p \mathbin{⋇} \partial M^k$ onto a domain in $\mathbb{R}^k$ only if $k = 2$. Here we should mention that Almgren [A] recently obtained the isoperimetric inequality for absolutely area minimizing integral currents.

We give some remarks and open problems in the last section.

Finally, we would like to thank Richard Schoen for his interest in this work.

1. - Isoperimetric inequality for cones

DEFINITION. Let $S \subset \mathbb{R}^n$ be a k-dimensional rectifiable set and p a point in $\mathbb{R}^n$. We define the *k-dimensional angle* of S from $p, A^k(S,p)$ to be the k-dimensional mass of $(p ⋇ S)^\infty \cap \partial B^n(p,1)$ counting multiplicity.

Note that

$$A^k(S,p) = (k+1)\omega_{k+1}\Theta^{k+1}(p ⋇ S, p),$$

where $\Theta^{k+1}(p ⋇ S,p)$ denotes the $(k+1)$-dimensional density of $p ⋇ S$ at p. Using this, we can also define the angle of a set in a Riemannian manifold. Note also that Gromov's definition of the *visual volume* of S from p $\mathrm{Vis}(S;p)$ is equivalent to the k-dimensional angle of S from p in that

$$\mathrm{Vis}(S;p) = \frac{A^k(S,p)}{(k+1)\omega_{k+1}}.$$

THEOREM 1. *Let C be a union of closed curves $C_i, 0 \le i \le m$, in $\mathbb{R}^n$ and $p \in \mathbb{R}^n$ a point not in C. If C is radially connected from p and $A^1(C,p) \ge 2\pi$, then*

$$\mathrm{Area}(p ⋇ C) \le \frac{1}{4\pi}\ \mathrm{Length}(C)^2.$$

PROOF. For any nonclosed space curve $\Lambda \subset \mathbb{R}^n$ and any $p \in \mathbb{R}^n$, $p ⋇ \Lambda$ is obviously flat. Therefore we can develop Λ onto a plane curve Λ' in a 2-plane Π containing p, preserving the distance from p and the 1-dimensional angle from p. Clearly this development also preserves the arclengths of Λ and Λ', and the areas of $p ⋇ \Lambda$ and $p ⋇ \Lambda'$. Moreover, we can develop Λ in such a way that under a suitable parametrization of Λ', Λ' winds around p counterclockwise (nowhere moving backward or clockwise) at all points of Λ' except for a (possibly empty) subset at which Λ' is a subset of rays emanating from p.

For each closed curve C_i, choose a point $q_i \in C_i$ such that $\mathrm{dist}(p,q_i) = \mathrm{dist}(p,C_i)$, and develop the nonclosed curve $C_i \sim \{q_i\}$ onto a curve C_i' in a 2-plane $\Pi \ni p$ as above. By hypothesis, $\mathrm{dist}(p,C_i') > 0$ for all i. We construct a connected plane curve $\overline{C}$ by cutting C_i' and inserting C_j' into C_i' as follows. Let q_i^1 and q_i^2 be the two end points of C_i'. Since C_i is closed, we have

$$\mathrm{dist}(p,q_i^1) = \mathrm{dist}(p,q_i^2) = \mathrm{dist}(p,C_i'). \tag{4}$$

Assume without loss of generality that

$$\mathrm{dist}(p,q_0^1) \le \mathrm{dist}(p,q_1^1) \le \ldots \le \mathrm{dist}(p,q_m^1).$$

It follows from radial connectedness of C that for each $1 \le i \le m$, there exists a point q_i^3 in a curve $C_{k(i)}', k(i) \in \{0,\ldots,i-1\}$, such that

$$\mathrm{dist}(p,q_i^3) = \mathrm{dist}(p,q_i^1). \tag{5}$$

First, let $C_i'(\theta) : [a_i, b_i] \to \Pi$ be a parametrization of $C_i', i = 0, 1$, by the angle θ taken with respect to a fixed ray from p in Π counterclockwise. For some values of θ $C_i'(\theta)$ may be a line segment. Then there exists $x_1 \in [a_0, b_0]$ such that $C_0'(x_1) = q_1^3$. Let ρ_θ be the map of counterclockwise rotation on Π around the origin p by the angle of θ. We then construct a curve $\overline{C}_1(\theta)$ in Π, $a_0 \leq \theta \leq b_0 + b_1 - a_1$, by

$$\overline{C}_1(\theta) = \begin{cases} C_0'(\theta) & \text{for } \theta \in [a_0, x_1], \\ \rho_{x_1 - a_1}(C_1'(\theta - x_1 + a_1)) & \text{for } \theta \in [x_1, x_1 + b_1 - a_1], \\ \rho_{b_1 - a_1}(C_0'(\theta - b_1 + a_1)) & \text{for } \theta \in [x_1 + b_1 - a_1, b_0 + b_1 - a_1]. \end{cases}$$

By (4) and (5) we see that $\overline{C}_1$ is a connected curve and that

$$\text{dist}(p, \overline{C}_1(a_0)) = \text{dist}(p, \overline{C}_1(b_0 + b_1 - a_1)). \tag{6}$$

Even if $C_i'(\theta)$ is a line segment for some θ, the above construction is well defined.

Secondly, let $C_2'(\theta)$ be a parametrization of C_2' by the angle θ with $a_2 \leq \theta \leq b_2$. Again by radial connectedness of C there exists $x_2 \in [a_0, b_0 + b_1 - a_1]$ such that

$$\text{dist}(p, \overline{C}_1(x_2)) = \text{dist}(p, q_2'). \tag{7}$$

Now we can construct a curve $\overline{C}_2(\theta) : [a_0, b_0 + \sum_{i=1}^{2}(b_i - a_i)] \to \Pi$ defined by

$$\overline{C}_2(\theta) = \begin{cases} \overline{C}_1(\theta) & \text{for } \theta \in [a_0, x_2], \\ \rho_{x_2 - a_2}(C_2'(\theta - x_2 + a_2)) & \text{for } \theta \in [x_2, x_2 + b_2 - a_2], \\ \rho_{b_2 - a_2}(\overline{C}_1(\theta - b_2 + a_2)) & \text{for } \theta \in [x_2 + b_2 - a_2, b_0 + \sum_{i=1}^{2}(b_i - a_i)]. \end{cases}$$

Then (4), (6), and (7) imply that $\overline{C}_2$ is a connected curve and that

$$\text{dist}(p, \overline{C}_2(a_0)) = \text{dist}(p, \overline{C}_2(b_0 + \sum_{i=1}^{2}(b_i - a_i))).$$

Continuing this process until we connect all the subarcs of $C_0', \ldots, C_m'$, we can obtain a connected curve $\overline{C}_m(\theta) : [a_0, b_0 + \sum_{i=1}^{m}(b_i - a_i)] \to \Pi$ such that

$$\text{dist}(p, \overline{C}_m) > 0,$$

$$\text{dist}(p, \overline{C}_m(a_0)) = \text{dist}(p, \overline{C}_m(b_0 + \sum_{i=1}^{m}(b_i - a_i))),$$

$$\text{Length}(C) = \text{Length}(\overline{C}_m) \text{ and } \text{Area}(p \divideontimes C) = \text{Area}(p \divideontimes \overline{C}_m).$$

Remember that

$$A^1(\overline{C}_m, p) = \sum_{i=0}^{m}(b_i - a_i) = \sum_{i=0}^{m} A^1(C_i', p)$$
$$= \sum_{i=0}^{m} A^1(C_i, p) = A^1(C, p) \geq 2\pi.$$

Therefore the following lemma will complete the proof of Theorem 1.

LEMMA 1. *Let* $\Gamma(\theta)$, $a \leq \theta \leq b$, *be a connected plane curve parametrized by the angle from the origin. If* $\operatorname{dist}(0, \Gamma) > 0, b - a \geq 2\pi$, *and*

$$\operatorname{dist}(0, \Gamma(a)) = \operatorname{dist}(0, \Gamma(b)), \tag{8}$$

then

$$\operatorname{Area}(0 \divideontimes \Gamma) \leq \frac{1}{4\pi} \operatorname{Length}(\Gamma)^2.$$

Here $\operatorname{Area}(0 \divideontimes \Gamma)$ *counts multiplicity, and for some values of* $\theta, \Gamma(\theta)$ *may be a line segment which is a subset of a ray emanating from the origin. Finally, equality holds if and only if* Γ *is a circle and* $b - a = 2\pi$.

PROOF. If $b - a = 2\pi$, then Γ is the boundary of a domain and the lemma follows from the classical isoperimetric inequality. Thus let us assume $b-a > 2\pi$. Define a subset Γ_α of Γ by

$$\Gamma_\alpha = \{q \in \Gamma : \operatorname{dist}(0, q) > \alpha\}.$$

Then the function $f(x) : \mathbb{R}^+ \to [0, b - a]$ defined by

$$f(x) = A^1(\Gamma_x, 0)$$

is monotonically nonincreasing and lower semicontinuous. Since $\lim_{x \to +0} f(x) > 2\pi$ and $\lim_{x \to \infty} f(x) = 0$, there exists $\beta > 0$ such that

$$\lim_{x \to \beta - 0} f(x) \geq 2\pi \geq f(\beta). \tag{9}$$

In case $f(x)$ is continuous at $x = \beta$, let $\Gamma^1 = \Gamma_\beta$ and $\Gamma^2 = \Gamma \sim \Gamma_\beta$. In case $f(x)$ is not continuous at $x = \beta$, let $\Gamma_{\beta,0} = \{q \in \Gamma : \operatorname{dist}(0, q) = \beta\}$. It follows from (9) that there exists a subset $\Gamma_{\beta,1}$ of $\Gamma_{\beta,0}$ such that

$$A^1(\Gamma_\beta \cup \Gamma_{\beta,1}, 0) = 2\pi. \tag{10}$$

Let $\Gamma^1 = \Gamma_\beta \cup \Gamma_{\beta,1}$ and $\Gamma^2 = \Gamma \sim \Gamma^1$. In either case, then, by (10) (and (8) if $\Gamma(a), \Gamma(b) \in \Gamma_\beta$) we can see that after suitable rotations around 0 components of Γ^1 fit together to form the boundary of a domain $D \ni 0$ with the property that

$$\operatorname{Length}(\partial D) = \operatorname{Length}(\Gamma^1) \text{ and } \operatorname{Area}(D) = \operatorname{Area}(0 \divideontimes \Gamma^1).$$

Hence, by the isoperimetric inequality for D, we have

$$\text{Area}(0 \divideontimes \Gamma^1) \leq \frac{1}{4\pi} \text{Length}(\Gamma^1)^2. \tag{11}$$

Moreover, since the circle $C(0, \beta)$ of radius β with center at 0 lies inside $\overline{D}$, we have

$$\text{Length}(\Gamma^1) \geq 2\pi\beta. \tag{12}$$

Therefore

$$\begin{aligned}
\frac{1}{4\pi} \text{Length}(\Gamma)^2 &= \frac{1}{4\pi}[\text{Length}(\Gamma^1) + \text{Length}(\Gamma^2)]^2 \\
&\geq \frac{1}{4\pi} \text{Length}(\Gamma^1)^2 + \frac{1}{2\pi} \text{Length}(\Gamma^1) \cdot \text{Length}(\Gamma^2) \\
&\geq \text{Area}(0 \divideontimes \Gamma^1) + \frac{1}{4\pi} \text{Length}(\Gamma^1) \cdot \text{Length}(\Gamma^2) \text{ (by (11))} \\
&\geq \text{Area}(0 \divideontimes \Gamma^1) + \frac{\beta}{2} \text{Length}(\Gamma^2) \text{ (by (12))} \\
&\geq \text{Area}(0 \divideontimes \Gamma^1) + \text{Area}(0 \divideontimes \Gamma^2) \\
&= \text{Area}(0 \divideontimes \Gamma),
\end{aligned}$$

where the last inequality follows from the fact that Γ^2 lies inside the circle $C(0, \beta)$. Here equality holds if and only if $\Gamma^2 = \emptyset$ and D is a disk, that is, Γ is a circle and $b - a = 2\pi$. This completes the proof.

We can easily see that if $0 \in \Gamma$, we do not need to assume $b - a \geq 2\pi$ in Lemma 1. Thus we get the following corollary of Theorem 1.

COROLLARY 1. *If C is a union of closed curves in $\mathbb{R}^n$ which is radially connected from a point p in C, then*

$$\text{Area}(p \divideontimes C) \leq \frac{1}{4\pi} \text{Length}(C)^2.$$

REMARK. In the isoperimetric inequality of Theorem 1 and Corollary 1, equality holds if and only if $p \divideontimes C$ can be developed, after cutting and inserting, $1-1$ onto a disk.

2. - Volume and angle via divergence theorem

PROPOSITION 1. *If $M^k \subset \mathbb{R}^n$ is a compact k-dimensional minimal submanifold with boundary and p is a point in $\mathbb{R}^n$, then*

$$\text{Volume}(M) \leq \text{Volume}(p \divideontimes \partial M).$$

PROOF. Let $r(x) = \text{dist}(p, x)$, $x \in M$. From the harmonicity of linear coordinate functions of $\mathbb{R}^n$ on M^k, we have

$$\Delta r^2 = 2k. \tag{13}$$

Integrating this over M, we get

$$k \cdot \text{Volume}(M) = \int_{\partial M} r \frac{\partial r}{\partial \nu}, \tag{14}$$

where ν is the outward unit conormal vector to ∂M. For $q \in \partial M$, let $\vec{r}(q)$ be the position vector of q from p. Then $\vec{r}(q)$ and the tangent space of ∂M at q determine the k-dimensional plane Π which contains $\vec{r}(q)$ and $T_q(\partial M)$. Let $\eta(q)$ be the unit vector in Π which is perpendicular to ∂M and satisfying $\eta \cdot \vec{r} \geq 0$. Let θ and φ be the angles between $\vec{r}$ and η, and ν respectively. Then we have

$$\theta \leq \varphi.$$

Hence

$$\frac{\partial r}{\partial \nu} = \cos\ \varphi \leq \cos\ \theta = \frac{\partial r}{\partial \eta}. \tag{15}$$

It follows from (14) and (15) that

$$\text{Volume}(M) \leq \frac{1}{k} \int_{\partial M} \vec{r} \cdot \eta = \text{Volume}(p \divideontimes \partial M).$$

LEMMA 2. (i) log r *is subharmonic on any minimal surface $M^2 \subset \mathbb{R}^n$.*
(ii) *For $k \geq 3$, r^{2-k} is superharmonic on any k-dimensional minimal submanifold $M^k \subset \mathbb{R}^n$.*

PROOF.

(i)
$$\begin{aligned}\Delta \log r &= \frac{1}{2}\Delta \log r^2 = \frac{1}{2}\,\text{div}(\frac{1}{r^2}\nabla r^2)\\ &= -\frac{\nabla r}{r^3} \cdot 2r\nabla r + \frac{1}{2r^2}\Delta r^2\end{aligned}$$

$$= -\frac{2}{r^2}|\nabla r|^2 + \frac{2}{r^2} \text{ (by (13))}$$
$$= \frac{2}{r^2}(1 - |\nabla r|^2) \geq 0$$

since $|\nabla r| \leq 1$ on M.

(ii)
$$\Delta r^{2-k} = \operatorname{div}((2-k)r^{1-k}\nabla r) = \frac{1}{2}(2-k)\operatorname{div}(r^{-k}\nabla r^2)$$
$$= \frac{1}{2}k(k-2)r^{-k-1}\nabla r \cdot 2r\nabla r + \frac{1}{2}(2-k)r^{-k}\Delta r^2$$
$$= k(k-2)r^{-k}|\nabla r|^2 + k(2-k)r^{-k}$$
$$= k(k-2)r^{-k}(|\nabla r|^2 - 1) \leq 0.$$

PROPOSITION 2. *Let $M^k \subset \mathbb{R}^n$ be a compact k-dimensional minimal submanifold with boundary and p an interior point of M. Then*

$$A^{k-1}(\partial M, p) \geq k\omega_k. \tag{16}$$

Moreover, equality holds if and only if M^k is a star-shaped domain in a k-plane with respect to p.

PROOF. Let $M_\varepsilon = M \sim B^n(p,\varepsilon)$ and $S_\varepsilon = M \cap \partial B^n(p,\varepsilon)$ for $\varepsilon < \operatorname{dist}(p, \partial M)$. Suppose $k = 2$. By Lemma 2(i),

$$0 \leq \frac{1}{2}\int_{M_\varepsilon} \Delta \log r^2 = \int_{\partial M_\varepsilon} \frac{1}{r}\frac{\partial r}{\partial \nu} = -\int_{S_\varepsilon} \frac{1}{r}\frac{\partial r}{\partial \nu} + \int_{\partial M} \frac{1}{r}\frac{\partial r}{\partial \nu}.$$

Since $\frac{\partial r}{\partial \nu} \to 1$ on S_ε as $\varepsilon \to 0$, we have

$$2\pi \leq 2\pi\Theta^2(M,p) = \lim_{\varepsilon\to 0}\int_{S_\varepsilon} \frac{1}{r}\frac{\partial r}{\partial \nu} \leq \int_{\partial M} \frac{1}{r}\frac{\partial r}{\partial \nu}.$$

It follows from (15) that

$$2\pi \leq \int_{\partial M} \frac{1}{r}\frac{\partial r}{\partial \eta} = \operatorname{Length}((p \mathbin{⋇} \partial M)^\infty \cap \partial B^n(p,1)) = A^1(\partial M, p).$$

Obviously, equality holds if and only if $\Theta^2(M,p) = 1$, $|\nabla r| = 1$ on M, and $\theta = \varphi$ on ∂M, that is, M is a star-shaped domain in a plane.

Now suppose $k \geq 3$. By Lemma 2(ii),

$$0 \geq \int_{M_\varepsilon} \Delta r^{2-k} = -(2-k)\int_{S_\varepsilon} r^{1-k}\frac{\partial r}{\partial \nu} + (2-k)\int_{\partial M} r^{1-k}\frac{\partial r}{\partial \nu}.$$

Since

$$\lim_{\varepsilon\to 0}\int\limits_{S_\varepsilon} r^{1-k}\frac{\partial r}{\partial \nu} = k\omega_k\Theta^k(M,p) \geq k\omega_k,$$

we obtain

$$\begin{aligned} k\omega_k &\leq \int\limits_{\partial M} r^{1-k}\frac{\partial r}{\partial \nu} \leq \int\limits_{\partial M} r^{1-k}\frac{\partial r}{\partial \eta} \\ &= \text{Volume}((p \divideontimes \partial M)^\infty \cap \partial B^n(p,1)) \\ &= A^{k-1}(\partial M, p), \end{aligned}$$

completing the proof.

THEOREM 2. *If Σ is a minimal surface in $\mathbb{R}^n$ whose boundary is radially connected from a point p on Σ, then Σ satisfies the isoperimetric inequality. And equality holds if and only if Σ is a planar disk.*

PROOF. Proposition 1, Theorem 1 (or Corollary 1 if $p \in \partial\Sigma$), Proposition 2, and Remark.

COROLLARY 2. *Every minimal surface Σ in $\mathbb{R}^n$ whose boundary consists of two components satisfies the isoperimetric inequality.*

PROOF. Choose any two points p and q, one from each component of $\partial\Sigma$. Then there exists a hyperplane $\Pi \subset \mathbb{R}^n$ which bisects and is perpendicular to the line segment $\overline{pq}$. By connectedness of Σ, $\Sigma \cap \Pi$ is nonempty. Clearly $\partial\Sigma$ is radially connected from any point in $\Sigma \cap \Pi$. Hence the corollary follows from Theorem 2.

3. - Remarks and open problems

(i) Dual to radial connectedness is spherical connectedness: A set $X \subset \mathbb{R}^n$ is said to be *spherically connected* from p if the image of X under the central projection of $\mathbb{R}^n \sim \{p\}$, from p, onto S^{n-1} is connected. In general, given a family $\{\pi_i\}_{i\in I}$ of maps π_i from $\mathbb{R}^n$ into a set U, we say that a set $X \subset \mathbb{R}^n$ is *$\{\pi_i\}_{i\in I}$-connected* if $\pi_i(X)$ is connected for every $i \in I$. Therefore X is weakly connected according to [LSY] if X is $\{\pi_{x_1},\ldots,\pi_{x_n}\}$-connected for some rectangular coordinate system $\{x_1,\ldots,x_n\}$ in $\mathbb{R}^n$, where $\pi_{x_i}((x_1,\ldots,x_n)) = x_i$.

(ii) Given a k-dimensional submanifold N^k of $\mathbb{R}^n$, how can one find a point $p \in \mathbb{R}^n$ such that Volume$(p \divideontimes N)$ = $\min_{q\in\mathbb{R}^n}$ Volume$(q \divideontimes N)$? Let $MC(N) = \{p \in \mathbb{R}^n :$ Volume$(p \divideontimes N) = \min_{q\in\mathbb{R}^n}$ Volume$(q \divideontimes N)\}$ and let us call $MC(N)$ the *minimizing center* of N. Is $MC(N)$ a subset of $H(N)$, the convex hull of N? Is it true that $MC(N) = H(N)$ if and only if N is a convex hypersurface of a $(k+1)$-plane in $\mathbb{R}^n$?

(iii) Let $C \subset \mathbb{R}^n$ be a closed curve. Then it is not difficult to prove that

$$A^1(C,p) \geq 2\pi$$

for any $p \in H(c)$ since every closed curve of length $< 2\pi$ on the unit sphere in $\mathbb{R}^n$ is contained in an open hemisphere (see [H]). In view of this fact and Proposition 2, it is tempting to conjecture that if $\Sigma \subset \mathbb{R}^n$ is a compact minimal surface with boundary then

$$A^1(\partial\Sigma,p) \geq 2\pi$$

for any $p \in H(\partial\Sigma)$. If this conjecture is true, we will be able to prove the isoperimetric inequality for some minimal surfaces with three or four boundary components. Indeed if we can choose points $p_1, \dots, p_j$, $j = 3$ or 4, one from each component of $\partial\Sigma$, in such a way that the point p which is the same distance away from $p_1, \dots, p_j$ lies inside $H(\partial\Sigma)$, then the isoperimetric inequality follows from Theorem 1 since $\partial\Sigma$ is radially connected from p. In particular, if there exist three points in $\partial\Sigma$, one from each component of $\partial\Sigma$, such that they make an acute triangle, then the orthocenter of the triangle lies inside $H(\partial\Sigma)$ and $\partial\Sigma$ is radially connected from the orthocenter.

(iv) How can one find a point p such that $A^k(N^k,p) = \max_{q\in\mathbb{R}^n} A^k(N,q)$?

(v) Proposition 1 and Proposition 2 are valid even for a stationary set S which is a connected union of k-dimensional minimal submanifolds of $\mathbb{R}^n$. Along sing(S), the singular set of S, whose Hausdorff dimension is $\dim S - 1$, several minimal submanifolds meet each other and the sum of conormal vectors on S to sing(S) vanishes because of stationarity of S. Therefore, when applying divergence theorem, we can regard sing(S) as a subset of the interior of S. In fact, if $p \in \text{sing}(S)$, (16) can be replaced by

$$A^{k-1}(\partial S,p) \geq k\omega_k\Theta^k(S,p).$$

Area minimizing 2-currents mod 3 and compound soap films are examples of stationary sets.

(vi) Prove Proposition 1 and Proposition 2 for minimal submanifolds in $\mathbb{S}^n$ or $\mathbb{H}^n$.

(vii) There are minimal surfaces constructed by F. Morgan which are neither weakly connected nor radially connected. Let Σ_1 and Σ_2 be area minimizing annular minimal surfaces in $\mathbb{R}^3$ which are spaced sufficiently far from each other in such a way that $\partial\Sigma_1$ and $\partial\Sigma_2$ lie in four parallel planes. Bridge Σ_1 and Σ_2 with a thin strip Σ_3 whose boundary lies in a plane perpendicular to the four planes. Then this bridged minimal surface Σ is not radially connected from any point of itself. However, Σ is weakly connected. Now, along $\partial\Sigma_3$ attach sufficiently many small minimal handles which are tilted in almost every direction of the space. Then one easily sees that the resulting multiply connected minimal surface is neither radially connected nor weakly connected. As this surface is area minimizing, it satisfies the isoperimetric inequality.

(viii) Both the method of developing a cone into a planar domain and the method of cutting and inserting are purely two-dimensional. For this reason, we are not able to extend our arguments to the case of higher dimensional minimal submanifolds. Moreover, by N. Smale's bridge principle [S], there exists a 3-dimensional minimal submanifold $M \subset \mathbb{R}^4$ obtained by joining, by a long thin bridge, boundaries of two compact minimal submanifolds with no Jacobi fields which are arbitrarily far apart from each other. Then Volume$(p \divideontimes \partial M)$ can be arbitrarily larger than Volume(M) and hence it is impossible to prove the isoperimetric inequality for the cone $p \divideontimes \partial M$ as in Theorem 1.

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