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1. - Notations

1.1. Let X be a complex analytic manifold and let $M \subset X$ be a real analytic submanifold. One denotes by $\pi : T^*X \to X$ the cotangent bundle to X, by $T^*_M X$ the conormal bundle to M in X and by α the canonical one-form on T^*X. One denotes by $s^{\pm}(M,p)$ the numbers of positive and negative eigenvalues for the Levi form of M at $p \in T^*_M X$. To a morphism $f : X \to Y$ one associates the natural mappings

$$T^*X \overset{{}^t f'}{\leftarrow} X \times_Y T^*Y \overset{f_\pi}{\to} T^*Y.$$

1.2. Let Λ be a real analytic homogeneous symplectic manifold of dimension $2n$ and let E be a complexification of Λ (endowed with the induced homogeneous complex symplectic structure). Let V be a regular smooth involutive complex submanifold of E of codimension l and let $\varphi_1 = \cdots = \varphi_l = 0$ be a system of local equations for V at $p \in V$ with $\{\varphi_i, \varphi_j\} \equiv 0$ (where $\{\cdot, \cdot\}$ denotes the Poisson braket);

DEFINITION 1.1. (cf. [S-K-K]). The *generalized Levi form* on $(T_pV)^\perp$ of V with respect to Λ at $p \in V \cap \Lambda$, denoted $L_\Lambda(V,p)$, is the Hermitian form of matrix

$$(\{\varphi_i, \varphi_j^c\}(p))_{1 \le i,j \le l}$$

where $(\cdot)^\perp$ denotes the symplectic orthogonal and φ_j^c denotes the holomorphic conjugate of φ_j with respect to Λ. (Notice that this definition does not depend on the choice of the system of equations for V).

Choose a system of local homogeneous symplectic coordinates $(z;\zeta)$ in E at $p \in V$ such that $\zeta_1 = \varphi_1, \cdots, \zeta_l = \varphi_l$ and set $X = \{\zeta = 0\}$, $Y = \{\zeta = z_1 = \cdots = z_l = 0\}$. Let $f : X \to Y$ be the projection $f(z) = (z_{l+1}, \ldots, z_n)$. Consider the natural mapping $f_\pi : V \to T^*Y$ given

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by the identification $V \cong X \times_Y T^*Y \subset T^*X$ and note that $f_\pi^{-1} f_\pi(p)$ is the bicharacteristic leaf of V issued from p.

DEFINITION 1.2. Let F be a sheaf on V and $W \subset V$ a locally closed subset. We will say that F is *isotropically flabby* on W if for every $p \in W$ there exists a neighbourhood Ω of p in V such that $F|_{W\cap\Omega} \cong (f_\pi^{-1}G)|_{W\cap\Omega}$ for a sheaf G on T^*Y which is conically flabby on $f_\pi(W \cap \Omega)$. (Notice that this definition does not depend on the choice of coordinates at p).

1.3. We will also recall some notations from the microlocal theory of sheaves as in [K-S1]. One denotes by $\mathrm{D}^b(X)$ the derived category of the category of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on a real analytic manifold X. For F an object of $\mathrm{D}^b(X)$, one denotes by $\mathrm{SS}(F)$ its micro-support, a closed conic involutive subset of T^*X.

If M is a real analytic submanifold of X, one denotes by $\mu_M(F)$ the Sato's microlocalization of F along M, an object of $\mathrm{D}^b(T^*_MX)$. Recall the identification $\mu_M(F) \cong \mu\mathrm{hom}(\mathbb{C}_M, F)$, where μhom denotes the microlocalization bifunctor of [K-S1] and, for a locally closed subset $A \subset X$, $\mathbb{C}_A$ denotes the sheaf which is 0 on $X\setminus A$ and the constant sheaf with fiber $\mathbb{C}$ on A.

For $\Omega \subset T^*X$, one denotes by $\mathrm{D}^b(X;\Omega)$ the localization of $\mathrm{D}^b(X)$ at Ω. Recall that the objects of $\mathrm{D}^b(X;\Omega)$ are the same as those of $\mathrm{D}^b(X)$ and that a morphism $h : F \to G$ in $\mathrm{D}^b(X)$ becomes an isomorphism in $\mathrm{D}^b(X;\Omega)$ if $\mathrm{SS}(H) \cap \Omega = \emptyset$, where H is the third term of a distinguished triangle $F \xrightarrow{h} G \to H \xrightarrow{+1}$.

Recall that μhom induces a functor from $\mathrm{D}^b(X;\Omega)^\circ \times \mathrm{D}^b(X;\Omega)$ to $\mathrm{D}^b(\Omega)$.

1.4. Let X be a complex analytic manifold. One denotes by $\mathcal{O}_X$ the sheaf of germs of holomorphic functions on X and by $\mathcal{E}_X$ the sheaf of finite order microdifferential operators on T^*X.

If X is a complexification of a real analytic manifold M, recall that the sheaf $\mathcal{C}_M$ of Sato's microfunctions is defined by $\mathcal{C}_M = \mu_M(\mathcal{O}_X) \otimes or_{M/X}[\dim^{\mathbb{R}} M]$, where $or_{M/X}$ denotes the relative orientation sheaf.

2. - Statement of the results

2.1. Let M be a real analytic manifold and X be a complexification of M. Let $\mathcal{M}$ be a left coherent $\mathcal{E}_X$-module defined on an open subset $U \subset T^*X$. Denote by $V \subset T^*X$ its characteristic variety and set $l = \mathrm{codim}^{\mathbb{C}} V$. We will make the following hypotheses:

$$(2.1) \qquad \begin{cases} \mathcal{M} \text{ has simple characteristics along } V, \\ V \cap T^*_MX \text{ is clean,} \\ \alpha|_{V\cap T^*_MX} \neq 0. \end{cases}$$

Note that (2.1) implies in particular that V is a smooth regular involutive submanifold of T^*X.

For $p \in V \cap T^*_M X$ we will denote by $s^\pm(M, V, p)$ the numbers of positive and negative eigenvalues for $L_{T^*_M X}(V, p)$.

Let us recall the following classical result (cf. also [D'A-Z] for further developments):

THEOREM 2.1. (cf. [K-S1, Theorem 3.1]). *Let $\mathcal{M}$ be a left coherent $\mathcal{E}_X$-module satisfying* (2.1) *and assume that for an open subset $W \subset V \cap T^*_M X$*

(i) *$s^-(M, V, p)$ is constant, say $s^-(M, V)$, on W.*

Then the complex $R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)$ of microfunction solutions of the system $\mathcal{M}$ is concentrated in degree $s^-(M, V)$ on W.

As we will recall in the next section, in the proof of Theorem 2.1 given in [K-S1], the complex $R\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)$ is "reduced" to a complex $\mu_N(\mathcal{O}_Y)[1-d]$ of microfunctions along a real analytic hypersurface N of a complex analytic manifold Y (for a shift $1-d$ which is calculated). Concerning such a complex of microfunctions, we will need the following result.

THEOREM 2.2. (cf. [S-T, Theorem 3.2]). *Let ω be an open subset of $T^*_N Y$ and assume:*

(i) *$s^-(N, q)$ is constant, say $s^-(N)$, on ω,*

(ii) *there is no germ of complex curve in ω.*

Then:

(a) *the complex $\mu_N(\mathcal{O}_Y)$ is concentrated in degree $s^-(N)$,*

(b) *the sheaf $H^{s^-(N)}(\mu_N(\mathcal{O}_Y))$ is conically flabby on ω.*

2.2. Set $\widetilde{T^*_M X} = \cup_{p \in V \cap T^*_M X} \Sigma_p$, where Σ_p denotes the leaf of V issued from p.

We can now state our main theorem.

THEOREM 2.3. *Let $\mathcal{M}$ be a left coherent $\mathcal{E}_X$-module satisfying* (2.1) *and assume that for an open subset $W \subset V \cap T^*_M X$:*

(i) *$s^-(M, V, p)$ is constant, say $s^-(M, V)$, on W,*

(ii) *for any germ γ of complex curve at $p \in W$ contained in $\widetilde{T^*_M X}$, one has $\gamma \subset \Sigma_p$.*

Then the sheaf $\mathcal{E}xt^{s^-(M,V)}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)$ is isotropically flabby on W.

In particular we will recover the following result.

COROLLARY 2.4. (cf. [S-K-K, Chapter III, Theorem 2.3.6]). *Let $\mathcal{M}$ be a left coherent $\mathcal{E}_X$-module satisfying* (2.1) *and assume that for an open subset $W \subset V \cap T^*_M X$:*

(i) *$L_{T^*_M X}(V)(p)$ is non degenerate.*

Then the sheaf $\mathcal{E}xt^{s^-(M,V)}_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_M)$ *is conically flabby on* W.

3. - Proof of the results

We keep the same notations as in §2.

3.1. We first need some preparation to the proof of Theorem 2.3. The first part of our proof will follow the argument used by Kashiwara and Schapira in [K-S1] in order to prove Theorem 2.1. Their method consists in "reducing" the germ of $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_M)$ at $p \in W$ to a germ of the complex $\mu_N(\mathcal{O}_Y)[1-d]$ of microfunctions along a real analytic hypersurface N of a complex analytic manifold Y (for a shift $1-d$ which is calculated).

We need to refine here their procedure since, due to the definition of isotropic flabbiness, one has to deal with the restriction of $\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_M)$ to an open neighbourhood Ω of p and not only with its germ at p.

3.2. It follows from the proof of [K-S1, Theorem 3.1] that one has:

PROPOSITION 3.1. *For each* $p \in W$ *there exist an open neighbourhood* $\Omega \subset T^*X$ *of* p, *a contact transformation* $\chi : T^*X \to T^*X$ *defined on* Ω *and a smooth map* $f : X \to Y$ *of complex analytic manifolds such that:*

$$\begin{cases} \chi(V) = X \times_Y T^*Y, \\ \chi(T^*_M X) = T^*_S X, \ \textit{S being the real analytic boundary of a strictly} \\ \textit{pesudo-convex set,} \\ f_\pi((X \times_Y T^*Y) \cap T^*_S X) = T^*_N Y, \ \textit{N being a real analytic hypersurface} \\ \textit{of Y.} \end{cases}$$

Moreover, a quantization of χ *gives an isomorphism:*

$$\chi_*(\mathrm{R}\mathcal{H}om_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_M)) \cong \mu\mathrm{hom}(\mathbb{C}_S, f^{-1}\mathcal{O}_Y)[1] \ \textit{in} \ \mathrm{D}^b(\chi(\Omega)).$$

Let $\tilde{p} = \chi(p)$ for $p \in W \cap \Omega$. If $\psi(x) = 0$ is a local equation for S at $x_\circ$ with $d\psi(x_\circ) = \tilde{p}$, we set $S_p = \{x \in X; \psi(x) \geq 0\}$. We similarly define N_q for $q = f_\pi(\tilde{p})$.

Let us use the following notations:

$$\begin{aligned} \lambda_0(\tilde{p}) &= T_{\tilde{p}}\pi^{-1}\pi(\tilde{p}), \\ \lambda_S(\tilde{p}) &= T_{\tilde{p}}T^*_S X, \\ \rho(\tilde{p}) &= (T_{\tilde{p}}(\chi(V)))^\perp, \\ \tilde{\lambda}_S(\tilde{p}) &= (\lambda_S(\tilde{p}) \cap \rho(\tilde{p})^\perp) + \rho(\tilde{p}). \end{aligned}$$

Moreover we set

$$d = \frac{1}{2}[\tau(\lambda_0(\tilde{p}), \tilde{\lambda}_S(\tilde{p}), \lambda_S(\tilde{p})) + 2l - \dim^{\mathbb{R}}(\lambda_S(\tilde{p}) \cap \rho(\tilde{p}))],$$

where $l = \operatorname{codim}^{\mathbb{C}} V = \dim^{\mathbb{C}} X - \dim^{\mathbb{C}} Y$ and $\tau(\cdot, \cdot, \cdot)$ denotes the Maslov index of three Lagrangian planes in $T_{\tilde{p}}T^*X$.

LEMMA 3.2. (cf. [K-S1, Proposition 7.4.2]). *There exists a fundamental system of open neighbourhoods $U \subset X$ of $x_\circ$ such that the natural morphism:*

$$\mathrm{R}f_!\mathbb{C}_{S_p \cap U} \to \mathbb{C}_{N_q \cap f(U)}[l - d]. \tag{3.1}$$

is an isomorphism in $\mathrm{D}^b(f(U))$.

3.3. Given $p \in W$ let us choose an open neighbourhood $\Omega \subset T^*X$ of p and an open neighbourhood $U \subset X$ of $f_\pi(\chi(p))$ such that:

(i) Ω satisfies the requirements of Proposition 3.1,
(ii) U satisfies the requirements of Lemma 3.2,
(iii) $\chi(\Omega) \subset \pi^{-1}(U)$,
(iv) $\chi(\Omega) \cap \mathrm{SS}(S_p)^a = \emptyset$,
(v) $\mathbb{C}_{S_p \cap U} \cong \mathbb{C}_S$ in $\mathrm{D}^b(X; \chi(\Omega))$,
(vi) $\mathbb{C}_{N_q \cap f(U)} \cong \mathbb{C}_N$ in $\mathrm{D}^b(Y; f_\pi(\chi(\Omega)))$,

where $(\cdot)^a$ denotes the antipodal map.

PROPOSITION 3.3. *The natural morphism in* $\mathrm{D}^b(\chi(\Omega))$*:*

$$f_\pi^{-1}\mu\mathrm{hom}(\mathbb{C}_N, \mathcal{O}_Y)[d] \to {}^t f'^{-1}\mu\mathrm{hom}(\mathbb{C}_S, f^{-1}\mathcal{O}_Y \otimes \omega_{X/Y})[l], \tag{3.2}$$

induced by (3.1), *is an isomorphism on* $\chi(W \cap \Omega)$.

Let us first explain how (3.2) is induced by (3.1). The natural morphism

$$\mathrm{R}{}^t f'_! f_\pi^{-1}\mu\mathrm{hom}(\mathrm{R}f_!\mathbb{C}_{S_p \cap U}, \mathcal{O}_Y) \to \mu\mathrm{hom}(\mathbb{C}_{S_p \cap U}, f^{-1}\mathcal{O}_Y \otimes \omega_{X/Y})$$

induces by adjunction

$$f_\pi^{-1}\mu\mathrm{hom}(\mathrm{R}f_!\mathbb{C}_{S_p \cap U}, \mathcal{O}_Y) \to {}^t f'^{-1}\mu\mathrm{hom}(\mathbb{C}_{S_p \cap U}, f^{-1}\mathcal{O}_Y \otimes \omega_{X/Y}). \tag{3.3}$$

From (3.1) and (3.3) we get the morphisms

$$\begin{aligned} f_\pi^{-1}\mu\mathrm{hom}(\mathbb{C}_{N_q \cap f(U)}, \mathcal{O}_Y)[d] &\to f_\pi^{-1}\mu\mathrm{hom}(\mathrm{R}f_!\mathbb{C}_{S_p \cap U}, \mathcal{O}_Y)[l] \\ &\to {}^t f'^{-1}\mu\mathrm{hom}(\mathbb{C}_{S_p \cap U}, f^{-1}\mathcal{O}_Y \otimes \omega_{X/Y})[l] \end{aligned}$$

which give (3.2) owing to (v) and (vi) above.

PROOF OF PROPOSITION 3.3. For each $p' \in \chi(W \cap \Omega)$ and $j \in \mathbb{N}$ we have to prove the isomorphism

$$H^j(f_\pi^{-1}\mu\mathrm{hom}(\mathbb{C}_N, \mathcal{O}_Y)[d])_{p'} \cong H^j(\mu\mathrm{hom}(\mathbb{C}_S, f^{-1}\mathcal{O}_Y \otimes \omega_{X/Y})[l])_{p'}.$$

Setting $q' = f_\pi(p')$ one concludes by considering the following chain of isomorphisms:

$$\begin{aligned} H^j(\mu\mathrm{hom}(\mathbb{C}_N, \mathcal{O}_Y)[d])_{q'} &\cong H^j(\mu\mathrm{hom}(\mathbb{C}_{N_q \cap f(U)}, \mathcal{O}_Y)[d])_{q'} \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(Y;q')}(\mathrm{R}f_!\mathbb{C}_{S_p \cap U}, \mathcal{O}_Y)[l+j] \\ &\cong \mathrm{Hom}_{\mathrm{D}^b(X;p')}(\mathbb{C}_{S_p \cap U}, f^!\mathcal{O}_Y)[l+j] \\ &\cong H^j(\mu\mathrm{hom}(\mathbb{C}_{S_p \cap U}, f^!\mathcal{O}_Y)[l])_{p'} \\ &\cong H^j(\mu\mathrm{hom}(\mathbb{C}_S, f^{-1}\mathcal{O}_Y \otimes \omega_{X/Y})[l])_{p'}. \end{aligned}$$

Q.E.D

3.4. Let $\omega = f_\pi(\chi(W \cap \Omega))$. For each $p \in W \cap \Omega$ let $q = f_\pi(\chi(p))$.

PROOF OF THEOREM 2.3. One has (cf. [K-S1]):

$$s^-(N,q) = s^-(M,V,p) - d. \tag{3.4}$$

The hypothesis (i) of Theorem 2.3 and the equality (3.4) imply that $s^-(N,q)$ is constant, say $s^-(N)$, for $q \in \omega$. Combining Proposition 3.1 and 3.3 we get:

$$\chi_*(\mathcal{E}xt^{s^-(M,V)}_{\mathcal{E}_X}(\mathcal{M}, \mathcal{C}_M)|_{W\cap\Omega}) \cong f_\pi^{-1}(H^{s^-(N)}\mu_N(\mathcal{O}_Y))|_{\chi(W\cap\Omega)}. \tag{3.5}$$

In order to prove Theorem 2.3 we are then reduced to show the flabbiness of $H^{s^-(N)}\mu_N(\mathcal{O}_Y)$ on ω.

The hypothesis (ii) of Theorem 2.3 is a symplectic invariant and, after application of χ, it reads:

(ii)$_\chi$ If γ is a germ of complex curve in $f_\pi^{-1}(T_N^*Y)$ at $\tilde{p} \in \chi(W \cap \Omega)$, then $\gamma \subset f_\pi^{-1}f_\pi(\tilde{p})$.

It follows that there is no germ of complex curve in ω and hence the conical flabbiness of $H^{s^-(N)}\mu_N(\mathcal{O}_Y)$ is obtained by applying Theorem 2.2.

Q.E.D.

PROOF OF COROLLARY 2.4. The totally isotropic space of $L_{T_M^*X}(V,p)$ has dimension $\dim^{\mathbb{R}}(\rho(p) \cap \lambda_M(p)) + \dim^{\mathbb{C}}(\lambda_N(q) \cap i\lambda_N(q))$ (cf. [K-S1, Lemma 3.4]).

Since $L_{T_M^*X}(V,p)$ is non-degenerate, one has

$$\lambda_N(q) \cap i\lambda_N(q) = \{0\}, \tag{3.6}$$

$$\rho(p) \cap \lambda_M(p) = \{0\}. \tag{3.7}$$

From (3.6) one deduces that the Levi form of $N \subset Y$ at q is non-degenerate. It follows that there is no germ of complex curve on ω, that $s^-(N,q)$ is constant (say $s^-(N)$) on ω and that, due to (3.4), $s^-(M,V,p)$ is constant (say $s^-(M,V)$) on W. The isotropic flabiness of $\mathcal{E}xt^{s^-(M,V)}_{\mathcal{E}_X}(\mathcal{M},\mathcal{C}_M)$ is then deduced as in the proof of Theorem 2.3.

One concludes noticing that (3.7) implies the injectivity of $f_\pi|_{X(W\cap\Omega)}$ and hence isotropically flabby means in fact conically flabby. Q.E.D.

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REFERENCES

[D'A-Z] A. D'Agnolo - G. Zampieri, *Generalized Levi forms for microdifferential systems*, in "D-modules and microlocal geometry", T. Monteiro Fernandes, K. Kashiwara, P. Schapira eds., Gruyter publ. (1992).

[K-S1] M. Kashiwara - P. Schapira, *A vanishing theorem for a class of systems with simple characteristics*, Invent. Math. **82** (1985), 579-592.

[K-S2] M. Kashiwara - P. Schapira, *Sheaves on manifolds*, 292, Springer-Verlag (1990).

[S-K-K] M. Sato - T. Kawai - M. Kashiwara, *Hyperfunctions and pseudo-differential equations*, Lecture Notes in Math., Springer-Verlag **287** (1973), 265-529.

[S-T] P. Schapira - J.M. Trépreau, *Microlocal pseudoconvexity and "edge of the wedge" theorem*, Duke Math. J. **61**, **1** (1990), 105-118.

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