ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

JINDŘICH NEČAS
VLADIMÍR ŠVERÁK

On regularity of solutions of nonlinear parabolic systems

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 18, nº 1 (1991), p. 1-11

<http://www.numdam.org/item?id=ASNSP_1991_4_18_1_1_0>

On Regularity of Solutions of Nonlinear Parabolic Systems

JINDŘICH NEČAS - VLADIMÍR ŠVERÁK

1. - Introduction

It is well-known that the full regularity of the elliptic systems

$$D_\alpha A_i^\alpha(Du) = 0$$

in two dimensions can (under standard assumptions) be proved by using $W^{1,2+\delta}$-estimates for linear elliptic systems with L^∞ coefficients. (See, for example, M. Giaquinta [2]). The purpose of this paper is to show that a similar method can be used when dealing with nonlinear parabolic systems

$$\frac{\partial u_i}{\partial t} = D_\alpha A_i^\alpha(Du).$$

The idea is to show that $\frac{\partial u}{\partial t}$ is bounded in $L^\infty(-T,0;L^{2+\delta}(\Omega))$ and then apply the theory of elliptic systems. The required estimate is obtained by using estimates for solutions of linear parabolic systems with L^∞-coefficients. (See Lemma 1). In the two-dimensional case we get full regularity.

2. - Preliminaries

Let $n \geq 2,\ N \geq 1$. We shall be dealing with open sets $Q = \Omega \times (-T,0) \subset \mathbb{R}^{n+1}$, where Ω is a bounded domain in $\mathbb{R}^n$ and $T > 0$. A typical point of $\mathbb{R}^{n+1}$ is denoted by $z = (x,t), x \in \mathbb{R}^n, t \in \mathbb{R}$.

For $\delta > 0$ we let

$$\Omega_\delta = \{x \in \Omega,\ \text{dist }(x,\partial\Omega) > \delta\}$$

Pervenuto alla Redazione il 16 Febbraio 1990.

and

$$Q_\delta = \Omega_\delta \times (-T + \delta, 0).$$

For $x \in \mathbb{R}^n$ and $\rho > 0$ we define

$$B_{x,\rho} = \{y \in \mathbb{R}^n, |x - y| < \rho\}.$$

If $a, b \in \mathbb{R}$, we denote by $a \wedge b$ the minimum of the two numbers.

The Sobolev spaces W_p^k, $\overset{\circ}{W}{}_p^k$ are defined in the standard way.

The space $L^2(-T, 0; W_2^1(\Omega))$ is denoted by $W_2^{1,0}(Q)$. The norm $[\cdot]_{2,Q}$ on $W_2^{1,0}(Q)$ is defined by

$$[u]_{2,Q} = \{\int_Q (|u|^2 + \sum_{i=1}^{n} |D_i u|^2)\}^{\frac{1}{2}}.$$

The spaces $L^\infty(-T, 0; L^p(\Omega))$, $p \geq 1$ will be denoted by $L^{p,\infty}(Q)$ and the corresponding norm is denoted by $\|\cdot\|_{p,\infty,Q}$.

The usual L^p-norm is denoted by $\|\cdot\|_{p,Q}$.

Let us consider the nonlinear parabolic system

$$\frac{\partial u_i}{\partial t} - D_\alpha A_i^\alpha(Du) = 0. \qquad (i = 1, 2, \ldots, N) \tag{1}$$

where $u = (u_1, \ldots, u_N), Du = (D_\alpha u_i)_{1 \leq i \leq N, 1 \leq \alpha \leq n} = (\frac{\partial u_i}{\partial x_\alpha})_{1 \leq i \leq N, 1 \leq \alpha \leq n}$ is the gradient matrix of u and the summation over repeated indexes is understood.

We shall suppose that the functions A_i^α have continuous derivatives satisfying

$$\left\{ \sum_{i,j} \sum_{\alpha,\beta} |\frac{\partial A_i^\alpha}{\partial \xi_\beta^j}(\xi)|^2 \right\}^{\frac{1}{2}} \leq M \tag{2}$$

and

$$\frac{\partial A_i^\alpha}{\partial \xi_\beta^j}(\xi) \pi_\alpha^i \pi_\beta^j \geq \nu |\pi|^2, \qquad \nu > 0,$$

for every $\xi, \pi \in \mathbb{R}^{nN}$.

(Of course, for higher regularity results we have to assume higher smoothness of A_i^α).

By a weak solution of (1) we mean a function $u \in W_2^{1,0}(Q)$ satisfying

$$\int_Q \left(u_i \frac{\partial \varphi_i}{\partial t} - A_i^\alpha(Du) D_\alpha \varphi_i \right) dz = 0$$

for every $\varphi \in \overset{\circ}{W}{}_2^1(Q)$.

We shall also be dealing with linear strongly parabolic systems

$$\frac{\partial u_i}{\partial t} - D_\alpha a_{ij}^{\alpha\beta} D_\beta u_j = 0 \qquad (i = 1, \dots, N) \tag{4}$$

where $a_{ij}^{\alpha\beta} = a_{ij}^{\alpha\beta}(z)$ are L^∞-functions in Q satisfying for almost every $z \in \Omega$ the conditions

$$\left\{ \sum_{i,j} \sum_{\alpha,\beta} |a_{ij}^{\alpha\beta}|^2 \right\}^{\frac{1}{2}} \leq M \tag{2'}$$

and

$$a_{ij}^{\alpha\beta} \xi_\beta^j \xi_\alpha^i \geq \nu |\xi|^2 \tag{3'}$$

for every $\xi \in \mathbb{R}^{nN}$. By a weak solution of (4) we mean a function $u \in W_2^{1,0}(Q)$ satisfying

$$\int_Q \left(u_i \frac{\partial \varphi_i}{\partial t} - a_{ij}^{\alpha\beta} D_\beta u_j D_\alpha \varphi_i \right) dz = 0 \tag{$*$}$$

for every $\varphi \in \overset{\circ}{W}{}_2^1(Q)$.

We shall use the following well-known results.

(i) If u is a weak solution of (1) or (4), then u is continuous in time with respect to the L^2-norm. More precisely, if $\Omega' \subset\subset \Omega$, then the map $t \to u(\cdot, t)$ from $(-T, 0)$ into $L^2(\Omega')$ is continuous. (See, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 3, Lemma 4.3).

(ii) We have the imbedding

$$L^{2,\infty}(Q) \cap W_2^{1,0}(Q) \hookrightarrow L^{q_0}(Q)$$

where

$$q_0 = \begin{cases} \dfrac{2(n+2)}{n}, & \text{if } n > 2 \\ \text{is any number } \in [1,4) & \text{if } n = 2. \end{cases}$$

(See, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 2).

We denote by c_i various constants. The value of these constants can depend on ν, M, Ω, T, n and N. The dependence on additional parameters will be indicated.

3. - L^p-estimates

The first statement of the following Lemma is well known (see, for example, O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4], Chap. 3).

The second statement will be used for the $L^\infty(-T,0;L^{2+\delta}(\Omega))$-estimate mentioned in the introduction.

LEMMA 1. *Let u be a weak solution of the linear system (4). Then, for any $\delta>0$,*

(i)
$$u\in L^{2,\infty}(Q_\delta)\cap W_2^{1,0}(Q_\delta)$$
and
$$\|u\|_{2,\infty,Q_\delta}+[u]_{2,Q_\delta}\le c_1(\delta)\|u\|_{2,Q}.$$

(ii) *For every $p\in[2,(2+\frac{\nu}{NM})\wedge q_0)$ the function u belongs to $L^{p,\infty}(Q_\delta)$ and*
$$\|u\|_{p,\infty,Q_\delta}\le c_2(\delta,p)\|u\|_{2,Q}.$$

PROOF. Let $\gamma\ge 1$ and let $k>0$ be such that $meas\{z\in Q,|u(z)|=k\}=0$. Define $g_k:[0,\infty)\to\mathbb{R}$ by
$$g_k(t)=\begin{cases} t^\gamma, & \text{if } 0\le t\le k,\\ k^\gamma+\gamma k^{\gamma-1}(t-k), & \text{if } t\ge k.\end{cases}$$
Clearly $g_k'(t)=\gamma(t\wedge k)^{\gamma-1}$ and
$$g_k''(t)=\begin{cases} \gamma(\gamma-1)t^{\gamma-2}, & \text{if } 0<t<k,\\ 0, & \text{if } t>k.\end{cases}$$

Define also the function $\eta^k:\mathbb{R}^N\to\mathbb{R}$ by
$$\eta^k(u)=g_k(|u|^2)$$
We have
$$\eta^k_{u_i}(u)=\frac{\partial\eta^k}{\partial u_i}(u)=2\gamma(|u|^2\wedge k)^{\gamma-1}u_i$$
$$\eta^k_{u_iu_j}(u)=\frac{\partial^2\eta^k}{\partial u_i\partial u_j}(u)=2\gamma(|u|^2\wedge k)^{\gamma-1}(\delta_{ij}+2(\gamma-1)d_{ij}(u)).$$
In the second formula we assume $|u|^2\ne k$ and
$$d_{ij}(u)=\begin{cases} 0, & \text{if } |u|^2>k,\\ \dfrac{u_iu_j}{|u|^2}, & \text{if } |u|^2<k.\end{cases}$$

Let ω_ϵ be a family of symmetric mollifying functions satisfying

$\omega_\epsilon \in \mathcal{D}(\mathbb{R}),\ \ \omega_\epsilon(t) \geq 0,$

$\omega_\epsilon(t) = \omega_\epsilon(-t),$

support $\omega_\epsilon \subset (\epsilon, -\epsilon),$

$\int_{\mathbb{R}} \omega_\epsilon = 1.$

For $f \in L^1(Q)$ let us denote by $(f)_\epsilon$ the function defined a.e. in Q by

$$(f)_\epsilon(x,t) = \int_{\mathbb{R}} f(x, t-s)ds.$$

(We extend f by zero outside Q). Let $\psi \in \overset{\circ}{W}{}_2^1(\Omega \times (-T+\epsilon, -\epsilon))$. Following E. Giusti, M. Giaquinta [3] we set $\varphi = (\psi)_\epsilon$ in $(*)$ and we see that

$$\int_Q \frac{\partial (u_i)_\epsilon}{\partial t} \psi_i dz = -\int_Q (a_{ij}^{\alpha\beta} D_\beta u_j)_\epsilon D_\alpha \psi_i dz. \tag{5}$$

Let $\theta \in \mathcal{D}(\Omega)$ with $0 \leq \theta \leq 1$ and $\theta = 1$ on Ω_δ and let $\rho \in \mathcal{D}(-T, 0)$, $\rho \geq 0$. For ϵ sufficiently small we can use (5) with

$$\psi_i = \eta_{u_i}^k(u_\epsilon)\theta^2\rho^2$$

to get

$$\int_Q \left(\frac{\partial}{\partial t}\eta^k(u_\epsilon)\right)\theta^2\rho^2 dz = -\int_Q \left(a_{ij}^{\alpha\beta} D_\beta u_j\right)_\epsilon D_\alpha(\eta_{u_i}^k(u_\epsilon))\theta^2\rho^2 dz$$
$$- \int_Q (a_{ij}^{\alpha\beta} D_\beta u_j)_\epsilon \eta_{u_i}^k(u_\epsilon) 2\theta D_\alpha\theta\rho^2 dz.$$

Integrating by parts on the left-hand side, letting $\epsilon \to 0$ and then using the chain rule for the derivative $D_\alpha(\eta_{u_i}^k(u))$ (which is legal) we see that

$$\begin{aligned}\int_Q \eta^k(u)\theta^2(\rho^2)' dz = &-\int_Q a_{ij}^{\alpha\beta} D_\beta u_j \eta_{u_i u_l}^k(u) D_\alpha u_l \theta^2\rho^2 dz \\ &- \int_Q a_{ij}^{\alpha\beta} D_\beta u_j \eta_{u_i}^k(u) 2\theta D_\alpha\theta\rho^2 dz.\end{aligned} \tag{6}$$

Since $|d_{ij}| \leq 1$ we see that if $0 \leq 2(\gamma-1) < \frac{\nu}{NM}$ then the matrix $\tilde{a}_{lj}^{\alpha\beta} = a_{ij}^{\alpha\beta}(\delta_{il} + 2(\gamma-1)d_{il}(u))$ satisfies the condition (2') with ν replaced by $\nu_1 = \nu - 2(\gamma-1)NM$.

We can estimate the right-hand side of (6) by

$$\begin{aligned}
\nu_1 \int_Q & 2\gamma(|u|^2 \wedge k)^{\gamma-1}|Du|^2\theta^2\rho^2 dz \\
& + 2M \left\{ \int_Q |Du|^2\theta^2\rho^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} dz \right\}^{\frac{1}{2}} \\
& \times \left\{ \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1}|D\theta|^2\rho^2 dz \right\}^{\frac{1}{2}} \\
& \leq -\frac{\nu_1}{2} \int_Q 2\gamma(|u|^2 \wedge k)^{\gamma-1}|Du|^2\theta^2\rho^2 dz \\
& + \frac{4M^2}{\nu_1} \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1}|D\theta|^2\rho^2 dz.
\end{aligned}$$

Let $t_1 \in (-T+\delta, 0)$. As we have remarked in Section 2, under our assumptions the function $t \to u(\cdot, t)$ is continuous mapping of $(-T, 0)$ into $L^2(\Omega_\delta)$. Hence we can use (6) with ρ defined by

$$\rho^2(t) = \begin{cases} 0 & \text{if } t \in (-T, -T + \frac{\delta}{2}) \\ \frac{2}{\delta}(t + T - \frac{\delta}{2}) & \text{if } t \in (-T + \frac{\delta}{2}, -T + \delta) \\ 1 & \text{if } t \in (-T + \delta, t_1) \\ 0 & \text{if } t \in (t_1, 0). \end{cases}$$

We get

$$\begin{aligned}
\int_Q & \eta^k(u(x, t_1))\theta(x)dx + \frac{\nu_1}{2} \int_Q |Du|^2\theta^2\rho^2 2\gamma(|u|^2 \wedge k)^{\gamma-1} dz \\
& \leq c_3 \int_Q |u|^2 2\gamma(|u|^2 \wedge k)^{\gamma-1}|D\theta|^2\rho^2 dz + \int_{\Omega\times(-T,t_1)} (\rho^2)'\eta^k(u)\theta^2 dz.
\end{aligned}$$

Letting $\gamma = 1$ we get (i).

We can use (i) and the imbedding

$$L^{2,\infty}(Q) \cap W_2^{1,0}(Q) \hookrightarrow L^{q_0}(Q)$$

to infer that $\|u\|_{q_0, Q_\delta} \leq c_4(\delta)\|u\|_{2,Q}$. Using this and letting $k \to \infty$ in (7) we get (ii) with $p = 2\gamma$.

LEMMA 2. *Let u be a weak solution of the nonlinear system* (1). *Then $u \in W_2^1(Q_\delta)$, the derivatives $D_i u$, $i = 1, \ldots, n$ and $D_{n+1}u = \frac{\partial u}{\partial t}$ belong to the space $L^{p,\infty}(Q_\delta) \cap W_2^{1,0}(Q_\delta)$ and for each $i = 1, \ldots, n, n+1$*

$$\|D_i u\|_{p,\infty,Q_\delta} + [D_i u]_{2,Q_\delta} \leq [u]_{2,Q}.$$

PROOF. As above, we denote by Du the vector $(D_1 u, \ldots, D_n u) \in \mathbb{R}^{nN}$. let us fix an index r, $1 \leq r \leq n+1$ and let $e_r \in \mathbb{R}^n \times \mathbb{R}$ be the r-th vector of the canonical basis. Let $\delta' > 0$. For $0 < h < \delta'$ let

$$u_h(z) = h^{-1}[u(z) - u(z - he_r)].$$

Define the functions $a_{hij}^{\alpha\beta} \in L^\infty(Q_{\delta'})$ for a.e. $z \in Q_{\delta'}$ by

$$a_{hij}^{\alpha\beta}(z) = \int_0^1 A_{i,\xi_\beta^j}^{\alpha}(Du(z) - hDu_h(z) + \tau h Du_h(z))d\tau.$$

It is not difficult to see that u_h is the weak solution of the linear system

$$\frac{\partial u_{hi}}{\partial t} - D_\alpha a_{hij}^{\alpha\beta} D_\beta u_{hj} = 0$$

in $Q_{\delta'}$. The functions $a_{hij}^{\alpha\beta}$ clearly satisfy the conditions (2') and (3'). Hence, by Lemma 1

$$\begin{aligned} &\|u_h\|_{p,\infty,Q_{2\delta'}} \leq c_6(\delta', p)\|u_h\|_{2,Q_{\delta'}} \\ &\|Du_h\|_{2,Q_{2\delta'}} \leq c_7(\delta')\|u_h\|_{2,Q_{\delta'}}. \end{aligned} \tag{8}$$

Suppose first $1 \leq r \leq n$. In this case the difference is taken in the direction of the space variables. Since $u \in W_2^{1,0}(Q)$, we have

$$\|u_h\|_{2,Q_{\delta'}} \leq \|D_r u\|_{2,Q}. \tag{9}$$

Using Nirenberg's Lemma we see from (8) that $Du \in W_2^{1,0}(Q_{\delta'})$ and

$$\begin{aligned} &\|D_r u\|_{p,\infty,Q_{2\delta'}} \leq c_6(\delta', p)\|D_r u\|_{2,Q_{\delta'}} \\ &\|DD_r u\|_{2,Q_{2\delta'}} \leq c_7(\delta')\|D_r u\|_{2,Q_{\delta'}} \end{aligned} \tag{10}$$

for every $1 \leq r \leq n$. Now let $r = n+1$. Following S. Campanato [1] we notice that we can use equation (1) and the L^2-estimate of $D_\alpha D_\beta u$ obtained above to infer that $\frac{\partial u}{\partial t} \in L^2(Q_{2\delta'})$ and

$$\left\|\frac{\partial u}{\partial t}\right\|_{2,Q_{2\delta'}} \leq c_8(\delta')\|Du\|_{2,Q}. \tag{11}$$

Now we can use (8) with Q replaced by $Q_{2\delta'}$ and using (11) we get by the same argument as above

$$\|\frac{\partial u}{\partial t}\|_{p,\infty,Q_{4\delta'}} \leq c_9(\delta', p)\|Du\|_{2,Q}$$

$$\|D\frac{\partial u}{\partial t}\|_{2,Q_{4\delta'}} \leq c_{9'}(\delta')\|Du\|_{2,Q}.$$

The proof is finished.

THEOREM 1. *Let u be a weak solution of the system (1) and let p be the exponent from Lemma 1. The for each $\delta > 0$*

$$\frac{\partial u}{\partial t} \in L^{p,\infty}(Q_\delta)$$

and

$$u \in L^\infty(-T+\delta, 0; W_q^2(\Omega_\delta))$$

for some $q = q(\nu, M, p, \delta)$ with $2 < q < p$. Moreover

$$\|u\|_{L^\infty(-T+\delta,0;W_q^2(\Omega_\delta))} + \|\frac{\partial u}{\partial t}\|_{p,\infty,Q_\delta} \leq c_{10}(\delta, p, q)\|u\|_{2,Q}.$$

PROOF. Let $\delta' > 0$. We notice that u can be considered as a weak solution of the linear system (4) with

$$a_{ij}^{\alpha\beta}(z) = \int_0^1 A^\alpha_{i,\xi^j_\beta}(\tau Du(z))d\tau.$$

(See, for example S. Campanato [1]). Using this and Lemma 1 we get estimates for the norms $\|u\|_{p,\infty,Q_{\delta'}}$ and $[u]_{2,Q_{\delta'}}$. Now we can use Lemma 2 to get estimates of the norms $\|Du\|_{p,\infty,Q_{2\delta'}}$, $\|\frac{\partial u}{\partial t}\|_{p,\infty,Q_{2\delta'}}$. Lemma 2 also implies $D_\alpha D_\beta u \in L^2(Q_{2\delta'})$, $(0 \leq \alpha,\ \beta \leq n)$. We see that equation (1) is satisfied pointwise almost everywhere in $Q_{2\delta'}$ and that for almost every $t \in (-T+2\delta', 0)$ the function $u(\cdot, t)$ belongs to $W_2^2(\Omega_{2\delta'})$ and is the weak solution of the elliptic system

$$D_\alpha A_i^\alpha(Dv) = \frac{\partial u_i}{\partial t}$$

in $\Omega_{2\delta'}$. We can now use well-known L^p-estimates for elliptic systems (see Lemma 3 below). The proof is finished.

LEMMA 3. *Let $p > 2$ and let $g \in L^p(\Omega)$. Let $u \in W_2^1(\Omega)$ be a weak solution of the elliptic system*

$$D_\alpha A_i^\alpha(Du) = g_i \qquad i = 1, \ldots, n \tag{12}$$

Then there exists $q = q(\nu, M, p) > 2$ *such that* $u \in W^2_{q,loc}(\Omega)$. *Moreover, for every* $\delta > 0$

$$\|u\|_{W^2_q(\Omega_\delta)} \leq c_{11}(\nu, M, p, q, \delta)(\|u\|_{W^1_2(\Omega)} + \|g\|_{p,\Omega}).$$

PROOF. Using the standard difference quotient technique, it is not difficult to verify that the following computations are legal.

Let $1 \leq s \leq n$. We let $v = D_s u$ and take the s-th derivative of (12). We get

$$D_\alpha a^{\alpha\beta}_{ij} D_\beta v_j = D_s g_i \tag{13}$$

where $a^{\alpha\beta}_{ij}(z) = A^\alpha_{i,\xi^j_\beta}(Du(x))$.

This implies

$$\frac{\nu}{2}\int_\Omega \varsigma^2 |Dv|^2 dx \leq c_{12}(\nu, M)\int_\Omega (|v|^2|D\varsigma|^2 + |g|^2)dx \tag{14}$$

for every $\varsigma \in \mathcal{D}(\Omega)$ (Cacciopoli's inequality). The required estimate can now be obtained by using the technique of reverse Hölder inequalities. (See, for example, M. Giaquinta [2], Chap. 5, Theorem 2.2). The proof is finished.

COROLLARY. *Let the assumptions of Theorem 1 be satisfied.*

(i) *If* $n \leq 4$, *then* u *is Hölder continuous in* Q.

(ii) *If* $n \leq 2$, *then* Du *is Hölder continuous in* Q.

(iii) *If* $n \leq 2$ *and the functions* A^α_i *are smooth, then the solution* u *is smooth.*

REMARK. If $n \geq 3$, then Du may not be continuous. Examples are provided by nonregular solutions of elliptic systems. These can be found in J. Nečas [5].

PROOF OF THE COROLLARY. Let $\delta > 0$.

(i) Since $W^2_q(\Omega_{\delta/2}) \hookrightarrow C^{0,\alpha}(\Omega_\delta)$ with $\alpha = (2 - \frac{n}{q}) \wedge 1$, we have $u \in L^\infty(-T + \delta, 0; C^{0,\alpha}(\Omega_\delta))$.

Since we have also $\frac{\partial u}{\partial t} \in L^2(Q_\delta)$, u is Hölder continuous by Lemma 4 below.

(ii) In this case we have $W^2_q(\Omega_{\delta/2}) \hookrightarrow C^{1,\beta}(\Omega_\delta)$ $\beta = 1 - \frac{n}{q}$.

Hence $Du \in L^\infty(-T + \delta, 0; C^{0,\beta}(\Omega_\delta))$. Using the Hölder continuity of u it is easy to see that in fact $Du(\cdot, t) \in C^{0,\beta}(\Omega_\delta)$ for every $t \in (-T + \delta, 0)$, the $C^{0,\beta}$-norm being bounded independently of t.

Now we can use Lemma 3.1, Chap. 2 from O.A. Ladyzenskaja, V.A. Solonnikov, N.N. Ural'ceva [4] to infer that Du is Hölder continuous in Q_δ.

(iii) The higher regularity follows in the standard way from the theory of linear equations.

LEMMA 4. *Let $\alpha > 0$, $q > 1$, $\delta > 0$ and suppose*

$$u \in L^\infty(-T,0;C^{0,\alpha}(\Omega)) \text{ and } \frac{\partial u}{\partial t} \in L^q(Q).$$

Denote $K_1 = \|u\|_{L^\infty(-T,0;C^{0,\alpha}(\Omega))}$, $K_2 = \|\frac{\partial u}{\partial t}\|_{q,Q}$. *Then there exists* $K = K(K_1, K_2, \delta)$ *such that*

$$|\tilde{u}(x,t_1) - \tilde{u}(x,t_2)| \le K|t_1 - t_2|^\beta$$

for every $x \in \Omega_\delta$ *and every* t_1, $t_2 \in (-T,0)$, *where* $\beta = \dfrac{\alpha/q'}{\alpha + n/q}$, $q' = \dfrac{q}{q-1}$ *and* $\tilde{u}$ *is a suitable representative of* u.

PROOF. Suppose first that u is continuous. Let $x \in \Omega_\delta$ and let $0 < \rho < \delta$. Define

$$w_\rho(t) = \frac{1}{|B_{x,\rho}|}\int_{B_{x,\rho}} u(y,t)dy.$$

It is easy to see that w'_ρ is bounded in $L^q(-T,0)$ by $c_{13}\rho^{-\frac{n}{q}}K_2$.

Let $t_1, t_2 \in (-T,0)$. We can write

$$\begin{aligned}|u(x,t_1) - u(x,t_2)| &\\ &\le |u(x,t_1) - w_\rho(t_1) + w_\rho(t_1) - w_\rho(t_2) + w_\rho(t_2) - u(x,t_2)| \\ &\le 2K_1\rho^\alpha + c_{12}K_2\rho^{-\frac{n}{q}}|t_1 - t_2|^{\frac{1}{q'}}.\end{aligned}$$

The proof is easily finished by using this inequality with $\rho = |t_1 - t_2|^{\frac{\beta}{\alpha}}$.

REMARK. It is not difficult to see that if the boundary of Ω is sufficiently regular (say, lipshitzian), then K can be chosen independent of δ.

Acknowledgement

We thank Oldřich John, Jan Malý and Jana Stará for helpful discussions.

REFERENCES

[1] S. CAMPANATO, *On the Nonlinear Parabolic Systems in Divergence Form. Hölder Continuity and Partial Hölder Continuity of the Solutions.* Ann. Mat. Pura Appl., **137** (1984).

[2] M. GIAQUINTA, *Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems*, Princeton University Press, 1983.

[3] M. GIAQUINTA - E. GIUSTI, *Partial Regularity for the Solutions to Nonlinear Parabolic Systems*, Ann. Mat. Pura Appl., **97** (1973).

[4] O.A. LADYZENSKAJA - V.A. SOLONNIKOV - N.N. URAL'CEVA, *Linear and Quasilinear Equations of Parabolic Type*, Moscow, 1966.

[5] J. NEČAS, *Introduction to the Theory of Nonlinear Elliptic Equations*, Teubner-Texte zur Math. **52**, Leipzig, 1983.

MFF UK
Sokolovská 83
18600 PRAHA 8
Czechoslovakia