ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

DANILO PERCIVALE

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Folded Shells: a Variational Approach

DANILO PERCIVALE

Introduction

Though many rigorous theories for elastic plate and shell problems have been given in some recent papers (see [1], [2], [3]), some further questions arise when the thin shell is not smooth. This is the case of folded shells, that is, the case of shells which are very smooth except on a set of zero two dimensional Hausdorff measure.

Some results in this direction have been obtained by Ciarlet, Le Dret and others (see [4], [5], [10]) when the regular part of the shell is flat (and the exceptional set is a straight line in $\mathbb{R}^3$). In this paper we propose a variational method which permits us to treat the case of a shell in which the regular part is a smooth two-dimensional surface in $\mathbb{R}^3$ (or a smooth curve in $\mathbb{R}^2$) and the singular set is a smooth curve (respectively a point) in $\mathbb{R}^3$ (or $\mathbb{R}^2$). The method is that of starting from a 3-dimensional elastic body and passing to the limit when one of the dimensions goes to zero. The strain energy of the limit problem takes into account the contribution of the regular part of the shell (which coincides with the result obtained in [1]) and some constraints due to the fact that the shell is folded along a (prescribed) curve. The result obtained here is the physical elucidation of the convergence of the minimizing sequence of the approximating problems to the solution of a limiting minimization problem. Moreover a convergence result of the (rescaled) energies is obtained.

The connection between the present result (and others of this kind) and those arising from application of the Γ-convergence theory constitutes another problem.

1. – Notation and statement of the result

In the following we denote by Σ_1 and Σ_2 two smooth compact $(n-1)$-dimensional manifolds of $\mathbb{R}^n$ and we suppose that $\Sigma_1 \cap \Sigma_2$ is a smooth $(n-2)$-dimensional manifold.

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For the sake of simplicity we make the following assumption, which may be dropped by a localization argument: that there is a single parametrization

$$\Phi : \omega \subset \mathbb{R}^{n-1} \to \mathbb{R}^n$$

of $\Sigma = \Sigma_1 \cup \Sigma_2$, where ω is a regular open set, in the sense that

i) there are two regular open sets $\omega_1, \omega_2 \subset \omega$ such that $\overline{\omega} = \overline{\omega}_1 \cup \overline{\omega}_2$, $\omega_1 \cap \omega_2 = \emptyset$;

ii) $\Phi_i = \Phi_{|\omega_i} \in C^3(\overline{\omega}_i)$ $i = 1, 2$.

We call $\gamma = \Sigma_1 \cap \Sigma = \Phi(\overline{\omega}_1 \cap \overline{\omega}_2)$ and we denote by $\nu^i (i = 1, 2)$ the unit normal vector to Σ_i and suppose that $\langle \nu^1, \nu^2 \rangle = 0$ on γ. Another assumption we make on Σ_i is that if $T^i_\alpha = \partial \Phi_i / \partial \xi^\alpha$, then $\{T^1_i, \dots, T^{n-1}_i\}$ is an orthogonal set of tangent vectors to Σ^2_i whose unit vectors are $\tau^i_\alpha = T^i_\alpha / \|T^i_\alpha\|$. This assumption is reasonable when $n = 2$ or $n = 3$ (the physical cases) (see [7]).

Since we have supposed that $\langle \nu^1, \nu^2 \rangle = 0$ on γ, we may set $\tau^1_{\alpha_1} = \nu^2$ and $\tau^2_{\alpha_2} = \nu^1$ on γ, and for all $\varepsilon > 0$ define

$$\Sigma^\varepsilon_i = \{\sigma + t\nu^i(\sigma) : \sigma \in \Sigma_i, |t| < \varepsilon\}.$$

The mapping $(\sigma, t) \to \sigma + t\nu(\sigma)$ is invertible on Σ^ε_i if ε is small enough, therefore the mapping $\sigma(x)$ and $N(x) = \nu(\sigma(x))$ are well defined on Σ^ε_i, and, consequently, $\Sigma^\varepsilon = \Sigma^\varepsilon_1 \cup \Sigma^\varepsilon_2$ will be a n-dimensional elastic body (with $n = 2$ or $n = 3$). Assuming now that Σ^ε_i is a strained elastic body, let us define its strain energy in the following way: for every square matrix A, we denote by A^* its symmetric part. Let $f : \mathbb{R}^{n^2} \to \mathbb{R}$ satisfy the conditions

$$f \text{ is convex and 2-homogeneous,} \tag{1.1}$$

$$f(A) = f(A^*), \tag{1.2}$$

$$|A^*|^2 \leq f(A) \leq c(1 + |A^*|^2). \tag{1.3}$$

For every $\varepsilon > 0$ and $u \in H^1(\Sigma^\varepsilon_1 \cup \Sigma^\varepsilon_2)$ we set

$$F_\varepsilon(u) = \int_{\Sigma^\varepsilon_1} f(e(u))\mathrm{d}x + \int_{\Sigma^\varepsilon_2} f(e(u))\mathrm{d}x,$$

where $e(u)$ is a tensor function, usually called "the strain" associated to u.

Let us now consider $g \in C(\mathbb{R}^n)$ and set

$$\phi_\varepsilon(u) = \varepsilon^2 \int_{\Sigma^\varepsilon_1 \cup \Sigma^\varepsilon_2} gu.$$

The boundaries of Σ_1^ε and Σ_2^ε can be distinguished in the following parts: $\partial_0\Sigma_1^\varepsilon = (\partial\Sigma_i\backslash\gamma) \times (-\varepsilon,\varepsilon)$, $\partial_0\Sigma_2^\varepsilon = (\partial\Sigma_2\backslash\gamma) \times (-\varepsilon,\varepsilon)$, $\partial_0\Sigma^\varepsilon = \partial_0\Sigma_1^\varepsilon \cup \partial_0\Sigma_2^\varepsilon$ and choosing $\Gamma_i \subset (\partial\Sigma_1\backslash\gamma)$, we set $\Gamma_i^\varepsilon = \Gamma_i \times (-\varepsilon,\varepsilon)$. We now introduce the sequence of functionals, defined on $H^1(\Sigma_1^\varepsilon \cup \Sigma_2^\varepsilon)$ by

$$G_\varepsilon(u) = \begin{cases} F_\varepsilon(u) - \phi_\varepsilon(u) & \text{on } \Gamma_i^\varepsilon \cup \Gamma_2^\varepsilon, \\ +\infty & \text{otherwise.} \end{cases}$$

For every $u \in H^1(\Sigma_i^\varepsilon)$ we set

$$\mathrm{D}_{\nu^i} u = \langle \mathrm{D}u, \nu^i \rangle \nu^i,$$

$$\delta^i u = \mathrm{D}u - \langle \mathrm{D}u, \nu^i \rangle \nu^i.$$

having the geometric meaning of normal and tangential derivatives respectively and we denote by δ_h^i the h-th component of δ^i. We say that $u \in H^{m,p}(\Sigma)$ if $u \circ \Phi_i \in H^{m,p}(\omega_i)$. If $u \in H^1(\Sigma_i)$ then u is the trace of some $\tilde{u} \in H^{3/2}(\Sigma_i^1)$ and we may define $\delta^i u$ as the trace of $\delta^i \tilde{u}$. This definition does not depend on $\tilde{u}$ since if $\tilde{u} = 0$ on Σ_i then $\delta^i \tilde{u} = 0$ on Σ_i.

We state here some elementary properties of the operator δ^i (see also [1]).

PROPOSITION 1.1. *Let f, g be smooth functions defined on Σ_i, and assume f has compact support in Σ_i. If we set*

$$(\delta_h^i)^{-1} f = -\delta_h^i f + f \nu_h \delta_k^i \nu_k,$$

then

$$\int_{\Sigma_i} f \delta_h^i g \mathrm{d}\sigma = \int_{\Sigma_i} g (\delta_h^i)^{-1} f \mathrm{d}\sigma.$$

PROPOSITION 1.2. *Let u be a smooth function define in a neighbourhood of Σ_i. Then*

$$\delta_h^i \left(\int_0^\varepsilon u(\sigma + t\nu^i(\sigma)) \mathrm{d}t \right) = \int_0^\varepsilon (\delta_h^i u + t\delta u \delta_h^i \nu^i) \mathrm{d}t.$$

For every $u \in H^1(\Sigma_1^\varepsilon, \mathbb{R}^n)$ or $u \in H^1(\Sigma_i, \mathbb{R}^n)$, we define

$$e_{\tau^i}(u) = [(I - \nu^i \otimes \nu^i)\delta u]^*,$$

and it is readly verified that

$$e_{\tau^i}(u) = (\langle \delta^i(u - \langle u, \nu^i \rangle \nu^i) + \langle u, \nu^i \rangle \delta^i \nu^i, \tau^i \otimes \tau^{i,\beta} \rangle \tau^{i,\alpha} \otimes \tau^{i,\beta})^*,$$

which shows that $e_{\tau^i}(u) \in L^2(\Sigma_i)$ if

$$\langle u, \nu^i \rangle \in L^2(\Sigma_i), \qquad u - \langle u, \nu^i \rangle \nu^i \in H^1(\Sigma_i).$$

Finally we define, for every $u \in H^1(\Sigma_i^\varepsilon)$,

$$\tilde{u}^i(\sigma) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} u(\sigma + t\nu^i(\sigma))\mathrm{d}t.$$

if $(u, \nu^i) \in H^2(\Sigma)$ then $\delta_h^i u_k \nu_k^i \in H^1(\Sigma_i)$ and therefore it has a trace on γ.

We are now able to introduce the following function space

$$X(\Sigma) = \{u \in H^1(\Sigma) : u_{|\gamma} = 0 = u_{|\partial\Sigma}, \quad e_{\tau^i}(u) = 0,$$
$$u \cdot \nu^i \in H_0^2(\Sigma_i), \quad \nu^1\delta^1 u^1 + \nu^2\delta^2 u^2 = 0 \text{ on } \gamma\}$$

and the functional defined by

$$G(u) = \begin{cases} \displaystyle\int_{\Sigma_1} f_0(\nu^1\delta^1\delta^1 u^1) + \int_{\Sigma_2} f_0(\nu^2\delta^2\delta^2 u^2) - \int_{\Sigma_1\cup\Sigma_2} g \cdot u & \text{if } u \in X(\Sigma), \\ +\infty & \text{otherwise.} \end{cases}$$

The aim of this paper is to prove the following

THEOREM. *Let u_ε be a solution of the problem*

$$\min\{G_\varepsilon(u) : u\varepsilon H^1(\Sigma^\varepsilon)\},$$

then the sequence $\tilde{u}_\varepsilon^i$ is compact in $L^2(\Sigma_i)$ and if u is one of its limit points then it is a solution of the problem

$$\min\{G(u) : u \in X(\Sigma)\}.$$

In addition the rescaled energies converge, that is

$$\lim_{\varepsilon\to 0} \varepsilon^{-3} G_\varepsilon(u) = G(u).$$

2. – Proof of the result

In order to prove result we need some preliminary lemmas. Take a sequence $u_\varepsilon \in H^1(\Sigma_\varepsilon)$. For every $\varepsilon > 0$, for every $i = 1, 2 u_\varepsilon^i = u_{\varepsilon|\Sigma_i^\varepsilon}$ and for every $\sigma \in \Sigma_i$ set

$$v_\varepsilon^i(\sigma) = \varepsilon^{-3} \int_{-\varepsilon}^{\varepsilon} t e_{\tau^i}(u_\varepsilon)\mathrm{d}t. \tag{2.1}$$

We state the following

LEMMA 2.1. *Assume that* $u_\varepsilon \in H^1(\Sigma^\varepsilon)$ *and* $f_\varepsilon(u_\varepsilon) \le c\varepsilon^3$. *Then:*

i) v^i_ε *is bounded in* $L^2(\Sigma_i)$ *for every* $i = 1, 2$*;*

ii) *if* $\tilde{u}^i_\varepsilon \to u^i$ *in* $L^2(\Sigma_i)$ *then* $e_{\tau^i}(u) = 0$ *and* $\langle u^i, \nu^i\rangle \in H^2(\Sigma_i)$ *and* $v^i_\varepsilon \to -\frac{2}{3}[(I - \nu^i \otimes \nu^i)(\nu^i \delta^i \delta^i u^i)]$ *in the weak topology of* $L^2(\Sigma_i)$.

PROOF. The proof is exactly the same of Lemma IV.1 of [1].

Set now, for every $u_\varepsilon \in H^1(\Sigma_\varepsilon)$ such that $u_\varepsilon = 0$ on $\Gamma^\varepsilon_1 \cup \Gamma^\varepsilon_2$, $u^i_\varepsilon = u_{\varepsilon|\Sigma^\varepsilon_i}$,

$$\tilde{u}_\alpha(\sigma) = \left\langle T^i_\alpha, \varepsilon^{-3}\int_{-\varepsilon}^{\varepsilon} t u^i_\varepsilon \mathrm{d}t\right\rangle \tag{2.2}$$

and $\mathcal{U}^i_\alpha = (\tilde{u}^i_\alpha \cdot \phi^i)$. Then $\mathcal{U}^i \in H^1(\Sigma_i, \mathbb{R}^2)$ and

$$2e(\mathcal{U}^i_\varepsilon)) = \frac{\partial \mathcal{U}^k_{\alpha,\varepsilon}}{\partial \xi_\beta} + \frac{\partial \mathcal{U}^j_{\beta,\varepsilon}}{\partial \xi_\alpha} = T^\beta_j \delta^i_j \tilde{u}^i_{\alpha,\varepsilon} + T^\alpha_k \delta^i_k \tilde{u}^i_{\beta,\varepsilon} \tag{2.3}$$

From Proposition 1.2, we get

$$\delta^i \tilde{u}^i_{\alpha,\varepsilon} = \left\langle \delta^i T^\alpha_i, \varepsilon^{-3}\int_{-\varepsilon}^{\varepsilon} tu\,\mathrm{d}t\right\rangle + \left\langle T_i\alpha, \varepsilon^{-3}\int_{-\varepsilon}^{\varepsilon}(t\delta u_\varepsilon + t^2 \delta u_\varepsilon \delta \nu^i)\mathrm{d}t\right\rangle$$

ant then

$$\begin{aligned} 2e(\mathcal{U}^i) = \varepsilon^{-3}\int_{-\varepsilon}^{\varepsilon} t\langle(\delta^i u_\varepsilon)^*, T^\alpha_i \otimes T^\beta_i\rangle \mathrm{d}t + \varepsilon^{-3}\int_{-\varepsilon}^{\varepsilon}\langle t u_\varepsilon, (T^\alpha_i \delta T^\beta_i)^*\rangle \mathrm{d}t \\ + \varepsilon^{-3}\int_{-\varepsilon}^{\varepsilon} t^2 \langle(\delta u_\varepsilon \delta \nu)^*, T^\alpha_i \otimes T^\beta_i\rangle \mathrm{d}t = S^\varepsilon_1 + S^\varepsilon_2 + S^\varepsilon_3. \end{aligned} \tag{2.4}$$

But

$$\begin{aligned} S^\varepsilon_1 &= \|T^i_\alpha\|\,\|T^i_\beta\|\varepsilon^{-3}\int_{-\varepsilon}^{\varepsilon} t\langle(\delta u_\varepsilon)^*, \tau^i_\alpha \otimes \tau^i_\beta\rangle \mathrm{d}t \\ &= \|T^i_\alpha\|\,\|T^i_\beta\|\varepsilon^{-3}\int_{\varepsilon}^{\varepsilon} t\langle e_\tau(u_\varepsilon), \tau^i_\alpha \otimes \tau^i_\alpha\rangle \mathrm{d}t. \end{aligned} \tag{2.5}$$

Moreover

$$S^\varepsilon_2 = \langle (2\varepsilon)^{-1}(T^\alpha_i \delta T^\beta)^*, \int_{-\varepsilon}^{\varepsilon} \mathrm{D}u_\varepsilon \nu^i \mathrm{d}t - \varepsilon^{-2}\int_{-\varepsilon}^{\varepsilon} t^2 \mathrm{D}u_\varepsilon \nu^i \mathrm{d}t\rangle \tag{2.6}$$

and then

$$\int_{\Sigma_i} |S_2^\varepsilon|^2 \le \frac{c}{\varepsilon}\int_{\Sigma_i}\int_{-\varepsilon}^{\varepsilon} |\mathrm{D}u_\varepsilon|^2 \le \frac{c}{\varepsilon^3}\int_{\Sigma_i^\varepsilon} |e(u_\varepsilon)|^2$$

by using a result of [8]. Of course an analogous estimate holds for S_3^ε.

LEMMA 2.2. *Assume* $u_\varepsilon \in H^1(\Sigma_\varepsilon)$, $u = 0$ *on* $\partial_0\Sigma_1^\varepsilon \cup \partial_0\Sigma_2^\varepsilon$ *and* $F_\varepsilon(u_\varepsilon) \le c\varepsilon^3$. *Then*

(1) $\tilde{u}^i_{\varepsilon,\alpha}$ *is compact in* $L^2(\Sigma_i)$.

(2) *if* $\tilde{u}_\varepsilon \to u^i$ *in* $L^2(\Sigma_i)$ *then* $\tau^i_\alpha \nu^i \delta^i u^i \in H^1(\Sigma_i)$, $\tau_\alpha \nu \delta u = 0$ *on* $\partial\Sigma_i \backslash \gamma$ *and* $\tilde{u}_{\varepsilon,\alpha} \to -\dfrac{2}{3}\tau^\alpha \nu \delta u$.

PROOF. To prove (1) we have only to prove that $\mathcal{U}^i_{\varepsilon,\alpha}$ is compact in $L^2(\omega_i)$. From (2.3), (2.4), (2.5), (2.6) we get

$$\int_{\omega_i} |e(\mathcal{U}^i_\varepsilon)|^2 \le \frac{c_1}{\varepsilon^3}\int_{\Sigma_i^\varepsilon} |e_{\tau^i}(u^i_\varepsilon)|^2 \le \frac{c_1}{\varepsilon}\int_{\Sigma_i^\varepsilon} |\mathrm{D}u_\varepsilon|^2 \le \frac{c}{\varepsilon^3}\int_{\Sigma_i^\varepsilon} |e(u^i_\varepsilon)| \le c. \tag{2.7}$$

Moreover as for (2.6) we have

$$\tilde{u}^i_{\varepsilon,\alpha} = \left\langle \tau^i_\alpha, \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \mathrm{D}u^i_\varepsilon \nu^i - \frac{1}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} t^2 \mathrm{D}u^i_\varepsilon \nu^i \right\rangle,$$

and therefore

$$\int_{\omega_i} |\mathcal{U}^i_\varepsilon|^2 \le \frac{c}{\varepsilon}\int_{\Sigma_i^\varepsilon} |\mathrm{D}u^i_\varepsilon|^2 \le \frac{c}{\varepsilon^3}\int_{\Sigma_i^\varepsilon} |e(u^i_\varepsilon)|^2 \le c, \tag{2.8}$$

again by the above mentioned result of [9]. Combining (2.7), (2.8) we get (1).

In order to prove (2) we observe that given $\theta \in C_0^\infty(\Sigma_i)$ we have

$$\frac{1}{\varepsilon^3}\int_{\Sigma_i} \theta \tau^i_\alpha \int_{-\varepsilon}^{\varepsilon} t u^i_\varepsilon \mathrm{d}t = \frac{1}{2\varepsilon}\int_{\Sigma_i} \theta \left\langle \tau^\alpha_i, \left[\int_{-\varepsilon}^{\varepsilon} \mathrm{D}u^i_\varepsilon \nu^i - \frac{1}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon} t^2 \mathrm{D}u^i_\varepsilon \nu^i\right]\right\rangle$$

$$= \frac{1}{2\varepsilon}\int_{\Sigma_i} \theta \left\langle \tau^\alpha_i, \int_{-\varepsilon}^{\varepsilon} (\mathrm{D}u^i_\varepsilon + {}^t\mathrm{D}u^i_\varepsilon)\nu^i - \frac{1}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon} t^2(\mathrm{D}u^i_\varepsilon + {}^t\mathrm{D}u^i_\varepsilon)\nu^i \right\rangle \tag{2.9}$$

$$-\frac{1}{2\varepsilon}\left\{\int_{\Sigma_i} \theta \left\langle \tau^\alpha_i, \int_{-\varepsilon}^{\varepsilon} {}^t\mathrm{D}u^i_\varepsilon \nu \frac{1}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon} t\,{}^t\mathrm{D}u^i_\varepsilon \nu \right\rangle\right\}.$$

Since

$$\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\langle e(u_\varepsilon^i),\nu^i\otimes\tau_\alpha\rangle \text{ and } \frac{1}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} t^2\langle e(u_\varepsilon^i),\nu^i\otimes\tau_\alpha\rangle$$

go to zero in $L^2(\Sigma_i)$ as ε goes to zero, equality (2.9) becomes

$$\begin{aligned}\frac{1}{\varepsilon^3}\int_{\Sigma_i}\theta\left\langle\tau_i^\alpha,\int_{-\varepsilon}^{\varepsilon}tu_\varepsilon^i\mathrm{d}t\right\rangle\\ = \omega_\varepsilon-\frac{1}{2\varepsilon}\int_{\Sigma_i}\theta\left\langle\tau_i^\alpha,\int_{-\varepsilon}^{\varepsilon}\nu^i\delta^iu_\varepsilon^i\right\rangle+\frac{1}{2\varepsilon^3}\int_{\Sigma_i}\theta\left\langle\tau_i^\alpha,\int_{-\varepsilon}^{\varepsilon}t^2\nu^i\delta^iu^i\right\rangle\end{aligned} \tag{2.10}$$

where ω_ε denotes from now on any quantity which goes to zero as ε goes to zero.

From Proposition 1.1 and 1.2 we deduce

$$\int_{\Sigma_i}\theta\tilde{u}_{\varepsilon,\alpha}=\omega_\varepsilon+\int_{\Sigma_i}(\delta^i)^{-1}(\theta\tau_i^\alpha\nu^i)\left[\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}u_\varepsilon^i-\frac{1}{2\varepsilon^\varepsilon}\int_{-\varepsilon}^{\varepsilon}t^2u_\varepsilon^i\right], \tag{2.11}$$

and therefore

$$\int_{\Sigma_i}\theta\tilde{u}_{\varepsilon,\alpha}^i=\omega_\varepsilon+\int_{\Sigma_i}\theta\tau_i^\alpha\nu^i\delta^iu^i.$$

Then $\tau_i^\alpha\nu^i\delta^iu^i\in L^2(\Sigma_i)$ and $\tilde{u}_{\varepsilon,\alpha}^i\to\tau_\alpha^i\nu^i\delta^iu^i$ in $L^2(\Sigma_i)$.

We recall now that $\tau_\alpha\nu\delta u=\tau_\alpha\delta(\langle u\cdot\nu\rangle)-\tau_\alpha u\delta\nu$; since by Lemma 2.2 $\langle u,\nu\rangle\in H^2(\Sigma_i)$ we get $\tau_\alpha\nu\delta u\in H^1(\Sigma_i)$ and since $u_\varepsilon=0$ on $\partial_0\Sigma_i^\varepsilon$ we have $\tau_\alpha\nu\delta u=0$ on $\partial\Sigma_i\backslash\gamma$ and (2) is completely proved. In the above lemma we have proved that for every $i=1,2$ and for every $\alpha\tau_\alpha^i\nu^i\delta^iu^i\in H^1(\Sigma_i)$ and $\tau_\alpha^i\nu^i\delta^iu^i=0$ on $\partial\Sigma_i\backslash\gamma$. Then it makes sense to ask what happens to the trace of $\tau_\alpha^i\nu^i\delta^iu^i$ on γ; this is show in the following

LEMMA 2.3. *Assume that all the hypotheses of Lemma 2.2 are satisfied. Then*

$$\tau_{\alpha_1}^1\nu^1\delta^1u^1+\tau_{\alpha_2}^2\nu^2\delta^2u^2=0 \tag{2.12}$$

H^{n-2}-a.e. on γ.

PROOF. For every $a\in\gamma$ let $S(a,\varepsilon)$ be the n-sphere centered in a and having radius ε and set $S_i^\varepsilon=S(a,\varepsilon)\cap\Sigma_i$. Then, by Lemma 2.2, we have

$$\tau_{\alpha_i}^i\nu^i\delta^iu_{|\gamma}^i(a)=-\frac{3}{2}\lim_{\varepsilon\to0}\int_{S_i^\varepsilon}\tilde{u}_{\varepsilon,\alpha_i}^i$$

for H^{n-2}-a.e. in γ. Since

$$\tilde{u}^i_{\varepsilon,\alpha_i} = \left\langle \tau^i_{\alpha_i}, \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} \mathrm{D}u_\varepsilon \nu^i - \frac{1}{2\varepsilon^3}\int_{-\varepsilon}^{\varepsilon} t^2 \mathrm{D}u_\varepsilon \nu^i \right\rangle, \tag{2.13}$$

denoting by $R_\varepsilon x + b_\varepsilon$ the projection of u_ε into the space of rigid displacements of $(\Sigma_1^\varepsilon \cup \Sigma_2^\varepsilon) \cap S(a, \lambda\varepsilon)$ and by x_0^ε the center of mass of Σ_1^ε, we have

$$R_\varepsilon x = \left(\frac{\int_{\Sigma_1^\varepsilon \cup \Sigma_2^\varepsilon} (x - x_0^\varepsilon) \wedge u_\varepsilon}{\int_{\Sigma_1^\varepsilon \cap \Sigma_2^\varepsilon} |x - x_0^\varepsilon|^2} \right) \wedge x, \tag{2.14}$$

and it readily verified that

$$|R_\varepsilon|^2 \le c \int_{\Sigma_\varepsilon} |u_\varepsilon|^2. \tag{2.15}$$

From (2.13) we get

$$\begin{aligned} \int_{S_i^\varepsilon} \tilde{u}^i_{\varepsilon,\alpha_1} &= \int_{S_i^\varepsilon} \left\langle \tau^i_{\alpha_i}, \int_{-\varepsilon}^{\varepsilon} (\mathrm{D}u^i_\varepsilon - R_\varepsilon)\nu^i - \frac{1}{\varepsilon^2}\int_{-\varepsilon}^{\varepsilon} t^2 (\mathrm{D}u^i_\varepsilon - R_\varepsilon)\nu^i \right. \\ &\quad \left. + \int_{S_i^\varepsilon} \left\langle \tau^i_{\alpha_i}, R_\varepsilon \nu^i \frac{1}{\varepsilon^2} \int_{S_i^\varepsilon} t^2 R_\varepsilon \nu^i \right\rangle \right. = I_1^\varepsilon + I_2^\varepsilon. \end{aligned} \tag{2.16}$$

Choosing $\lambda > 0$ such that $(\Sigma_1^\varepsilon \cup \Sigma_2^\varepsilon) \cap S(a, \lambda\varepsilon) \supset S_i^\varepsilon \times (-\varepsilon, \varepsilon)$ we obtain

$$\begin{aligned} |I_{1,i}^\varepsilon|^2 &\le \frac{c}{\varepsilon} H^{n-1}(S_i^\varepsilon) \int_{S_i^\varepsilon} \int_{-\varepsilon}^{\varepsilon} |\mathrm{D}u^i_\varepsilon - R_\varepsilon|^2 \le \frac{c}{\varepsilon} \int_{(\Sigma_1^\varepsilon \cup \Sigma_2^\varepsilon) \cap S(a,\varepsilon)} |\mathrm{D}^i_\varepsilon - R_\varepsilon|^2 \\ &\le \frac{c}{\varepsilon} \int_{\Sigma_1^\varepsilon \cup \Sigma_2^\varepsilon} |e(u^1_\varepsilon)|^2 = \omega_\varepsilon. \end{aligned}$$

Moreover by using (2.15) we get for H^{n-2}-almost every $a \in \gamma$

$$I_{2,i}^\varepsilon = \omega_2 + \frac{2}{3} \langle \tau^i_{\alpha_i}(a), R_\varepsilon \nu^i(a) \rangle.$$

Then

$$\sum_{i=1}^{2} \int_{S_i^\varepsilon} \tilde{u}^i_{\varepsilon,\alpha_i} = \omega_\varepsilon + \frac{2}{3} \sum_{i=1}^{2} \langle \tau^i_{\alpha_i}(a), R_\varepsilon \nu^i(a) \rangle.$$

Since R_ε is skew-symmetric and $\tau^1_{\alpha_1}(a) = \nu^2(a)$, $\tau^2_{\alpha_2}(a) = \nu^1(a)$ we have

$$\sum_{i=1}^{2} \langle \tau^i_{\alpha_i}(a), R_\varepsilon \nu^i(a)\rangle = 0$$

and the lemma is completely proved.

Since i_ε is defined in $\Sigma_1^\varepsilon \cup \Sigma_2^\varepsilon$ it remains defined a function $\overline{u}_\varepsilon$ in $\Sigma_1^1 \cup \Sigma_2^1$ as

$$\overline{u}_\varepsilon(\sigma, z) = u_\varepsilon(\sigma + \varepsilon z \nu^1) \text{ in } \Sigma_1^1,$$

$$\overline{u}_\varepsilon(\sigma, s) = u_\varepsilon(\sigma + \varepsilon s \nu^2) \text{ in } \Sigma_2^1.$$

It is readily verified that if $F_\varepsilon(u_\varepsilon) \leq c\varepsilon^3$ then $\overline{u}_\varepsilon$ is compact in $L^2(\Sigma_1^1 \cap \Sigma_2^1)$ and

$$\int_{\Sigma_1^1 \cap \Sigma_2^1} |e(\overline{u}_\varepsilon)|^2 \leq \frac{c}{\varepsilon} \int_{\Sigma_\varepsilon} |e(u_\varepsilon)|^2 \tag{2.18}$$

and therefore $e(\overline{u}_\varepsilon) \to 0$ in $L^2(\Sigma_1^1 \cap \Sigma_2^1)$. Moreover we have

$$\int_{\partial(\Sigma_1^1 \cap \Sigma_2^1)} |\overline{u}_\varepsilon|^2 \leq c \int_{\Sigma_i^1} |e(\overline{u}_\varepsilon)|^2 \leq \frac{c}{\varepsilon^2} \int_{\Sigma_i^\varepsilon} |e(u_\varepsilon)|^2. \tag{2.19}$$

Since $e(\overline{u}_\varepsilon) \to 0$ in $L^2(\Sigma_1^1 \cap \Sigma_2^1)$, (2.19) implies that $\overline{u}_\varepsilon \to 0$ in $L^2(\Sigma_1^1 \cap \Sigma_2^1)$.

Now we are able to prove following

LEMMA 2.4. *Assume that all the hypotheses of Lemma 2.2 are satisfied. Then*

$$u^1_{|\gamma} = u^2_{|\gamma} = 0 \qquad H^{n-2}\text{-a.e. on } \gamma.$$

PROOF. It is enought to observe that

$$\int_{\Sigma_1^\varepsilon \cap \Sigma_2^\varepsilon} u_\varepsilon = \varepsilon^2 \int_{\Sigma_1^1 \cap \Sigma_2^1} \overline{u}_\varepsilon + \omega_\varepsilon$$

Since $\overline{u}_\varepsilon \to 0$ in $L^2(\Sigma_1^1 \cap \Sigma_2^1)$ the thesis easily follows.

Combining the two previous lemmas we easily obtain

LEMMA 2.5. *Assume that all the hypotheses of Lemma 2.2 are satisfied. Then*

$$\nu^1 \delta^1 u^1 + \nu^2 \delta^2 u^2 = 0 \qquad H^{n-2}\text{-a.e. on } \gamma.$$

PROOF. Since Lemma 2.4 $u^i = 0$ on γ, if t is the unit tangent vector to γ then

$$t\nu^1 \delta^1 u^1 = t\nu^2 \delta^2 u^2 = 0 \qquad H^{n-2}\text{-a.e. on } \gamma.$$

Combining the last equality with (2.12) we get the thesis.

We need now some remarks about boundary data, namely we state the following

LEMMA 2.6. *Suppose that* $F_\varepsilon(u_\varepsilon) \leq c\varepsilon^3$ *and* $\tilde{u}^i_\varepsilon \to u^i$ *in* $L^2(\Sigma_i)$. *Then* $u = 0$ *on* $\partial\Sigma$ *and* $\delta^i(u^i \cdot \nu) = 0$ *on* $\partial\Sigma\backslash\gamma$.

PROOF. It is enough to observe that

$$\delta^i_k(u^i_h \cdot \nu^i_h) = \nu^i_h \delta^i u^i_h + u^i_h \delta^i_k \nu^i_h$$

and then Lemmas 2.1, 2.2 permit us to conclude the proof.

A crucial step in proving our theorem is the following:

PROPOSITION 2.7. *For every sequence* $\{u_\varepsilon\}_{\varepsilon>0} \subset H^1(\Sigma^\varepsilon)$ *such that* $\tilde{u}^i_\varepsilon \to u^i$ *in* $H^1(\Sigma_i)$ *we have*

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^3} G_\varepsilon(u_\varepsilon) \geq G(u). \tag{2.20}$$

PROOF. We may suppose $G_\varepsilon(u_\varepsilon) \leq c\varepsilon^3$, since otherwise (2.20) is obvious. Then we have

$$G_\varepsilon(u_\varepsilon) = F_\varepsilon(u_\varepsilon) - \phi_\varepsilon(u_\varepsilon) \leq c\varepsilon^3,$$

which gives

$$F_\varepsilon(u_\varepsilon) \leq c\varepsilon^3 + \phi_\varepsilon(u_\varepsilon) \tag{2.21}$$

and therefore $F_\varepsilon(u_\varepsilon) < +\infty$ and $u_\varepsilon = 0$ on $\partial_0\Sigma^\varepsilon_i (i = 1, 2)$.

Since for every $\lambda > 0$

$$\phi_\varepsilon(u_\varepsilon) \leq \varepsilon^2\lambda \int_{\Sigma^\varepsilon} |u_\varepsilon|^2 + \frac{\varepsilon^2}{\lambda} \int_{\Sigma^\varepsilon} |g|^2$$

we obtain, by using the already mentioned result of [9] and condition (1.3),

$$\begin{aligned} F_\varepsilon(u_\varepsilon) &\leq c\varepsilon^3 + \frac{\varepsilon^2}{\lambda} \int_{\Sigma^\varepsilon} |g|^2 + \varepsilon^2\lambda \int_{\Sigma^\varepsilon} |u_\varepsilon|^2 \\ &\leq c\varepsilon^3 + \lambda \int_{\Sigma^\varepsilon} |e(u_\varepsilon)|^2 \leq c\varepsilon^3 + \lambda \int_{\Sigma^\varepsilon} f(e(u_\varepsilon)) \end{aligned}$$

(here c denotes various constants). A suitable choice of λ yields

$$F_\varepsilon(u_\varepsilon) \leq c\varepsilon^3.$$

Then we may apply Lemmas 2.1, 2.2, 2.3, 2.4 and deduce that the function $u : \Sigma \to \mathbb{R}^3$ defined by $u_{|\Sigma_i} = u^i$ belongs to $X(\Sigma)$. To complete the proof it is enough to proceed as in [1] Theorem II.5.

To conclude the proof of our main theorem we have to prove that $G(u)$, the limit functional, is, is some sense, as good as possible. This will be done in the following

PROPOSITION 2.8. *For every $u \in X^1(\Sigma)$ there exists a sequence*

$$\{u_\varepsilon\}_{\varepsilon>0} \subset H^1(\Sigma^\varepsilon)$$

such that $u_\varepsilon = 0$ on $\partial_0\Sigma_\varepsilon$ $\tilde{u}_\varepsilon \to u$ weakly in $H^1(\Sigma)$ and

$$\lim G_\varepsilon(u_\varepsilon) = G(u). \tag{2.22}$$

PROOF. Since $u = 0$ on Γ and

$$\nu_j^2\nu_i^1\delta_j^1u_i^1 + \nu_j^1\nu_i^1\delta_j^2u_i^2 = 0$$

on γ, it is possible to construct a sequence $\{v_\varepsilon^i\}_{\varepsilon>0} \subset X(\Sigma)$ such that $\nu^i\delta^i\delta^iv_\varepsilon^i \to \nu^i\delta^i\delta^iu$ in $L^2(\Sigma_i)$, $v_\varepsilon^i \to u$ in $H^1(\Sigma^i)$ and $x \in \Sigma_i^\varepsilon \cap \Sigma_2^\varepsilon x = \sigma + t\nu^1 = \sigma' + t'\nu^2$ then

$$v_\varepsilon^1 - t\nu^1\delta v_\varepsilon^1 = v_\varepsilon^2 - t'\nu^2\delta^2v_\varepsilon^2.$$

Let η^i $(i = 1, 2)$ be an arbitrary smooth function having compact support in Σ_i, then there exist sequences $\{\eta_\varepsilon^i\}_{\varepsilon>0} \subset C_0^\infty(\Sigma_i)$, $\eta_\varepsilon^i \to \eta^i$ in $H^2(\Sigma_i)$, such that for every $x \in \Sigma_1^\varepsilon \cap \Sigma_2^\varepsilon$, $x = \sigma + t\nu^1 = \sigma' + t'\nu^2$,

$$t^2\eta_\varepsilon^1(\sigma) = t'^2\eta_\varepsilon^2(\sigma').$$

We put, on Σ_i^ε

$$u_\varepsilon^i = v_\varepsilon^i - t\nu^i\delta^iv_\varepsilon^i + \frac{t^2}{2}\eta_\varepsilon^i$$

and setting $\phi_\varepsilon = -\nu^i\delta^iv_\varepsilon^i$ simple computations (see [1]) show that

$$\begin{aligned}
\frac{1}{\varepsilon^3}\int_{\Sigma_i^\varepsilon} f(e(u_\varepsilon^i))\mathrm{d}x &= \frac{1}{\varepsilon^3}\int_{\Sigma_i^\varepsilon} f(\delta v_\varepsilon^i + \varphi_\varepsilon \otimes \nu^i + t(-\delta v_\varepsilon^i\delta\nu^i + \delta\varphi_\varepsilon + \eta_\varepsilon^i \otimes \nu^i)) + \omega_\varepsilon \\
&= \frac{1}{\varepsilon^3}\int_{\Sigma_i^\varepsilon} t^2 f(-\delta v_\varepsilon^i\delta\nu^i + \delta\varphi_\varepsilon + \eta_\varepsilon^i \otimes \nu^i) \\
&= \frac{2}{3}\int_{\Sigma_i} f(-\delta v_\varepsilon^i\delta\nu^i + \delta\varphi_\varepsilon + \eta_\varepsilon^i \otimes \nu) + \omega_\varepsilon \\
&= \frac{2}{3}\int_{\Sigma_i} f(-\delta u^i\delta\nu^i - \delta(\nu^i\delta^iu^i) + \eta^i \otimes \nu) + \omega_\varepsilon.
\end{aligned}$$

Since η^i is arbitrary we have

$$\frac{1}{\varepsilon^3}\int\limits_{\Sigma_i^\varepsilon} f(e(u_\varepsilon^i))\mathrm{d}x = \omega_\varepsilon + \frac{2}{3}\int\limits_{\Sigma_i} \min_{\xi\in\mathbb{R}^3} f(-\delta u^i\delta\nu^i - \delta(\nu^i\delta^i u^i) + \xi\otimes\nu^i). \tag{2.23}$$

Recalling that $e_{r^i} = 0$ and $\nu\delta\nu = \delta\nu\nu = 0$, the argument of f may be reduced to

$$-\nu^i\otimes(\nu^i\delta^i u^i\delta^i\nu^i - \xi) - \nu^i\delta^i\delta^i u^i,$$

so that (2.23) yields

$$\frac{1}{\varepsilon^3}\int\limits_{\Sigma_i^\varepsilon} f(e(u_\varepsilon^i))dx = \omega_\varepsilon + \frac{2}{3}\int\limits_{\Sigma_i} f_0(\nu^i\delta^i\delta^i u^i),$$

which concludes the proof.

END OF THE PROOF. Suppose that u_ε solves the problem

$$\min(G_\varepsilon(u) : u\in H^1(\Sigma^\varepsilon)).$$

We have then

$$G_\varepsilon(u_\varepsilon)\le G_\varepsilon(0) = 0,$$

and therefore

$$F_\varepsilon(u_\varepsilon)\le\phi_\varepsilon(u_\varepsilon) = \varepsilon^2\int\limits_{\Sigma_\varepsilon} g\overline{u}_\varepsilon.$$

The same argument used in the proof of Proposition 2.6 shows that $\tilde{u}_\varepsilon^i$ is compact in $L^2(\Sigma_i)$ (indeed in the weak topology of $H^1(\Sigma_1)$). If $\overline{u}$ is one of its limit points then $u\in X(\Sigma)$ and we want to prove that

$$G(v)\ge G(\overline{u})$$

for any other $v\in X(\Sigma)$. By Proposition 2.7 we may find a sequence $\{v_\varepsilon\}_{\varepsilon>0}\subset H^1(\Sigma^\varepsilon)$ such that $v_\varepsilon = 0$ on $\partial_0\Sigma^\varepsilon$, $\tilde{v}_\varepsilon\to v^i$ in $L^2(\Sigma_i)$ and

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^3}G_\varepsilon(v_\varepsilon) = G(v).$$

But $G_\varepsilon(v_\varepsilon)\ge G_\varepsilon(u_\varepsilon)$ and from Proposition 2.6 we deduce

$$\liminf\frac{1}{\varepsilon^3}G_\varepsilon(u_\varepsilon)\ge G(\overline{u}),$$

which shows $G(\overline{u})\le G(v)$ and concludes the proof.

3. – Examples

In this section we want to give explicitly the limit functional in some particular cases when the strain energy density of the three-dimensional body has the form

$$f(A) = \frac{\lambda}{2}(\operatorname{tr} A^*)^2 + \mu|A^*|^2$$

where λ, μ are the usual Lamé constants.

EXAMPLE 3.1. *The flat case.* Assume that Σ_1 is a portion of the plane $\{z = 0\}$, Σ_2 is a portion of the plane $\{y = 0\}$ and $\Sigma_1 \cap \Sigma_2$ is contained in the x-axis. Then denoting by w_i the vertical displacement in Σ_i and substituting the Lamé coefficients λ, μ with their expression in terms of the Young modulus E and the Poisson coefficients σ, we have for the limit energy of the folded plate

$$\int_{\Sigma_1} \frac{E}{3(1-\sigma^2)}[|\Delta w_1|^2 - 2(1-\sigma) \det \mathrm{D}^2 w_1]\mathrm{d}x\mathrm{d}y$$

$$+ \int_{\Sigma_2} \frac{E}{3(1-\sigma^2)}[|\Delta w_2|^2 - 2(1-\sigma) \det \mathrm{D}^2 w_2]\mathrm{d}x\mathrm{d}z,$$

with the constraints that the horizontal displacements are zero on each plate, $w_i = 0$ on $\Sigma_1 \cap \Sigma_2$, and

$$\frac{\partial w_2}{\partial z} + \frac{\partial w_1}{\partial y} = 0$$

on $\Sigma_1 \cap \Sigma_2$.

EXAMPLE 3.2. *The cylindrical-flat case.* Assume Σ_1 is the cylinder with radius R and axis $\{x = y = 0\}$ lying in $\{y \geq 0,\ 0 \leq z \leq 1\}$, $\Sigma_2 = \{(x, y, z) : y = 0,\ R \leq x \leq R',\ 0 \leq z \leq 1\}$.

Denote by ρ, θ, z the cylindrical coordinates, by U_ρ, u_θ, U_z the components of the displacements of Σ_1 and by w the (vertical) displacement of Σ_2. We obtain for the limit energy the following expression

$$\int_{\Sigma_1} \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}[\operatorname{tr}(A_0))^2 - 2\mu \det A_0 + \int_{\Sigma_2} \frac{E}{3(1-\sigma^2)}[|\Delta w|^2 - 2(1-\sigma) \det \mathrm{D}^2 w],$$

where

$$A_0 = \begin{pmatrix} \mathrm{D}_{zz}U_\rho & R^{-1}\cdot(\mathrm{D}_{z\theta}U_\rho - \mathrm{D}_z U_\theta) \\ R^{-1}\cdot(\mathrm{D}_{z\theta}U_\rho - \mathrm{D}_z U_\rho) & R^{-2}\cdot(U_\rho + \mathrm{D}_{\theta\theta}U_\rho) \end{pmatrix}.$$

The constraints are:

$$\mathrm{D}_z U_z = U_\rho + \mathrm{D}_\theta U_\theta = R\mathrm{D}_z U_\theta + \mathrm{D}_\theta U_z = 0 \qquad \text{on } \Sigma_1,$$

the tangential displacements are zero on Σ_2, and $$w = U_\rho = U_\theta = U_z = 0 \qquad \text{on } \Sigma_1 \cap \Sigma_2;$$

while condition (2.12) becomes

$$\frac{\partial U_\rho}{\partial \theta} + \frac{\partial w}{\partial x} = 0 \qquad \text{on } \Sigma_1 \cap \Sigma_2.$$

4. – Generalization

It is possible to extend our result to the case in which the two shells are not orthogonal.

The only change necessary is in condition (2.12); in fact it is easily seen that the rest of the proof of our theorem does not depend on the scalar product $\langle \nu^1, \nu^2 \rangle$. To show this, suppose that $\nu^1 \wedge \nu^2 \neq 0$ H^1-a.e. on γ and set

$$\tau^i = \nu^i \wedge \frac{\nu^1 \wedge \nu^2}{|\nu^1 \wedge \nu^2|},$$

(observe that if $\langle \nu^1, \nu^2 \rangle = 0$ then $\tau^i = \tau^{\alpha_i}$) and for every $a \in \gamma$ we may consider a system of coordinates whose axes are τ^1, τ^2 and $\nu_1 \wedge \nu_2$ and the induced metric on the $\nu^1 - \nu^2$ plane is given by $\langle \tau^i, \tau^j \rangle$. A calculation in these coordinates shows that (2.12) becomes

$$\tau^1_h \tau^2_k \delta^1_h u^1_k + \tau^2_h \tau^1_h \delta^2_h u^2_k = 0. \tag{4.1}$$

EXAMPLE 4.1. Assume Σ_1 is a portion of the plane $\{z = 0\}$, Σ_2 a portion of the plane $\{z = (\mathrm{tg}\alpha)y\}$ with $0 < \alpha < \pi$. Thus $\Sigma_1 \cap \Sigma_2 = \{z = y = 0\}$ and $\tau^1 = (0, 1, 0)$, $\tau^2 = (\cos\alpha, \sin\alpha, 0)$ so that setting $\mathbf{U}_1 = \langle u, \tau^1 \rangle$, $\mathbf{U}_2 = \langle u, \tau^2 \rangle$ condition (4.1) becomes

$$\frac{\partial \mathbf{U}^1_1}{\partial \tau_2} + \frac{\partial \mathbf{U}^2_2}{\partial \tau_1} = 0. \tag{4.2}$$

We remark that $\alpha = \dfrac{\pi}{2}$ then (4.2) is equivalent to (3.1); in the case $\alpha = \pi$, (4.2) is an identity.

The result can be extended also to non-homogeneous materials, that is the case in which f depends also on x: this can be done by using the same technique as in [1] modifying f_0 to

$$f_0(\sigma, z) = \min_{\xi \in \mathbb{R}^n} f(\sigma, z + \xi \otimes \nu).$$

As in [1] the result is still valid (with slight modifications) when the width of Σ^ε is not constant (see again [1] for further details).

REFERENCES

[1] E. Acerbi - G. Buttazzo - D. Percivale, *Thin inclusions in linear elasticity; a variational approach*, J. Reine Angew. Math. **300** (1988), pp. 1-16.

[2] D. Caillerie, *The effect of a thin inclusion of high rigidity in an elastic body*, Math. Methods Appl. Sci. **2** (1980), pp. 251-270.

[3] P.G. Ciarlet - P. Destuynder, *A justification of the two-dimensional linear plate model*, J. Méc. Théor. Appl. **18** (1979), pp. 315-344.

[4] P.G. Ciarlet - H. Le Dret - R. Nzengwa, *Modelisation de la jonction entre un corps élastique tridimensionnel et une plaque*, C.R. Acad. Sci. Paris Sér. I Math. **305** (1987), pp. 55-58.

[5] P.G. Ciarlet - H. Le Dret, *Justification de la condition aux limite d'encastrement d'une plaque par une méthode asymptotique*, C.R. Acad. Sci. Paris Sér. I Math. **307** (1988), pp. 1015-1018.

[6] E. De Giorgi - G. Dal Maso, *Γ-convergence and Calculus of Variations*, Lecture Notes in Math. **979**, Springer Verlag. (1983).

[7] M.P. Do Carmo, *Differential geometry of curves and surfaces*, Englewood-Cliff. (1976).

[8] I. Ekeland - R. Temam, *Convex Analysis and Variational problems*, North Holland (1976).

[9] R.V. Kohn - M. Vogelius, *A new model for thin plates with rapidly varying thichness II; a convergence proof*, Quart. Appl. Math. **43** (1985), pp. 1-23.

[10] H. Le Dret, *Modelisation d'une plaque pliee*, (preprint).

Dipartimento di Matematica
Politecnico di Torino
Corso Duca degli Abruzzi
10129 Torino