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# Regularized Semigroups and Systems of Linear Partial Differential Equations

M. HIEBER\* - A. HOLDERRIETH\*\* - F. NEUBRANDER

## 1. - Introduction

The class of “semigrupperi regularizzabili” was introduced by G. Da Prato [DaP1] in 1966. About twenty years later, this class was rediscovered independently by B. Davies and M. Pang [D-P] who called it “exponentially bounded  $C$ -semigroups”. Because Da Prato’s notion of a regularized (or regularizable) semigroup is more descriptive than  $C$ -semigroups, this term will be used here. Over the last few years the theory of regularized semigroups was further developed by R. deLaubenfels, I. Miyadera and N. Tanaka (see e.g. [deL1]–[deL4], [M-T], [Ta]). Further generalizations and extensions can be found in [DeL5], [Lu] and [T-O].

In the Sections 2 and 3 of this paper the Laplace transform approach is taken to introduce the concept and basic theory of regularized (or regularizable) semigroups. Although some of the results in these sections are not surprising to the expert, the presentation of the material clarifies, as we believe, the concepts substantially.

One of the main reasons to study regularized semigroups is their flexibility in applications to evolution equations (see e.g. [Ar2], [A-K], [D-P], [deL1]–[deL4], [Hi1], [Hi2], [H-R], [Ne2], [Pa], [Ta]). We will demonstrate this in Section 4, where it will be shown that there is a one-to-one correspondence between constant coefficient differential operators generating regularized semigroups and the Petrovskii correctness of the associated system. Hence, in studying evolution equations and, in particular, partial differential equations by functional analytic means, the theory of regularized semigroups appears to be an appropriate tool.

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**2. - Regularized semigroups**

Instead of starting with a definition, we first try to convince the sceptical reader that regularized semigroups are natural objects to look at if one studies abstract Cauchy problems

$$(ACP) \quad u'(t) = Au(t), \quad u(0) = x,$$

for closed operators  $A$  with domain  $D(A)$  and range  $Im(A)$  in a Banach space  $E$ .

It has to be emphasized that the operator  $A$  may have empty resolvent set  $\rho(A)$  and that  $D(A)$  may not be dense in  $E$ . Such operators occur frequently in the study of systems of linear partial differential equations (see Section 4).

The following theorem is the main result of this section. Assuming only the closedness of the operator  $A$ , it characterizes the existence of a global, exponentially bounded "integral solution" of  $(ACP)$ . Integrating  $(ACP)$  with respect to time one obtains

$$(ACP_0) \quad u(t) = A \int_0^t u(s)ds + x.$$

Clearly, a solution of  $(ACP_0)$  has to be less regular in time compared to a solution of  $(ACP)$ . Integrating once more and setting  $v(t) := \int_0^t u(s)ds$  yields

$$(ACP_1) \quad v(t) = A \int_0^t v(s)ds + tx.$$

Continuing in this manner gives the  $(n + 1)$ -times integrated Cauchy problem

$$(ACP_n) \quad v(t) = A \int_0^t v(s)ds + \frac{t^n}{n!} x.$$

Any function  $v(\cdot) \in C([0, \infty), E)$  satisfying  $(ACP_n)$  for all  $t \geq 0$  is called an "integral solution" of  $(ACP)$ . Clearly, if  $u(\cdot)$  solves  $(ACP)$ , then the  $n$ -th antiderivative

$$v(t) := \int_0^t \frac{(t - s)^{(n-1)}}{(n - 1)!} u(s)ds$$

solves  $(ACP_n)$ . Also, if  $v(\cdot)$  solves  $(ACP_n)$  and is  $(n + 1)$ -times continuously differentiable, then  $u(\cdot) := v^{(n)}(\cdot)$  solves  $(ACP)$ .

**THEOREM 2.1.** *Let  $A$  be a closed operator on a Banach space  $E$  and let  $n \in \mathbb{N}_0$ . Let  $v(\cdot) \in C([0, \infty), E)$  with  $\|v(t)\| \leq Me^{wt}$  for some  $M, w \geq 0$  and all*

$t \geq 0$ . Then  $v(\cdot)$  solves the integrated Cauchy problem  $(ACP_n)$  for the initial value  $x \in E$  if and only if

- (a) For all  $\lambda \in H_w := \{\lambda \in \mathbb{C}; \operatorname{Re} \lambda > w\}$  there exists a solution  $y = y(\lambda)$  of the pointwise resolvent equation  $\lambda y - Ay = x$ , and
- (b)  $y(\lambda) = \lambda^n \int_0^\infty e^{-\lambda t} v(t) dt$  for all  $\lambda \in H_w$ .

PROOF. “ $\Leftarrow$ ”. Define  $v_0(t) := \int_0^t v(s) ds$  and  $v_1(t) := \int_0^t v_0(s) ds$ . Integrating by parts yields  $y(\lambda)/\lambda^{n+j} = \lambda \int_0^\infty e^{-\lambda t} v_j(t) dt$  for all  $\lambda > \tilde{w}$ , where  $\tilde{w} := w + \epsilon$  and  $\epsilon > 0$  ( $j = 0, 1$ ). Because the functions  $v_j(\cdot)$  are Lipschitz continuous, the complex inversion theorem of Laplace transform theory is applicable (see e.g. [H-N; Corollary 3.3]). This implies that  $v_j(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} y(\lambda)/\lambda^{n+1+j} d\lambda$  for all  $t \geq 0$ , where  $\Gamma$  is the path  $\{\tilde{w} + i\tau, \tau \in \mathbb{R}\}$ . Since  $A$  is closed we obtain  $\int_{\tilde{w}-iN}^{\tilde{w}+iN} e^{\lambda t} Ay(\lambda)/\lambda^{n+2} d\lambda = A \int_{\tilde{w}-iN}^{\tilde{w}+iN} e^{\lambda t} y(\lambda)/\lambda^{n+2} d\lambda$  for all  $N > 0$ . Again, the closedness of  $A$  implies that that  $v_1(t) \in D(A)$  for all  $t \geq 0$  and

$$\begin{aligned} Av_1(t) &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \frac{Ay(\lambda)}{\lambda^{n+2}} d\lambda \\ &= \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} \frac{1}{\lambda^{n+2}} (\lambda y(\lambda) - x) d\lambda = v_0(t) - \frac{t^{n+1}}{(n+1)!} x. \end{aligned}$$

Using the differentiability of  $v_j(\cdot)$  and the closedness of  $A$ , one obtains

$$\frac{d}{dt} v_1(t) = v_0(t) = \int_0^t v(s) ds \in D(A) \quad \text{and} \quad A \int_0^t v(s) ds = v(t) - \frac{t^n}{n!} x$$

for all  $t \geq 0$ .

“ $\Rightarrow$ ”. Let  $\lambda \in H_w$ . Since  $A$  is closed and

$$\frac{1}{\lambda} \int_0^\infty e^{-\lambda t} v(t) dt = \int_0^\infty e^{-\lambda t} \int_0^t v(s) ds dt$$

it follows that  $\int_0^\infty e^{-\lambda t} v(t) dt \in D(A)$  and

$$\begin{aligned}
 A \left( \frac{1}{\lambda} \int_0^\infty e^{-\lambda t} v(t) dt \right) &= \int_0^\infty e^{-\lambda t} A \int_0^t v(s) ds dt \\
 &= \int_0^\infty e^{-\lambda t} \left( v(t) - \frac{t^n}{n!} x \right) dt = \int_0^\infty e^{-\lambda t} v(t) dt - \frac{1}{\lambda^{n+1}} x.
 \end{aligned}$$

Define  $y(\lambda) := \lambda^n \int_0^\infty e^{-\lambda t} v(t) dt$ . Then  $y(\lambda) \in D(A)$  and  $\lambda y(\lambda) - Ay(\lambda) = x$  for all  $\lambda \in H_w$ . □

REMARK. Using the complex inversion formula of Laplace transform theory (see [A-K, Proposition 3.1]), the statements (a) and (b) of Theorem 2.1 can be reformulated. They hold for some  $n \in \mathbb{N}_0$  if and only if there exists a polynomial  $p(\cdot)$  such that, for all  $\lambda \in H_w$ , there exists a solution  $y = y(\lambda)$  of the equation  $\lambda y - Ay = x$  such that  $\|y(\lambda)\| \leq p(|\lambda|)$  for all  $\lambda \in H_w$ .

As Theorem 2.1 shows, the existence of global, exponentially bounded integral solutions of (ACP) depends on two main factors:

- (1) The initial value  $x$  has to be in the intersection of the images of the operators  $(\lambda - A)$  for  $\lambda$  in the halfplane  $H_w$ ; that is,  $x \in \bigcap_{\lambda \in H_w} \text{Im}(\lambda - A)$ . This is a *range condition* for the operator  $A$ .
- (2) The pointwise resolvent  $y(\lambda)$  has to be Laplace representable if it is dampened by a large enough polynomial factor  $\lambda^n$ . This is a *growth or representability condition* on the pointwise resolvent  $y(\lambda)$ .

We are only interested in closed operators  $A$  for which the solution of (ACP) is unique. Again it follows from Theorem 2.1 that uniqueness of global, exponentially bounded integral solutions is implied by the following *point spectral condition*.

- (3) There exists  $w \in \mathbb{R}$  such that the intersection of the point spectrum  $p\sigma(A)$  of the operator  $A$  with the right halfplane  $H_w$  is empty.

In order to put the range condition (1) in a functional analytic framework, it is useful to introduce the notion of a “regularizing operator” (see also [DaP1], [D-P], [deL4]).

DEFINITION 2.2. Let  $A$  be a closed, linear operator on a Banach space  $E$  and let  $\Omega$  be an open, nonempty subset of  $\mathbb{C}$  with  $p\sigma(A) \cap \Omega = \emptyset$ . A bounded operator  $C$  is called a *regularizing operator for  $A$  on  $\Omega$*  if

- (a)  $C$  is injective,  $CAx = ACx$  for all  $x \in D(A)$ , and  $\text{Im } C \subset \bigcap_{\lambda \in \Omega} \text{Im}(\lambda - A)$ .
- (b) The  $\mathcal{L}(E)$ -valued function  $\lambda \mapsto (\lambda - A)^{-1}C$  is holomorphic on  $\Omega$ .

REMARK. Note that  $(\lambda - A)^{-1}$  is a closed operator with domain

$$D((\lambda - A)^{-1}) = \text{Im}(\lambda - A)$$

for all  $\lambda \in \Omega$ . Since  $C \in \mathcal{L}(E)$  and  $\text{Im} C \subset D((\lambda - A)^{-1})$  it follows that  $(\lambda - A)^{-1}C \in \mathcal{L}(E)$  for all  $\lambda \in \Omega$ . Also,  $(\lambda - A)^{-1}Cx = C(\lambda - A)^{-1}x$  for all  $x \in \text{Im}(\lambda - A)$ .

LEMMA 2.3. *Let  $C$  be a regularizing operator for  $A$  on  $\Omega$ . Then the following holds for all  $\lambda, \lambda_0 \in \Omega, \lambda \neq \lambda_0$ .*

$$(a) \quad \begin{aligned} (\lambda_0 - A)^{-1}(\lambda - A)^{-1}C &= (\lambda - A)^{-1}(\lambda_0 - A)^{-1}C \\ &= \frac{(\lambda_0 - A)^{-1}C - (\lambda - A)^{-1}C}{\lambda - \lambda_0}. \end{aligned}$$

(b)  $Cx \in D((\lambda - A)^{-n})$  for all  $x \in E$  and all  $n \in \mathbb{N}$ .

(c)  $(\lambda - A)^{-n}C \in \mathcal{L}(E)$  for all  $n \in \mathbb{N}$ .

PROOF. For  $x \in E$  define

$$y := \frac{(\lambda_0 - A)^{-1}Cx - (\lambda - A)^{-1}Cx}{\lambda - \lambda_0}.$$

Then  $y \in D(A)$ . It follows from  $A(\lambda - A)^{-1}C = \lambda(\lambda - A)^{-1}C - C \in \mathcal{L}(E)$  that  $(\lambda - A)y = (\lambda_0 - A)^{-1}Cx$  and  $(\lambda_0 - A)y = (\lambda - A)^{-1}Cx$ . Hence,  $(\lambda_0 - A)^{-1}Cx \in \text{Im}(\lambda - A) = D((\lambda - A)^{-1})$  and  $y = (\lambda_0 - A)^{-1}(\lambda - A)^{-1}Cx = (\lambda - A)^{-1}(\lambda_0 - A)^{-1}Cx$ . This proves (a). The closedness of the operator  $(\lambda_0 - A)^{-1}$  implies that  $\frac{d}{d\lambda}(\lambda - A)^{-1}Cx = -(\lambda - A)^{-2}Cx$  for all  $x \in E$ . The assumed analyticity of  $\lambda \mapsto (\lambda - A)^{-1}C$  implies that  $\frac{d}{d\lambda}(\lambda - A)^{-1}C = -(\lambda - A)^{-2}C \in \mathcal{L}(E)$  for all  $\lambda \in \Omega$ . This proves the statements (b) and (c) for  $n = 1, 2$ .

Next define  $C_1 : x \mapsto (\lambda_0 - A)^{-1}Cx$ . Then  $C_1$  is a regularizing operator for  $A$  on  $\Omega$ . The same arguments as above yield  $(\lambda_0 - A)^{-1}C_1x = (\lambda_0 - A)^{-2}Cx \in \text{Im}(\lambda - A)$  for all  $\lambda \in \Omega$  and the validity of the resolvent equation (a) in terms of the operator  $C_1$ . Therefore,  $Cx \in D((\lambda_0 - A)^{-3})$ . The analyticity of  $\lambda \mapsto (\lambda - A)^{-1}C_1$  implies that  $\frac{d}{d\lambda}(\lambda - A)^{-1}C_1 = -(\lambda - A)^{-2}C_1 \in \mathcal{L}(E)$  for all  $\lambda \in \Omega$ . In particular,  $(\lambda_0 - A)^{-2}C_1 = (\lambda_0 - A)^{-3}C \in \mathcal{L}(E)$ . Because  $\lambda_0$  was chosen arbitrarily, the statements (b) and (c) are proven for  $n = 1, 2, 3$ . Continuing this way by defining  $C_{n+1} : x \mapsto (\lambda_0 - A)^{-1}C_nx$  finishes the proof.  $\square$

We now return to Theorem 2.1 and the abstract Cauchy problem (ACP). In order to meet the *point spectral condition* (3), we assume from now on that there exists a regularizing operator  $C$  for  $A$  on an open right halfplane  $H_w$ . Then, for all  $x \in E$  and  $\lambda \in H_w$ , the equation  $\lambda y - Ay = Cx$  has a unique solution  $y = y(\lambda) = (\lambda - A)^{-1}Cx$ .

Assume that there exists an  $n \in \mathbb{N} \cup \{0\}$  such that, for all  $x \in E$  and  $\lambda \in H_w$ , the regularized resolvent  $\lambda \mapsto (\lambda - A)^{-1}Cx$  has a Laplace representation

$$(\lambda - A)^{-1}Cx = \lambda^n \int_0^\infty e^{-\lambda t} v(t, x) dt,$$

where  $v(\cdot, x)$  is a continuous function with  $\|v(t, x)\| \leq M_x e^{wt}$ .

Then, by Theorem 2.1, the function  $v(\cdot, x)$  is the solution of  $(ACP_n)$  for the initial value  $Cx$ . Define, for all  $t \geq 0$ , linear operators  $S(t)$  by  $S(t)x := v(t, x)$ . In order to obtain continuous dependence of the integral solutions  $v(\cdot, x)$  on the initial values  $x$ , we have to assume that the operators  $S(t)$  are in  $\mathcal{L}(E)$  for all  $t \geq 0$ . The continuity of  $t \mapsto v(t, x)$  implies the strong continuity of the operator family  $(S(t))_{t \geq 0}$ . The exponential boundedness of  $v(t, x)$  leads to  $\|S(t)\| \leq M e^{wt}$  for all  $t \geq 0$  (by the principle of uniform boundedness).

We summarize the above assumptions in the following definition (see also [deL3; Definition 4.1] and [Mi]).

DEFINITION 2.4. Let  $A$  be a closed operator on a Banach space  $E$  with a regularizing operator  $C$  on a right halfplane  $H_w$ . Let  $(S(t))_{t \geq 0}$  be a strongly continuous family of bounded operators with  $\|S(t)\| \leq M e^{wt}$  for some  $M, w > 0$  and all  $t \geq 0$ . If, for some  $n \in \mathbb{N} \cup \{0\}$ , all  $\lambda \in H_w$ , and all  $x \in E$ ,

$$(2.1) \quad (\lambda - A)^{-1}Cx = \lambda^n \int_0^\infty e^{-\lambda t} S(t)x dt,$$

then  $A$  is called the *generator of an  $n$ -times integrated,  $C$ -regularized semigroup*. If  $n = 0$ , then  $(S(t))_{t \geq 0}$  is called a *regularized (or  $C$ -regularized) semigroup*.

REMARK. The theory of integrated, regularized semigroups comprises all classes of semigroups connected with “correctly posed” Cauchy problems. The following operators are generators of regularized semigroups.

- (a) Generators of strongly continuous semigroups ( $C = I, n = 0$ ).
- (b) Generators of  $n$ -times integrated semigroups or, equivalently, generators of regular, exponentially bounded distribution semigroups ( $C = I, n \in \mathbb{N} \cup \{0\}$ ; see e.g. [Ar1], [Ne1]).
- (c) Generators of semigroups of growth order  $\alpha$  ( $C = (\lambda_0 - A)^{-(k+1)}$  where  $k$  is the integral part of  $\alpha > 0$ , and  $n = 0$  (see e.g. [DaP2], [D-P])).

The crucial properties of integrated, regularized semigroups are collected in the following lemma (see also [Ar1; Proposition 3.3], [deL3], [Ne1; Lemma 5.1], [Mi]).

LEMMA 2.5. Let  $A$  be the generator of an  $n$ -times integrated,  $C$ -regularized semigroup  $(S(t))_{t \geq 0}$ . Then the operators  $S(t)$  have the following properties for all  $t \geq 0$ .

(P<sub>0</sub>)  $S(t)C = CS(t)$ .

(P<sub>1</sub>)  $S(t)x \in D(A)$  and  $AS(t)x = S(t)Ax$  for all  $x \in D(A)$ .

(P<sub>2</sub>)  $\int_0^t S(s)x ds \in D(A)$  and  $A \int_0^t S(s)x ds = S(t)x - \frac{t^n}{n!}Cx$  for all  $x \in E$ .

(P<sub>3</sub>)  $S(t)x = \int_0^t S(s)Ax ds + \frac{t^n}{n!}Cx$  for all  $x \in D(A)$ .

(P<sub>4</sub>)  $S(0) = C$  if  $n = 0$  and  $S(0) = 0$  if  $n \geq 1$ .

PROOF. (P<sub>0</sub>): Let  $x \in E$ . Since  $C(\lambda - A)^{-1}Cx = (\lambda - A)^{-1}CCx$ , we obtain

$$\lambda^n \int_0^\infty e^{-\lambda t} CS(t)x dt = \lambda^n \int_0^\infty e^{-\lambda t} S(t)Cx dt.$$

The uniqueness theorem for Laplace transforms (see e.g. [H-N; Corollary 1.4]) implies that  $CS(t)x = S(t)Cx$ .

(P<sub>2</sub>): This follows immediately from Theorem 2.1.

(P<sub>3</sub>): Let  $x \in D(A)$ . Then the uniqueness theorem for Laplace transforms and

$$\begin{aligned} \lambda^{n+1} \int_0^\infty e^{-\lambda t} \frac{t^n}{n!} Cx dt &= Cx = (\lambda - A)^{-1}C(\lambda - A)x \\ &= \lambda^n \int_0^\infty e^{-\lambda t} S(t)(\lambda - A)x dt \\ &= \lambda^{n+1} \int_0^\infty e^{-\lambda t} S(t)x dt - \lambda^{n+1} \int_0^\infty e^{-\lambda t} \int_0^t S(s)Ax ds dt \end{aligned}$$

gives  $\frac{t^n}{n!}Cx = S(t)x - \int_0^t S(s)Ax ds$ .

(P<sub>1</sub>): This statement follows from (P<sub>2</sub>), (P<sub>3</sub>) and the closedness of  $A$ .

(P<sub>4</sub>): This statement follows immediately from (P<sub>2</sub>). □

### 3. - Characterizations

If  $A$  generates an integrated,  $C$ -regularized semigroup, then it follows from the property (P<sub>2</sub>) that (ACP) has integral solutions for all initial values in

$\text{Im}(C)$  which are contained in  $\bigcap_{\lambda \in H_w} \text{Im}(\lambda - A)$ . Next, using Theorem 2.1 and Lemma 2.5, the existence and uniqueness of classical, exponentially bounded solutions of (ACP) can be characterized. The following uniqueness result is only a slight modification of a classical result of Ju.I. Ljubic. The proof (see e.g. [Paz; Section 4.1]) extends to the more general statement below by replacing the resolvent  $R(\lambda, A)$  by  $(\lambda - A)^{-1}C$  and is therefore omitted.

**PROPOSITION 3.1.** *Let  $A$  be a closed operator on a Banach space  $E$  and suppose that  $\Omega$  satisfies the assumptions in Definition 2.2. Let  $(w, \infty) \subset \Omega$  for some  $w \in \mathbb{R}$ . If there exists a regularizing operator  $C$  on  $\Omega$  and a polynomial  $p(\cdot)$  such that  $\|(\lambda - A)^{-1}C\| \leq p(\lambda)$  for all  $\lambda > w$ , then (ACP) has at most one solution  $u(\cdot) \in C^1([0, T], E)$  for any  $0 < T < \infty$ .*

Besides characterizing regularized semigroups in terms of the abstract Cauchy problem (see also [deL3; Chapter 3], or [Ne1; Theorem 4.2]), it is shown next that the class of generators of integrated regularized semigroups coincides with the class of generators of regularized semigroups. For a similar result, see [deL3; Theorem 4.2].

**THEOREM 3.2.** *Let  $A$  be a closed operator on a Banach space  $E$  with  $p\sigma(A) \cap H_w = \emptyset$  for some  $w > 0$ . Then the following statements are equivalent.*

- (i)  *$A$  generates an integrated, regularized semigroup.*
- (ii) *There exists  $n \in \mathbb{N} \cup \{0\}$ ,  $\omega > w$ , and a regularizing operator  $C$  for  $A$  on  $H_w$  such that  $u'(t) = Au(t)$ ,  $u(0) = Cx$  has unique classical solutions for all  $x \in D(A^{n+1})$  which are all  $O(e^{\omega t})$ .*
- (iii)  *$A$  generates a regularized semigroup.*

**PROOF.** “(i)  $\Rightarrow$  (ii)”. Let  $A$  be the generator of an  $n$ -times integrated,  $C$ -regularized semigroup  $(S(t))_{t \geq 0}$ . It follows from  $(P_3)$  that  $S(\cdot)x$  is continuously differentiable for all  $x \in D(A)$  and that  $S'(t)x = S(t)Ax + \frac{t^{n-1}}{(n-1)!}Cx$  if  $n \geq 1$  or  $S'(t)x = AS(t)x$  if  $n = 0$ . It follows that  $S(\cdot)x$  is  $n$ -times continuously differentiable for all  $x \in D(A^n)$  and that

$$(3.1) \quad S^{(n)}(t)x = S(t)A^n x + \sum_{k=0}^{n-1} \frac{t^k}{k!} CA^k x$$

for all  $n \geq 1$ . Moreover,  $S(\cdot)x$  is  $(n+1)$ -times continuously differentiable for all  $x \in D(A^{n+1})$  and  $S^{(n+1)}(t)x = AS^{(n)}(t)x$ ,  $S^{(n)}(0)x = Cx$ .

It remains to be shown that the solutions of (ACP) are unique. It follows from the Laplace representation of  $(\lambda - A)^{-1}Cx$  that  $\|(\lambda - A)^{-1}Cx\|$  is polynomially bounded for all  $x \in E$  and all  $\lambda > 2w$ . Now the uniqueness follows from Proposition 3.1.

“(ii)  $\Rightarrow$  (iii)”. Define  $C_1 := C(\lambda_0 - A)^{-n}C$ . Then  $C_1$  is a regularizing operator for  $A$  on  $H_w$  and the initial value problem  $u'(t) = Au(t)$ ,  $u(0) = C_1x$ , has unique solutions  $u(\cdot, C_1x)$  for all  $x \in E$  which are  $O(e^{\omega t})$ . Let  $T > 0$  and

$[D(A)]$  be the Banach space  $D(A)$  endowed with the graph- norm. Define an operator  $K : E \rightarrow C([0, T], [D(A)])$  by  $Kx := u(t, C_1x)$ . Then  $K$  is closed, thus bounded. It follows that there exists  $M_T > 0$  such that

$$\|Kx\| = \sup_{t \in [0, T]} \{\|u(t, C_1x)\| + \|Au(t, C_1x)\|\} \leq M_T \|x\|$$

for all  $x \in E$ . Define linear operators on  $E$  by  $S(t) : x \mapsto u(t, C_1x)$ . Then  $S(t) \in \mathcal{L}(E)$  and  $\|S(t)\| \leq M_T$  for all  $t \in [0, T]$ . By assumption, for all  $x \in E$  there exists  $M_x > 0$  such that  $\|e^{-\omega t} S(t)x\| \leq M_x$  for all  $t \geq 0$ . By the principle of uniform boundedness, we obtain a constant  $M > 0$  such that  $\|S(t)\| \leq M e^{\omega t}$  for all  $t \geq 0$ . Clearly,  $(S(t))_{t \geq 0}$  is strongly continuous. It follows immediately from Theorem 2.1 that  $(S(t))_{t \geq 0}$  is a  $C_1$ - regularized semigroup generated by  $A$ . □

REMARK 3.3. We actually proved a more precise result than the statement of the theorem indicates. In fact, if  $A$  generates an  $n$ -times integrated,  $C$ -regularized semigroup  $(S(t))_{t \geq 0}$ , then  $u'(t) = Au(t)$ ,  $u(0) = Cx$ , has a unique solution  $u(\cdot)$  for all  $x \in D(A^{n+1})$  which is given by (3.1). In particular,  $\|u(t)\| \leq \tilde{M} e^{\omega t} (\|A^n x\| + \|Cx\|_{n-1})$  and  $\|u'(t)\| \leq \tilde{M} e^{\omega t} (\|A^{n+1} x\| + \|Cx\|_n)$  for all  $t \geq 0$ , where  $\|z\|_k := \|z\| + \|Az\| + \dots + \|A^k z\|$ .

Next, two resolvent type characterizations of generators of regularized semigroups will be given. The first one follows from the following considerations.

Let  $C$  be a regularizing operator for  $A$  on  $H_w$  for some  $w > 0$ . Set  $F(\lambda) := (\lambda - A)^{-1}C$ . Extending the statements of Lemma 2.3, it can be shown that

$$(-1)^k \frac{1}{k!} F^{(k)}(\lambda) = (\lambda - A)^{-(k+1)} C$$

for all  $\lambda \in H_w$  and  $k \in \mathbb{N} \cup \{0\}$ . Assume that the growth conditions

$$\left\| \frac{1}{k!} (\lambda - w)^{k+1} F^{(k)}(\lambda) \right\| = \|(\lambda - w)^{k+1} (\lambda - A)^{-(k+1)} C\| \leq M$$

are satisfied for all  $\lambda > w$  and all  $k \in \mathbb{N} \cup \{0\}$ . By Widder's representation theorem for Laplace transforms (see [Ar1] or [H-N]), there exists an exponentially bounded, normcontinuous operator family  $(W(t))_{t \geq 0}$  such that

$$(\lambda - A)^{-1} C = \lambda \int_0^\infty e^{-\lambda t} W(t) dt$$

for all  $\lambda > w$ . The analyticity of  $\lambda \mapsto (\lambda - A)^{-1} C$  implies that the above equality holds for all  $\lambda \in H_w$ . It follows from Theorem 3.2 that  $A$  generates a regularized semigroup.

Similarly, if  $C$  is a regularizing operator for  $A$  on  $H_w$  such that  $\|(\lambda - A)^{-1} C\| \leq p(|\lambda|)$  for some polynomial  $p(\cdot)$  and all  $\lambda \in H_w$ , then it

follows from the complex representation theorem for Laplace transforms (see [A-K], Proposition 3.1) that there exists  $n \in \mathbb{N}$  and an exponentially bounded, normcontinuous operator family  $(W(t))_{t \geq 0}$  such that

$$(\lambda - A)^{-1}C = \lambda^n \int_0^\infty e^{-\lambda t} W(t) dt$$

for all  $\lambda \in H_w$ . Again it follows from Theorem 3.2 that  $A$  generates a regularized semigroup. These observations prove the following *Hille-Yosida type characterization* of generators of regularized semigroups (see also [DaP1], [D-P], and [deL3], Theorem 5.7).

**THEOREM 3.4.** *Let  $A$  be a closed operator on a Banach space  $E$  with  $p\sigma(A) \cap H_w = \emptyset$  for some  $w > 0$ . Then the following statements are equivalent.*

- (i)  *$A$  generates a regularized semigroup.*
- (ii) *There exists a regularizing operator  $C$  for  $A$  on  $H_w$  and a polynomial  $p(\cdot)$  such that  $\|(\lambda - A)^{-1}C\| \leq p(|\lambda|)$  for all  $\lambda \in H_w$ .*
- (iii) *There exists a regularizing operator  $C_1$  for  $A$  on  $H_w$  and a constant  $M > 0$  such that  $\|(\lambda - A)^{-k}C_1\| \leq M/(\lambda - w)^k$  for all  $\lambda > w$  and all  $k \in \mathbb{N}$ .*

An immediate consequence of Theorem 3.4 is the following *Lumer-Phillips type characterization* of generators of regularized semigroups.

**COROLLARY 3.5.** *Let  $A$  be a closed operator on a Banach space  $E$  with  $p\sigma(A) \cap H_w = \emptyset$  for some  $w > 0$ . Then  $A$  generates a regularized semigroup if and only if*

- (a) *There exists a regularizing operator  $C$  on  $H_w$ .*
- (b) *There exists a polynomial  $p(\cdot)$  and a function  $F(\cdot) : E \rightarrow [0, \infty)$  such that, for all  $x \in D(A)$  and  $\lambda \in H_w$ ,  $p(|\lambda|)F(\lambda x - Ax) \geq \|Cx\|$ .*

**PROOF.** Let  $z \in E$ . Then  $x := (\lambda - A)^{-1}Cz \in D(A)$  and  $p(|\lambda|)F(Cz) = p(|\lambda|)F(\lambda x - Ax) \geq \|Cx\| = \|(\lambda - A)^{-1}C^2z\|$ . It follows from the uniform boundedness principle that there exists  $M > 0$  such that  $\|(\lambda - A)^{-1}C^2\| \leq Mp(|\lambda|)$  for all  $\lambda \in H_w$ . Since  $C^2$  is a regularizing operator, the statement follows from Theorem 3.4. □

For generators of regularized semigroups there exists a variety of interpolation and extrapolation results (the most general ones can be found in [deL4]; for others, see [A-N-S] and [M-T]). We mention one of them. Using the above characterizations one can reformulate Theorem 1 in [M-T] to the following characterization of regularized semigroups in terms of strongly continuous semigroups.

**THEOREM 3.6.** *Let  $A$  be a closed operator on a Banach space  $E$  with  $p\sigma(A) \cap H_w = \emptyset$  for some  $w > 0$ . Then  $A$  generates a regularized semigroup if and only if there exists a regularizing operator  $C$  for  $A$  on  $H_w$  and a*

continuously embedded Banach space  $Y \hookrightarrow E$  with  $C \in \mathcal{L}(E, Y)$  such that the part of  $A$  in  $Y$  generates a strongly continuous semigroup in  $Y$ .

#### 4. - Systems of partial differential equations in $\mathbb{R}^n$

In this section we investigate initial value problems of the form

$$(4.1) \quad \begin{aligned} \frac{\partial}{\partial t} u(x, t) &= \sum_{|k| \leq m} A_k D^k u(x, t) \quad x \in \mathbb{R}^n, t \geq 0 \\ u(x, 0) &= u_0(x), \end{aligned}$$

where  $D^k$  is defined by  $\left(\frac{\partial}{\partial x_1}\right)^{k_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{k_n}$  and  $A_k \in M_N(\mathbb{C})$ , the ring of all constant  $N \times N$ -matrices over  $\mathbb{C}$ . We are primarily interested in the case where the solution  $u$  belongs to one of the function spaces  $E := L^p(\mathbb{R}^n)^N$  ( $1 \leq p < \infty$ ),  $C^b(\mathbb{R}^n)^N$  or  $C^\alpha(\mathbb{R}^n)^N$  ( $0 < \alpha < 1$ ). Therefore, it is natural to use Fourier methods in order to investigate this problem. We start with some notation.

We denote by  $S^N$  the space of all functions from  $\mathbb{R}^n$  to  $\mathbb{C}^N$  having each component in  $S$ , the space of rapidly decreasing functions. The dual space  $(S^N)'$  of  $S^N$  is the space of tempered distributions. Note that the Fourier transform of matrix valued distributions is defined by applying the transform elementwise. We call an  $L^\infty$ -function  $M : \mathbb{R}^n \rightarrow M_N(\mathbb{C})$  a Fourier multiplier for  $L^p(\mathbb{R}^n)^N$  ( $1 \leq p \leq \infty$ ) if  $\mathcal{F}^{-1}(M\hat{\phi}) \in L^p(\mathbb{R}^n)^N$  for all  $\phi \in S^N$  and if

$$\|M\|_{\mathcal{M}_p^N} := \sup\{\|\mathcal{F}^{-1}(M\hat{\phi})\|_{L^p(\mathbb{R}^n)^N}; \phi \in S^N, \|\phi\|_{L^p(\mathbb{R}^n)^N} \leq 1\} < \infty.$$

The space of all such matrix-valued functions  $M$  is a Banach algebra, denoted by  $\mathcal{M}_p^N$ . The norm of  $\mathcal{M}_p^N$  is the above supremum. For details, we refer to [Hö] or [St]. With a given differential operator  $\sum_{|k| \leq m} A_k D_k$  on  $E$  and its symbol

$$P(\xi) := \sum_{|k| \leq m} A_k (i\xi)^k, \text{ we associate a linear operator } \mathcal{A}_E \text{ on } E \text{ as follows. Set}$$

$$(4.2) \quad \begin{aligned} D(\mathcal{A}_E) &:= \{f \in E; \mathcal{F}^{-1}(P\hat{f}) \in E\} \quad \text{and define} \\ \mathcal{A}_E f &:= \mathcal{F}^{-1}(P\hat{f}) \quad \text{for all } f \in D(\mathcal{A}_E). \end{aligned}$$

Then  $\mathcal{A}_E$  is a closed operator.

We note that the operator  $\mathcal{A}_{L^p}$  ( $p \neq \infty$ ) generates a  $C_0$ -semigroup on  $L^p(\mathbb{R}^n)^N$  if and only if  $e^{tP} \in \mathcal{M}_p^N$  for all  $t \geq 0$  and  $\|e^{tP}\|_{\mathcal{M}_p^N} \leq M e^{\omega t}$  for all  $t \geq 0$  and suitable constants  $M, \omega \in \mathbb{R}$ . In particular,  $\mathcal{A}_{L^2}$  generates a  $C_0$ -semigroup on  $L^2(\mathbb{R}^n)^N$  if and only if  $\sup_{\xi} \|e^{tP(\xi)}\| \leq M e^{\omega t}$  for all  $t \geq 0$ .

Symbols satisfying this condition have been completely characterized by Kreiss [Kr] in terms of properties of  $P(\xi)$ . Note that systems which are wellposed in  $L^2$  need not to be wellposed in  $L^p$ . Nevertheless, if  $\mathcal{A}_{L^2}$  generates a  $C_0$ -semigroup

on  $L^2(\mathbb{R}^n)^N$ , then  $\mathcal{A}_{L^p}$  generates, under suitable assumptions on the matrices  $A_k$ , a  $k$ -times integrated semigroup on  $L^p(\mathbb{R}^n)^N$  whenever  $k > n|1/2 - 1/p|$  (see [Hi2]).

We now prefer to express our conditions for wellposedness of (4.1) in terms of the eigenvalues of the matrix  $P(\xi)$  rather than  $e^{tP(\xi)}$ . More precisely, for an  $N \times N$ -matrix  $M$  with eigenvalues  $\lambda_j, j = 1, \dots, N$ , we define its spectral bound  $\Lambda(M)$  by  $\Lambda(M) := \max_j \operatorname{Re} \lambda_j$ . Then we observe first that a necessary condition for  $\mathcal{A}$  to be the generator of a  $C_0$ -semigroup on  $L^p(\mathbb{R}^n)^N$  is that there exists a constant  $\omega$  such that  $\Lambda(P(\xi)) \leq \omega$  for all  $\xi \in \mathbb{R}^n$ . Considering the operator  $\mathcal{A}$  on  $L^2(\mathbb{R}^2)^2$  given by  $\mathcal{A} = \begin{pmatrix} D^{(1,0)} & D^{(0,1)} \\ 0 & D^{(1,0)} \end{pmatrix}$ , we notice that this condition is far from being sufficient. We even have  $\rho(\mathcal{A}) = \emptyset$ . Nevertheless, we will investigate the problem (4.1) by semigroup methods. Instead of considering  $C_0$ -semigroups on interpolation or intermediate spaces, we will apply the preceding theory.

Petrovskii [Pe] proved in 1938 that  $e^{tP}$  satisfies an estimate  $\|e^{tP(\xi)}\| \leq Ce^{\omega t}(1 + |\xi|^q)$  for all  $\xi \in \mathbb{R}^n$  and suitable  $C, \omega, q$  if and only if  $\Lambda(P(\xi)) \leq C_1 \log(1 + |\xi|) + C_2$  for all  $\xi \in \mathbb{R}^n$  and suitable constants  $C_1$  and  $C_2$ . Later Gårding showed that one can choose  $C_1 = 0$ . Hence, by Parseval’s formula, we can conclude that the solution  $u$  of (4.1) exists for  $u_0 \in H^q$  (the Sobolev space of order  $q$ ) and satisfies  $\|u(t, \cdot)\|_{L^2} \leq Me^{\omega' t}\|u_0\|_{H^q}$  ( $q$  suitable,  $\omega' > \omega$ ) if and only if  $\Lambda P((\xi)) \leq \omega$  for all  $\xi \in \mathbb{R}^n$ .

The following theorem now shows that a differential operator on  $E$ , defined as in (4.2), is the generator of a regularized semigroup with  $C = R(1, \Delta)^\tau$  ( $\tau$  suitable) if and only if  $\Lambda(P(\xi)) \leq \omega$  for all  $\xi \in \mathbb{R}^n$  and some  $\omega$ . Here  $\Delta$  denotes the Laplacian on  $\mathbb{R}^n$ . In order to prove such a result, we make use of the following multiplier results.

For the time being, let  $j, n \in \mathbb{N}, j > n/2, 0 < \alpha < 1$  and  $f \in C^j(\mathbb{R}^n)$ . Assume that there exist constants  $M, L > 0$  such that

- a)  $|D^k f(\xi)| \leq M|\xi|^{-|k|-\beta}$  for some  $\beta > 0$  and all  $\xi \in \mathbb{R}^n$  with  $|\xi| > L$  and all  $k$  with  $|k| \leq j$ . Then  $f \in \mathcal{FL}^1 \subset M_1$  (see [Hi1]; Lemma 3.2).
- b)  $|D^k f(\xi)| \leq M(1 + |\xi|)^{-|k|}$  for all  $\xi \in \mathbb{R}^n$  with  $|\xi| > L$  and all  $k$  with  $|k| \leq j$ . Then there exists a constant  $C_\alpha$  such that, for all  $g \in C^\alpha(\mathbb{R}^n)$ , we have  $\|\mathcal{F}^{-1}(f\hat{g})\|_{C^\alpha} \leq C_\alpha\|g\|_{C^\alpha}$  (see [Tr], p. 30, p. 93).

We also make use of the following matrix-estimate. Let  $\Lambda(P(\xi)) \leq \omega$ . Then, for  $\omega' > \omega$ , there exists a constant  $C$  such that

$$(4.3) \quad \|e^{tP(\xi)}\| \leq Ce^{\omega' t}(1 + |\xi|)^{(N-1)m}$$

for all  $\xi \in \mathbb{R}^n$  and all  $t \geq 0$ . For a proof, see [Fr; p. 168].

Let  $E$  be one of the Banach spaces  $L^p(\mathbb{R}^n)^N$  ( $1 \leq p \leq \infty$ ),  $C_0(\mathbb{R}^n)^N$ ,  $C^b(\mathbb{R}^n)^N$  or  $C^\alpha(\mathbb{R}^n)^N$  ( $0 < \alpha < 1$ ). Define  $q_{E,m} \in \mathbb{R}_+$  by

$$q_{E,m} := \begin{cases} (N - 1)m & \text{for } E = L^2(\mathbb{R}^n)^N \\ Nm + [n/2]m + m & \text{for } E = C^\alpha(\mathbb{R}^n)^N \end{cases}$$

and let

$$q_{E,m} > \begin{cases} (N-1)m + mn|1/2 - 1/p| & \text{for } E = L^p(\mathbb{R}^n)^N \ (1 \leq p < \infty, p \neq 2) \\ (N-1)m + mn/2 & \text{for } E = C_0(\mathbb{R}^n)^N \\ (N-1)m + mn/2 + m & \text{for } E = L^\infty(\mathbb{R}^n)^N, C^b(\mathbb{R}^n)^N. \end{cases}$$

Then the following holds.

**THEOREM 4.1.** *Let  $E$  be one of the spaces listed above. Assume  $P : \mathbb{R}^n \rightarrow M_N(\mathbb{C})$  is given by  $P(\xi) = \sum_{|k| \leq m} A_k(i\xi)^k$  and let  $A_E$  be the operator defined as in (4.2). Then  $A_E$  generates a  $C$ -regularized semigroup on  $E$  with  $C := R(1, \Delta)^{q_{E,m}/2}$  if and only if there exists a constant  $\omega \in \mathbb{R}$  such that  $\Lambda(P(\xi)) \leq \omega$  for all  $\xi \in \mathbb{R}^n$ .*

**PROOF.** “ $\Leftarrow$ ”: For  $q \in \mathbb{R}$  define the function  $w_q : \mathbb{R}^n \rightarrow \mathbb{C}$  by  $w_q(\xi) := (1 + |\xi|^2)^{-q/2}$ . We will show that the family  $(S_E(t))_{t \geq 0}$  of operators on  $E$  given by

$$S_E(t)f = \mathcal{F}^{-1}(e^{tP} w_{q_{E,m}} \hat{f})$$

is the  $C$ -regularized semigroup generated by  $A_E$ , where  $C := R(1, \Delta)^{q_{E,m}/2}$ .

We claim first that the function  $u_t : \mathbb{R}^n \rightarrow M_N(\mathbb{C})$  defined by  $u_t(\xi) := e^{tP(\xi)} w_{q_{E,m}}(\xi)$  is a Fourier multiplier for  $E$ . Using the estimate (4.3), this is clear for  $E = L^2(\mathbb{R}^n)^N$ . Consider next the case  $E = L^p(\mathbb{R}^n)^N$  ( $p \neq 2$ ). Following Hörmander [Hö; Lemma 2.3] we define a positive  $C_c^\infty$ -function  $\psi$  on  $\mathbb{R}^n$  with  $\text{supp } \psi \subset \{x \in \mathbb{R}^n; 1/2 < |x| < 2\}$  such that  $\sum_{l=-\infty}^\infty \psi(2^{-l}x) = 1$  for all  $x \neq 0$ . Moreover, for  $l \in \mathbb{Z}$ , define  $\psi_l$  and  $g_l^t$  by  $\psi_l(x) := \psi(2^{-l}x)$  and  $g_l^t(x) := \psi_l(x)u_t(x)$ , respectively. In order to prove the claim, it suffices to show that  $\sum_{l=1}^\infty \|g_l^t\|_{M_p^N} < \infty$  for all  $t \geq 0$ . To this end, assume that  $1 \leq p \leq 2$  and note that from (4.3) we obtain

$$\|D^k(e^{tP} w_r)(\xi)\| \leq M_0 e^{\omega t} |\xi|^{Nm + (|k|-1)m - |k| - r}$$

for large enough  $\xi$ , all  $t \geq 0$ , all  $r \in \mathbb{R}_+$  and suitable constants  $M_0 > 0$  and  $\omega' > \omega$ . Hence,

$$\|D^k g_l^t\|_{L^2} \leq M_1 e^{\omega' t} 2^{l(Nm + (|k|-1)m - |k| - q_{E,m})} 2^{ln/2}$$

for some constant  $M_1$ , all multiindices  $k$  and all  $l > 1$ . Bernstein’s theorem (see, e.g. [Hi1; Lemma 2.1]) implies now that

$$\|g_l^t\|_{M_1} \leq \|g_l^t\|_{\mathcal{FL}^1} \leq M_2 e^{\omega' t} 2^{l(Nm - m + mn/2 - q_{E,m})}.$$

Let  $1 < p < 2$  and  $\theta := 2(1 - 1/p)$ . By the Riesz-Thorin convexity theorem,

we obtain

$$\|g_t^\sharp\|_{M_p} \leq \|g_t^\sharp\|_{M_1}^{1-\theta} \|g_t^\sharp\|_{M_2}^\theta \leq M_3 e^{\omega t} 2^{l((N-1)m+mn(1/p-1/2)-q_{E,m})}.$$

Finally, the inequality  $\|u_t\|_{M_p} \leq M_4 + \sum_{l>1}^\infty \|g_l^\sharp\|_{M_p}$  and the fact that  $M_p^N = M_{p'}^N$ , where  $1/p - 1/p' = 1$ , yield the claim.

The corresponding assertion for  $E = C^b(\mathbb{R}^n)^N$  follows from the case  $p = \infty$  by Lebesgue’s convergence theorem. Finally, the case  $E = C^\alpha(\mathbb{R}^n)^N$  is implied by the cited result b).

In order to show that  $t \mapsto S_E(t)$  is strongly continuous on  $E$  for all  $t \geq 0$ , consider first the cases  $E = L^p$  ( $p \neq \infty$ ) and  $C_0$ . Define, for  $\lambda > \omega$ , the function  $r_\lambda$  by

$$r_\lambda(\xi) = (\lambda - P(\xi))^{-1} w_{q_{E,m}}(\xi) = \int_0^\infty e^{-\lambda t} u_t(\xi) dt.$$

Then  $u_{t+h}(\xi)r_\lambda(\xi) - u_t(\xi)r_\lambda(\xi) = (\lambda r_\lambda(\xi) - w_q(\xi)) \int_t^{t+h} u_s(\xi) ds$ . Therefore  $t \mapsto \mathcal{F}^{-1}(u_t r_\lambda)$  is continuous with respect to the multiplier norm. In particular, this implies that the family  $(S_E(t))_{t \geq 0}$  is strongly continuous on the range of the mapping  $f \mapsto \mathcal{F}^{-1}(r_\lambda \hat{f})$  and therefore by density on  $E$ .

In order to prove the remaining cases, let  $r > m$  if  $E = L^\infty$  or  $C_b$  and let  $r = m$  if  $E = C^\alpha$ . Writing  $u_{t+h}(\xi) - u_t(\xi) = P(\xi)w_r(\xi) \int_t^{t+h} u_s(\xi)w_{-r}(\xi) ds$  and using the results a) and b), we obtain

$$\|u_{t+h} - u_t\|_{M_E} \leq \|Pw_r\|_{M_E} |h| \sup_{t \leq \tau \leq t+h} \|u_\tau w_{-r}\|_{M_E}.$$

Thus  $(S_E(t))_{t \geq 0}$  is strongly continuous on  $E$ .

Finally, let  $f \in S^N$  and  $\lambda > \omega$ . By Fubini’s theorem,

$$(\lambda - A_E)^{-1} C f = \mathcal{F}^{-1}(r_\lambda \hat{f}) = \int_0^\infty e^{-\lambda t} \mathcal{F}^{-1}(u_t \hat{f}) dt = \int_0^\infty e^{-\lambda t} S_E(t) f dt.$$

This proves the assertion for  $E = L^p$  ( $1 \leq p < \infty$ ) and  $E = C_0$ .

For the remaining spaces  $E = L^\infty, C^b$  and  $C^\alpha$ , let  $f \in E$  and note that, since  $\mathcal{F}^{-1}(u_t) \in L^1$ , we can apply Fubini’s theorem and obtain

$$\int_0^\infty e^{-\lambda t} S_E(t) f dt = \int_0^\infty e^{-\lambda t} (\mathcal{F}^{-1}(u_t) * f) dt = \int_0^\infty e^{-\lambda t} \mathcal{F}^{-1}(u_t) dt * f.$$

Using the definition of the Fourier transform in the distributional sense and

again Fubini's theorem, we obtain  $\int_0^\infty e^{-\lambda t} \mathcal{F}^{-1}(u_t) dt = \mathcal{F}^{-1}(r_\lambda)$  and therefore

$$\int_0^\infty e^{-\lambda t} S_E(t) f dt = \mathcal{F}^{-1}(r_\lambda) * f = (\lambda - A_E)^{-1} C f.$$

" $\Rightarrow$ ": Let  $C := R(1, \Delta)^{q_{E,m}/2}$  and let  $A$  be the generator of the  $C$ -regularized semigroup on  $E$ . Then  $w_r e^{tP} \in M_2^N = L^\infty(\mathbb{R}^n)^N$  for suitable  $r \in \mathbb{N}$  and all  $t \geq 0$  and the multiplier norm of  $w_r e^{tP}$  is exponentially bounded. Hence there exist constants  $M$  and  $\omega$  such that  $\|e^{tP(\xi)}\| \leq M e^{\omega t} (1 + |\xi|^2)^r$  for all  $\xi \in \mathbb{R}^n$ . Now the inequality  $e^{t\Lambda(P(\xi))} \leq \|e^{tP(\xi)}\|$  implies that

$$\Lambda(P(\xi)) \leq C_1 \log(1 + |\xi|) + C_2$$

holds for all  $\xi \in \mathbb{R}^n$  and suitable constants  $C_1$  and  $C_2$ . The Seidenberg-Tarski theorem (see [Fr; Sections 14, 15]) finally yields the assertion.  $\square$

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