ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

MASSIMO CICOGNANI
LUISA ZANGHIRATI

On a class of unsolvable operators

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série, tome 20, nº 3 (1993), p. 357-369
<http://www.numdam.org/item?id=ASNSP_1993_4_20_3_357_0>

On a Class of Unsolvable Operators

MASSIMO CICOGNANI - LUISA ZANGHIRATI

0. - Introduction

If P is an unsolvable differential operator, can one characterize the data f for which the equation $Pu = f$ has local (or microlocal) solutions?

This problem is considered here when P is a linear analytic partial differential operator of the following type:

$$P = A^l + \text{ lower order terms, } l \in \mathbb{N}, \tag{0.1}$$

where A is an analytic pseudodifferential operator of complex principal type which is microlocally modelled by the Mizohata operator $M = D_n + ix_n^h D_1$, h an *odd* positive integer.

The problem of the analytic-Gevrey hypoellipticity for operators with multiple characteristics of the previous type has been investigated by L. Cattabriga, L. Rodino and the second author in [3], essentially by reducing the operator P to the Mizohata operator M in Gevrey classes G^s with $1 \leq s < l/(l-1)$. The same reduction, together with the results about the model operator M of [12], is used here to deduce the local s-unsolvability of the operator P for every $1 < s \leq \infty$ ($G^\infty := C^\infty$) and to construct a projector along the image of P modulo microanalytic terms when $1 < s < l/(l-1)$ (see Theorem 1.3). This microlocal construction can be carried to the local level when P is a differential operator in $\mathbb{R}^2$ (this is done throughout Propositions 2.1, 2.2, 2.4); locally analytic rests can be neglected in view of the Cauchy-Kovalevsky theorem.

We recall the paper of N. Hanges [11] where the image of an "almost Mizohata operator" (see for example [20]) is completely determined. Concerning the model operator M, we mention the results in $\mathbb{R}^2$ of H. Ninomiya [14] where a property equivalent to $Q_+ f$ being locally analytic is presented (Q_+ is the projector along the image of M described in Proposition 26.3.7 of [12]).

Pervenuto alla Redazione il 24 Marzo 1992.

With respect to the sole unsolvability of the operator P in (0.1), we point out a recent independent proof given by T. Gramchev in [8] and quote the previous results in the C^∞ category of F. Cardoso, F. Treves [2], P. Popivanov [16] for $l = 2$ and R. Goldman [7] ($l \geq 3$ non vanishing subprincipal symbol) and F. Cardoso [1] (for every $1 < s < \infty$, $l = 2$). It is well known that the operator P is not s-solvable for $1 < s \leq \infty$ in the case of simple characteristics (i.e. $l = 1$): cf. [6], [15], [17].

When the operator P is given by (0.1) but A is microlocally reducible to $D_n + ix_n^h D_1$ with *h even*, the techniques that we use here lead to the construction of a local two-sided inverse of P modulo analytic terms in $\mathbb{R}^n$, $n \geq 2$ (see Remark 2.5). In this way one shows the local s-solvability of P in $\mathbb{R}^n$ for $1 < s < l/(l-1)$ (cf. [9] for the two-dimensional case) and obtains again the properties of analytic-Gevrey hypoellipticity for such an operator P already found in [3].

1. - Microlocal constructions

We recall that for $\Omega \subset \mathbb{R}^n$ an open set and $1 \leq s < \infty$, the space of Gevrey functions of index s, $G^s(\Omega)$, consists of all $f(x) \in C^\infty(\Omega)$ such that for every $K \subset\subset \Omega$ there exists a constant $C = C_{f,K} > 0$ with

$$\sup_{x \in K} |D^\alpha f(x)| \leq C^{|\alpha|+1}(\alpha!)^s, \qquad \alpha \in \mathbb{Z}_+^n. \tag{1.1}$$

The space $G^1(\Omega)$ coincides with the space of all real analytic functions in Ω. If $s > 1$ we denote by $G_0^s(\Omega)$ the set $G^s(\Omega) \cap C_0^\infty(\Omega)$ and write $G_0^{(s)'}(\Omega)$ (resp. $G^{(s)'}(\Omega)$) for the space of all s-ultradistributions (resp. with compact support), i.e. the dual space of $G_0^s(\Omega)$ (resp. $G^s(\Omega)$). We refer to [13] for an exhaustive exposition about these topics. We shall also write $G^\infty(\Omega)$ for $C^\infty(\Omega)$ in order to have uniform notations.

If $\gamma_0 = (x_0, \xi_0) \in \dot{T}^*(\Omega)$ and $f \in G_0^{(s)'}(\Omega)$ we say that γ_0 does not belong to the analytic wave front set of f, $WF_A(f)$, if and only if there exist a conic neighborhood $\Gamma = V \times C$ of γ_0 in $\dot{T}^*(\Omega)$ and a bounded sequence $f_j \in G^{(s)'}(\Omega)$ which is equal to f in V and satisfies

$$|\hat{f}_j(\xi)| \leq B(Bj/|\xi|)^j, \qquad j = 1, 2, \ldots, \xi \in C \tag{1.2}$$

for some constant B. The projection of $WF_A(f)$ in Ω is equal to the analytic singular support of f, i.e. the complement in Ω of the largest open set where f is analytic (see [12] Chap. VIII for this and other properties of analytic-Gevrey wave front sets).

DEFINITION 1.1. If $P(x, D)$, $x \in \Omega$, is a linear analytic partial differential operator and $f \in G_0^{(s)'}(\Omega)$, we say that f is *admissible* for P at $x_0 \in \Omega$ if there exists $u \in G_0^{(s)'}(\Omega)$ such that $Pu = f$ in a neighborhood $V \subset \Omega$ of x_0. The operator P is said to be *s-solvable* at x_0 if every $f \in G^s(\Omega)$ is admissible for P at x_0.

We also need to give a microlocal version of Definition 1.1. In order to do this, if Γ is an open conic set in $\dot{T}^*(\Omega)$, we introduce the following equivalence relation $\sim$ in $G_0^{(s)'}(\Omega)$:

$$f \sim g \Leftrightarrow WF_A(f-g) \cap \Gamma = \emptyset;$$

we write $A^s(\Gamma)$ for the quotient space $G_0^{(s)'}(\Omega)/\sim$. The support of $u \in A^s(\Gamma)$, here denoted by $[u]$, is the closed set in Γ equal to $WF_A(f) \cap \Gamma$ for any $f \in G_0^{(s)'}(\Omega)$ in the equivalence class u. In particular if $V \subset \Omega$ is an open set, the support of $u \in A^s(\dot{T}^*(V))$ is equal to $WF_A(f) \cap \dot{T}^*(V)$ for any $f \in G_0^{(s)'}(\Omega)$ in the equivalence class u.

DEFINITION 1.2. An ultradistribution $f \in G_0^{(s)'}(\Omega)$ is said to be *admissible* for the operator P at $\gamma_0 \in \dot{T}^*(\Omega)$ if there are an open conic neighborhood Γ of γ_0 and $u \in A^s(\Gamma)$ with $Pu = f$ in $A^s(\Gamma)$. P is called *s-solvable* at γ_0 if every $f \in G_0^{(s)'}(\Omega)$ is admissible for P at γ_0.

Clearly, if $f \in G_0^{(s)'}(\Omega))$ is admissible for P at x_0 then it is admissible for P also at every $\gamma \in \dot{T}^*_{x_0}$ in the microlocal meaning. The converse is true if $\Omega \subset \mathbb{R}^2$ and the principal symbol of P does not vanish identically in $\dot{T}^*_{x_0}$ (see Proposition 2.1). Another useful remark is that the s-solvability of P at x_0 implies the s_1-solvability of P at x_0 for every $1 < s_1 < s$, i.e. if P is not s-solvable at x_0 then P is not s_1-solvable at x_0 for every $s < s_1 \leq \infty$. This fact justifies the interpolation between results of non-solvability in the C^∞ category and the Cauchy-Kovalevsky theorem.

For the Mizohata operator $M = D_n + ix_n^h D_1$ with h an *odd* positive integer, operators E, Q_+, Q_- from $E'(\mathbb{R}^n)$ to $D'(\mathbb{R}^n)$ are constructed in [12], Par. 26.3, with the following properties:

$$ME = I - Q_+, \quad EM = I - Q_-, \tag{1.3}$$

$$Q_+M = MQ_- = 0, \tag{1.4}$$

$$\begin{aligned} WF_A(E) = \{(x,\xi,y,\eta) \in \dot{T}^*(\mathbb{R}^n) \times \dot{T}^*(\mathbb{R}^n) :& \\ (x,\xi) = (y,-\eta) \text{ or}& \\ (x_2,\dots,x_{n-1}) = (y_2,\dots,y_{n-1}), \xi = -\eta, \xi_1 = \xi_n = 0\},& \end{aligned} \tag{1.5}$$

$$\begin{aligned} WF_A(Q_\pm) = \{(x,\xi,y,\eta) \in \dot{T}^*(\mathbb{R}^n)) \times \dot{T}^*(\mathbb{R}^n) :& \\ (x,\xi) = (y,-\eta), x_n = \xi_n = 0, \xi_1 \gtrless 0 \text{ or}& \\ (x_2,\dots,x_{n-1}) = (y_2,\dots,y_{n-1}), \xi = -\eta, \xi_1 = \xi_n = 0\},& \end{aligned} \tag{1.6}$$

The operators Q_+ and Q_- are projectors along the image and on the kernel of M respectively. (If the integer h is *even* then property (1.3) is valid with $Q_+ = Q_- = 0$, i.e. there is a fundamental solution E, which satisfies (1.5), for the operator M.) It follows from (1.3)-(1.6) that $f \in G^{(s)'}(\mathbb{R}^n)$ is admissible for M at $\gamma_0 \in \dot{T}^*(\mathbb{R}^n)\backslash\{\xi_1 = \xi_n = 0\}$ if and only if $\gamma_0 \notin WF_A(Q_+f)$. For every $s > 1$ there is a function $f \in G_0^s(\mathbb{R}^n)$ such that $(0, \varepsilon_1) \in WF_A(Q_+f)$, $\varepsilon_1 = (1, 0, \ldots, 0)$: thus for every $1 < s \leq \infty$ the operator M is not s-solvable at $(0, \varepsilon_1)$ (see [17]). When $n = 2$, by using (1.3)-(1.6) again and the Cauchy-Kovalevsky theorem, one obtains that f is admissible at x_0 if and only if Q_+f is analytic at x_0.

We shall now prove that properties similar to (1.3)-(1.6) hold, from the microlocal point of view, for a linear analytic partial differential operator P in $\Omega \subset \mathbb{R}^n$ of order $m \geq 2$, whose principal symbol p_m satisfies the following hypothesis at a point $\gamma_0 = (x_0, \xi_0) \in \dot{T}^*(\Omega)$:

(1.7) there exist an open conic neighborhood Γ of γ_0 and a positive integer $l \leq m$ such that

$$p_m(x, \xi) = q_{m-l}(x, \xi)(a_1(x, \xi))^l, \quad (x, \xi) \in \Gamma,$$

where q_{m-l} is an analytic elliptic symbol of order $m - l$ in Γ, a_1 is a complex-valued first order analytic symbol of principal type, i.e. $d_\xi a_1(x, \xi) \neq 0$ on $\Sigma_\Gamma = \{(x, \xi) \in \Gamma; a_1(x, \xi) = 0\}$, $\gamma_0 \in \Sigma_\Gamma$. Without loss of generality we can suppose $d_\xi \operatorname{Re} a_1(x, \xi) \neq 0$ in Γ. We assume further that $\operatorname{Im} a_1(x, \xi)$ has a zero of fixed odd order h and changes sign from $-$ to $+$ along every bicharacteristic of $\operatorname{Re} a_1(x, \xi)$ near γ_0.

THEOREM 1.3. *If the operator P satisfies condition* (1.7) *then, shrinking the conic neighborhood Γ of γ_0, we can determine two linear operators F and F^+ from $A^s(\Gamma)$ to $A^s(\Gamma)$ for every $s \in]1, l/(l-1)[$, (for every $s \in]1, \infty]$ when $l = 1$), such that:*

$$FP = I, \ PF = I - F^+, \tag{1.8}$$

$$PA^s(\Gamma) = \operatorname{Ker} F^+ := \{g \in A^s(\Gamma); F^+g = 0\}. \tag{1.9}$$

Concerning the supports we have:

$$[Fg] \subset [g], \ [F^+g] \subset [g] \cap \Sigma_\Gamma, \ g \in A^s(\Gamma). \tag{1.10}$$

An ultradistribution $g \in G_0^{(s)'}(\Omega)$ is admissible for P at γ_0 if and only if $\gamma_0 \notin [F^+g]$ in $A^s(\Gamma)$. In particular, for every $s_1 \in]1, \infty]$, P is not locally s_1-solvable at x_0.

PROOF. We first assume that operators F and F^+ satisfying (1.8) exist. Then PFP is equal both to P and to $P - F^+P$: this fact yields $F^+P = 0$. Thus $Pu = g$ implies $F^+g = 0$. Conversely if $F^+g = 0$ then from (1.8) we have that

$u = Fg$ is the unique solution of $Pu = g$ in $A^s(\Gamma)$. In this way we have proved that $PA^s(\Gamma) = \operatorname{Ker} F^+$ and that F is the inverse of P from $\operatorname{Ker} F^+$ to $A^s(\Gamma)$.

If the operators F and F^+ satisfy properties (1.8)-(1.10) in $A^s(\Gamma)$, then they are well-defined as operators in $A^s(\Gamma_1)$ for any open conic set $\Gamma_1 \subset \Gamma$ and they still satisfy (1.8)-(1.10) with Γ replaced by Γ_1. Thus an ultradistribution $g \in G_0^{(s)'}(\Omega)$ is admissible for P at γ_0 if and only if $\gamma_0 \notin [F^+g]$. It will follow from the subsequent construction of the operators F and F^+ that there exists a function $f \in G^s(\Omega)$ which is not admissible for P at γ_0. As a consequence we have that P is not s-solvable at x_0, therefore P is not s_1-solvable at x_0 for every $s < s_1 \leq \infty$. The last statement of the Proposition follows from the fact that one can arbitrarily choose s in $]1, l/(l-1)[$.

We shall now perform the construction of the operators F and F^+. In doing so, we begin by recalling that if condition (1.7) is fulfilled then there exist an analytic homogeneous canonical transformation $\chi : \Gamma' \to \Gamma$, Γ' being an open conic neighborhood of $(0, \varepsilon_1)$, $\varepsilon_1 = (1, 0, \ldots, 0)$, and an analytic elliptic symbol e in Γ such that $\chi(0, \varepsilon_1) = \gamma_0$ and $\chi^*(ea_1) = \xi_n + ix_n^h \xi_1$. This transformation can be lifted to the analytic Fourier integral operators level (see [12], Vol. III and IV). Taking this fact into account together with Proposition 2.1 in [3], it is then sufficient to construct the operators F and F^+ in the following situation:

$$\gamma_0 = (0, \varepsilon_1), \ \Gamma \text{ conic neighborhood of } (0, \varepsilon_1),$$

$$P(x, D) = M^l + \sum_{r=0}^{l-1} P_r(x, D')M^r, \ D' = (D_1, \ldots, D_{n-1}),$$

where each P_r is an analytic pseudodifferential operator and $\operatorname{ord}(P_r) \leq l - r - 1$.

Let us take the elliptic operator $\Lambda = |D_1|^{(l-1)/l}$ in Γ and denote by $A^s(\Gamma; l)$ the space of all l-dimensional vectors with components in $A^s(\Gamma)$. We consider the operator S from $A^s(\Gamma; l)$ to $A^s(\Gamma; l)$ defined by:

$$S = MI + B, \ B = \begin{pmatrix} 0 & -\Lambda & 0 & \ldots & 0 \\ & & \ldots & & \\ 0 & & \ldots & & -\Lambda \\ P_0\Lambda^{1-l} & P_1\Lambda^{2-l} & \ldots & \ldots & P_{l-1} \end{pmatrix}; \tag{1.11}$$

$B = B(x, D')$ is a $l \times l$ matrix of pseudodifferential operators of order $(l-1)/l$. If $U = {}^t(u_1, \ldots, u_l)$, $V = {}^t(v_1, \ldots, v_l) \in A^s(\Gamma; l)$ then one has $SU = V$ if and only if

$$\begin{aligned} P\Lambda^{1-l}u_1 &= (M^{l-1} + P_{l-1}M^{l-2} + \ldots + P_2M + P_1)\Lambda^{1-l}v_1 \\ &+ (M^{l-2} + P_{l-1}M^{l-3} + \ldots + P_2)\Lambda^{2-l}v_2 + \ldots \\ &+ (M + P_{l-1})\Lambda^{-1}v_{l-1} + v_l \end{aligned} \tag{1.12}$$

and $u_{j+1} = \Lambda^{-1}(Mu_j - v_j)$ for $j = 1, \ldots, l-1$.

After shrinking the conic neighborhood Γ, two linear operators H and H' from $A^s(\Gamma;l)$ to $A^s(\Gamma;l)$, $1 < s < l/(l-1)$, are constructed in Lemma 2.3 of [3] in such a way that the following properties are satisfied:

$$[HU] = [H'U] = [U],\ U \in A^s(\Gamma;l),\ ([U] := \bigcup_{j=1}^{l} [u_j]); \tag{1.13}$$

$$HH' = H'H = I; \tag{1.14}$$

$$H'SH = MI. \tag{1.15}$$

H and H' are matrices of pseudodifferential operators of infinite order, i.e. with symbols exponentially growing at infinity (see [4], [18], [21]).

Now we can show the existence of a function $g \in G^s(\Omega)$ which is not admissible for P at γ_0. Let us take $f \in G_0^s(\Omega)$ such that $\gamma_0 \in [Q_+ f]$ (i.e. $f \in G_0^s(\Omega)$ which is not admissible for M at γ_0) and set $H\,^t(0,\dots,f) = {}^t(v_1,\dots,v_l)$: with this choice, modulo microanalytic terms, the right member in equality (1.12) defines a function $g \in G^s(\Omega)$ such that the equation $Pu = g$ cannot be solved in $A^s(\Gamma_1)$ for any conic neighborhood $\Gamma_1 \subset \Gamma$ of γ_0.

It follows from (1.5) and (1.6) that E and Q_+ are well-defined as operators from $A^s(\Gamma)$ to $A^s(\Gamma)$ and equalities (1.3) become $ME = I - Q_+$, $EM = I$ in $A^s(\Gamma)$. Let us set

$$\tilde{F} = HEH', \qquad \tilde{F}^+ = HQ_+H'$$

and write $\tilde{F} = (f_{ij})$, $\tilde{F}^+ = (f_{ij}^+)$, $i,j = 1,\dots,l$. From (1.14) and (1.15) we obtain:

$$S\tilde{F} = I - \tilde{F}^+,\ \tilde{F}S = I \qquad \text{in } A^s(\Gamma;l). \tag{1.16}$$

The above construction for the operator S can be carried over to the scalar operator P.

To do this, we first choose $U = {}^t(\Lambda^{l-1}u, \Lambda^{l-2}Mu, \dots, \Lambda M^{l-2}u, M^{l-1}u)$, $u \in A^s(\Gamma)$, in order to have $SU = {}^t(0,\dots,Pu)$. Thus the second equality in (1.16) gives $f_{1l}Pu = \Lambda^{l-1}u$ and leads us to define

$$F = \Lambda^{1-l} f_{1l}$$

so that $FP = I$ in $A^s(\Gamma)$ can be satisfied. Next we take $G = {}^t(0,\dots,g)$, $g \in A^s(\Gamma)$, and from the first equality in (1.16) we obtain

$$S\,^t(f_{1l}g, f_{2l}g, \dots, f_{ll}g) = {}^t(-f_{1l}^+g, f_{2l}^+g, \dots, g - f_{ll}^+g);$$

hence, in view of (1.12),

$$\begin{aligned} P\Lambda^{1-l}f_{1l}g = &-(M^{l-1} + P_{l-1}M^{l-2} + \dots + P_2M + P_1)\Lambda^{1-l}f_{1l}^+g \\ &-(M^{l-2} + P_{l-1}M^{l-3} + \dots + P_2)\Lambda^{2-l}f_{2l}^+g - \dots \\ &-(M + P_{l-1})\Lambda^{-1}f_{l-1l}^+g - f_{ll}^+g + g. \end{aligned}$$

We can now define

$$F^+ = (M^{l-1} + P_{l-1}M^{l-2} + \ldots + P_2M + P_1)\Lambda^{1-l}f_{1l}^+ +$$
$$+ (M^{l-2} + P_{l-1}M^{l-3} + \ldots + P_2)\Lambda^{2-l}f_{2l}^+ + \ldots +$$
$$+ (M + P_{l-1})\Lambda^{-1}f_{l-1l}^+ + f_{ll}^+$$

so that we also have $PF = I - F^+$ in $A^s(\Gamma)$.

Only the statements concerning the supports remain to be proved. By (1.13), (1.5) and (1.6) we have

$$[f_{1l}g] \subset [\tilde{F}G] = [HEH'G] = [EH'G] \subset [H'G] = [G] = [g],$$

$$[f_{il}^+g] \subset [\tilde{F}^+G] = [HQ^+H'G] = [Q^+H'G]$$
$$\subset [H'G] \cap \Sigma_\Gamma = [G] \cap \Sigma_\Gamma = [g] \cap \Sigma_\Gamma, \qquad i = 1, \ldots, l,$$

which completes the proof.

EXAMPLE 1.4. The foregoing proof shows how to obtain the operators F and F^+ from E and Q_+. The symbol of the intertwining operators H and H' may be computed by solving transport equations as done in [3]. A simple example is given by operators of the type $P = M^l + \sum_{r=0}^{l-1} P_r(D')M^r$ with $H(x_n, D') = \exp(-iB(D')x_n)$ (the operator B is defined in (1.11)). However operators F and F^+ satisfying (1.8), (1.9) and (1.10) may be not unique. For a power M^l, $l \geq 2$, of the Mizohata operator in a conic neighborhood Γ of $(0, \varepsilon_1)$, by an iterative use of (1.3), we obtain

$$E^lM^l = I, \;\; M^lE^l = I - \sum_{r=0}^{l-1} M^rQ_+E^r \text{ in } A^s(\Gamma).$$

On the other hand, following the proof of the above Proposition, we have in this case

$$F = \sum_{r=0}^{l-1} \frac{(ix_n)^r}{r!} E \frac{(-ix_n)^{l-1-r}}{(l-1-r)!}, \qquad F^+ = \sum_{r=0}^{l-1} M^rQ_{+,r+1}$$

with

$$Q_{+,r} = \sum_{s=0}^{r-1} \frac{(ix_n)^s}{s!} Q_+ \frac{(-ix_n)^{r-1-s}}{(r-1-s)!}$$

and it is not difficult to prove that $E^l \neq F$ and $F^+ \neq \sum_{r=0}^{l-1} M^rQ_+E^r$.

2. - Local constructions

PROPOSITION 2.1. *Let Ω be an open set in $\mathbb{R}^2$, $x_0 \in \Omega$ and assume that $p_m(\gamma_0) \neq 0$ for a certain $\gamma_0 \in \dot{T}^*_{x_0}$. Then an ultradistribution $g \in G_0^{(s)'}(\Omega)$ is admissible for P at x_0 if and only if g is admissible for P at every $\gamma \in \dot{T}^*_{x_0}$.*

PROOF. Let $g \in G_0^{(s)'}(\Omega)$ be admissible for P at every $\gamma \in \dot{T}^*_{x_0}$. By a compactness argument we may choose an open neighborhood $V \subset \Omega$ of x_0, a finite covering

$$\{\Gamma_j = V \times C_j : C_j \text{ open cone in } \mathbb{R}^2 \backslash \{0\}, j = 1, \dots, h\}$$

of $\dot{T}^*(V)$ and $u_j \in A^s(\Gamma_j)$ such that $Pu_j = g$ in $A^s(\Gamma_j)$, $j = 1, \dots, h$.

Since $\{\gamma = (x, \xi) \in \dot{T}^*_{x_0} : p_m(\gamma) = 0, |\xi| = 1\}$ is a finite set we may also assume $p_m(x, \xi) \neq 0$ in $\Gamma_i \cap \Gamma_j$, $i \neq j$, which implies $u_i = u_j$ in $A^s(\Gamma_i \cap \Gamma_j)$.

Let $\{\chi_j : j = 1, \dots, h\}$ be a partition of unity subordinate to the covering $\{C_j : j = 1, \dots, h\}$ of $\mathbb{R}^2 \backslash \{0\}$ with the properties indicated in [19], Chap. V, Sec. 1, and let $V' \subset V$ be a smaller neighborhood of x_0. If we set $\Gamma'_j = V' \times C_j$ then it is possible to choose the elements of such a partition of unity in order to have well-defined linear operators $\chi_j(D) : A^s(\Gamma_j) \to A^s(\Gamma'_j)$. Thus setting $u = \Sigma_j \chi_j u_j$ we obtain $u = u_j$ in $A^s(\Gamma'_j)$: property $Pu = g$ in $A^s(\dot{T}^*(V'))$ follows from the fact that $Pu_j = g$ in $A^s(\Gamma'_j)$ for every Γ'_j in the covering of $\dot{T}^*(V')$.

An application of the Cauchy-Kovalevsky theorem completes the proof.

We note that for every $\varepsilon > 0$ it is possible to take the operators χ_j in the above proof in order to have also

$$[\chi_j u] \subset [u]_\varepsilon := \{(x + y, \xi) \in \dot{T}^*(\Omega) : (x, \xi) \in [u], |y| < \varepsilon\}, u \in A^s(\Gamma_j)$$

(cf. Corollary 1.2 in [19], Chap. V).

Let us define $N^s_{x_0} = \{\gamma \in \dot{T}^*_{x_0} : P \text{ is not } s\text{-solvable at } \gamma\}$. As a consequence of the previous Proposition, if $\Omega \subset \mathbb{R}^2$ and the principal symbol of P does not vanish identically in $\dot{T}^*_{x_0}$ then any $g \in G_0^{(s)'}(\Omega)$ with $N^s_{x_0} \cap WF_A(g) = \emptyset$ is admissible for P at x_0. Since $N^s_{x_0} \subset \operatorname{Char} P$, $N^s_{x_0} \cap \{|\xi| = 1\}$ is a finite set when $\Omega \subset \mathbb{R}^2$.

Let us denote by Σ_+ the set of all $\gamma \in \operatorname{Char} P$ such that P satisfies hypothesis (1.7) at γ. It follows from Theorem 1.3 that $\Sigma_+ \cap \dot{T}^*_{x_0} \subset N^s_{x_0}$ for every $s \in]1, \infty]$. We shall now prove that the microlocal statement (1.9) in Theorem 1.3 can be expressed in a local form when $N^s_{x_0} = \Sigma_+ \cap \dot{T}^*_{x_0}$ and $\Omega \subset \mathbb{R}^2$.

PROPOSITION 2.2. *Let Ω be an open set in $\mathbb{R}^2$, $x_0 \in \Omega$, and let r denote the highest multiplicity of the characteristics of P in $\dot{T}^*_{x_0}$. Assume that $p_m(\gamma_0) \neq 0$ for a certain $\gamma_0 \in \dot{T}^*_{x_0}$ and that $N^s_{x_0} = \Sigma_+ \cap \dot{T}^*_{x_0}$ for an index $s \in]1, r/(r-1)[$. Then we can find open neighborhoods V, V' of x_0 with $V' \subset V \subset \Omega$, and a* *linear operator F^+ from $A^s(\dot{T}^*(V))$ to $A^s(\dot{T}^*(V'))$ such that:*

$$F^+Pu = 0 \text{ in } A^s(\dot{T}^*(V')) \text{ for every } u \in A^s(\dot{T}^*(V));$$

$$[F^+g] \cap N^s_{x_0} = \emptyset, g \in G_0^{(s)'}(\Omega)) \Rightarrow g \text{ is admissible for } P \text{ at } x_0.$$

Moreover for any fixed $\varepsilon > 0$ it is possible to find F^+ in order to satisfy also $[F^+g] \subset ([g] \cap \Sigma_+)_\varepsilon$.

PROOF. Let us write $N^s_{x_0} \cap \{|\xi| = 1\} = \{\gamma_j = (x_0, \xi_j) : j = 1, \dots, h\}$. By Theorem 1.3, for every $j = 1, \dots, h$ there exist an open conic neighborhood $\Gamma_j = V \times C_j$ of γ_j and operators F_j, $F_j^+ : A^s(\Gamma_j) \to A^s(\Gamma_j)$ which satisfy properties (1.8), (1.9), (1.10) with Γ replaced by Γ_j. We may assume $C_i \cap C_j = \emptyset$ for $i \neq j$. As in the proof of Proposition 2.1, next we take linear operators $\chi_j(D) : A^s(\Gamma_j) \to A^s(\Gamma'_j)$, $\Gamma'_j = V' \times C_j$, with supp $\chi_j \subset C_j$ and $\chi_j = 1$ in a smaller conic neighborhood C'_j of ξ_j in $\mathbb{R}^2 \backslash \{0\}$, and we define

$$F^+ = \Sigma_j \chi_j F_j^+$$

in order to obtain that F^+P is the null operator from $A^s(\dot{T}^*(V))$ to $A^s(\dot{T}^*(V'))$.

Take now $g \in G_0^{(s)'}(\Omega)$. Since F^+ coincides with F_j^+ as an operator from $A^s(V \times C'_j)$ to $A^s(V \times C'_j)$, in view of the last statement in Theorem 1.3, from $[F^+g] \cap N^s_{x_0} = \emptyset$ it follows that g is admissible for P at every γ_j, $j = 1, \dots, h$. Hence g is admissible for P at x_0 by Proposition 2.1.

We have already observed that for every $\varepsilon > 0$ it is possible to take the operators χ_j in order to have $[\chi_j u] \subset [u]_\varepsilon$, so the proof is complete.

In Proposition 2.2 the operator P is requested to be s-solvable at every point of $\operatorname{Char} P \backslash \Sigma_+$ over x_0. We shall now describe some sufficient conditions for microlocal solvability. Let us assume that the principal symbol p_m of P satisfies

$$p_m(x, \xi) = q_{m-l}(x, \xi)(a_1(x, \xi))^l, \qquad (x, \xi) \in \Gamma,$$

in a conic neighborhood Γ of $\gamma_0 \in \operatorname{Char} P \cap \dot{T}^*_{x_0}$, where q_{m-l} is an analytic elliptic symbol of order $m - l$ in Γ, a_1 is a first order analytic symbol of principal type, $d_\xi \operatorname{Re} a_1(x, \xi) \neq 0$ in Γ (cf. (1.7)). If one of the following three conditions is fulfilled then P is s-solvable at γ_0 (also in dimension $n > 2$), $s \in]1, l/(l-1)[$:

(2.1) a_1 is complex-valued, $\operatorname{Im} a_1(x, \xi)$ has a zero of fixed odd order h and changes sign from + to − along every bicharacteristic of $\operatorname{Re} a_1(x, \xi)$ near γ_0;

(2.2) a_1 is complex-valued, $\operatorname{Im} a_1(x, \xi)$ has a zero of fixed even order h along every bicharacteristic of $\operatorname{Re} a_1(x, \xi)$ near γ_0;

(2.3) a_1 is a symbol of real principal type.

If (2.1) is satisfied then we can repeat the arguments that we have used in the proof of Theorem 1.3 and reduce the operator P microlocally to the Mizohata operator $M = D_n + ix_n^h D_1$ in a neighborhood of $(0, -\varepsilon_1)$ (instead of $(0, \varepsilon_1)$). In this case, the s-solvability of P at γ_0 is deduced from the s-solvability of M at $(0, -\varepsilon_1)$.

The same reduction can be performed when (2.2) holds. Then P is s-solvable at γ_0 since $M = D_n + ix_n^h D_1$ with h even is locally solvable at the origin.

The s-solvability of P at γ_0 when condition (2.3) is fulfilled has been proved in [18].

Let us set

$$\Sigma_- = \{\gamma \in \operatorname{Char} P : P \text{ satisfies hypothesis (2.1) at } \gamma\},$$

$$\Sigma_e = \{\gamma \in \operatorname{Char} P : P \text{ satisfies hypothesis (2.2) at } \gamma\},$$

$$\Sigma_r = \{\gamma \in \operatorname{Char} P : P \text{ satisfies hypothesis (2.3) at } \gamma\}.$$

Then, the following assumption

$$\operatorname{Char} P \cap \dot{T}^*_{x_0} = (\Sigma_+ \cup \Sigma_- \cup \Sigma_e \cup \Sigma_r) \cap \dot{T}^*_{x_0} \tag{2.4}$$

implies $N^s_{x_0} = \Sigma_+ \cap \dot{T}^*_{x_0}$ as requested in Proposition 2.2.

Hypothesis (2.4) concerns only the principal symbol p_m of P while the set $N^s_{x_0}$ may generally depend on the lower order terms too (note that in Theorem 1.3 the construction of F^+ does depend on the lower order terms of P). Thus the class of all operators which satisfy $N^s_{x_0} = \Sigma_+ \cap \dot{T}^*_{x_0}$ is larger than the class of all operators whose principal symbol can be factorized in the way suggested by (2.4). We shall discuss this situation in the following example.

EXAMPLE 2.3. Let us set $R_\lambda = D_2^2 + x_2^2 D_1 + \lambda D_1$, $\lambda \in \mathbb{R}$. The operator R_λ is s-solvable at the origin of $\mathbb{R}^2$ for every $s \in]1, \infty]$ if and only if $\lambda \neq \pm 1$, ± 3, ± 5, ... (see [10] and [17]). Let us then consider

$$P_1 = (D_1 + i\alpha(x_1)D_2)^2 R_0, \qquad P_2 = R_{-1}(D_1 + i\alpha(x_1)D_2)^2$$

where $\alpha(x_1)$ denotes a real valued analytic function with a zero of odd order and sign that changes form $-$ to $+$ at $x_1 = 0$. The operators P_1 and P_2 have the same principal part and both do not fulfill condition (2.4) at the origin. However P_1 is not s-solvable only at $(0,0,0,1)$ over $(0,0)$ and all the hypotheses of Proposition 2.2 are satisfied by this operator. On the contrary P_2 is not s-solvable also at $(0,0,1,0)$ and $(0,0,-1,0)$ where assumption (1.7) does not hold.

It is not difficult to prove that

$$Q_+ x_2^r f \text{ (resp. } Q_- x_2^r f) \text{ analytic at the origin for } r = 0, \dots, l$$

are sufficient conditions for f to be an admissible datum at the origin for R_{-2l-1} (resp. R_{+2l+1}). It is also easy to prove that the same conditions are necessary and sufficient in order to have an admissible datum f for the operator M^{l+1} (resp. $(M^*)^{l+1}$); cf. Example 1.4.

We shall now prove that a sharper version of Proposition 2.2 holds when the condition $N^s_{x_0} = \Sigma_+ \cap \dot{T}^*_{x_0}$ is replaced by a stronger assumption of kind (2.4). In doing so, we shall construct local operators F and F^-, besides F^+, corresponding to the operators E and Q_- in (1.3)-(1.6).

PROPOSITION 2.4. *Let Ω be an open set in $\mathbb{R}^2$, $x_0 \in \Omega$. Assume that*

$$\operatorname{Char} P \cap \dot{T}^*_{x_0} = (\Sigma_+ \cup \Sigma_- \cup \Sigma_e) \cap \dot{T}^*_{x_0}.$$

Then we can determine open neighborhoods V, V' of x_0 with $V' \subset V \subset \Omega$ and linear operators F, F^+, F^- from $A^s(\dot{T}^*(V))$ to $A^s(\dot{T}^*(V'))$ for every $s \in]1, r/(r-1)[$, r being as in Proposition 2.2, such that

$$PF = I - F^+, \qquad FP = I - F^-, \tag{2.5}$$

where $I : A^s(\dot{T}^*(V)) \to A^s(\dot{T}(V'))$ is the map induced by the identity in $G_0^{(s)'}(\Omega)$.

If ε is any fixed positive number then it is possible to find F, F^+, F^- in order to have also

$$[Fg] \subset [g]_\varepsilon, \ [F^\pm g] \subset ([g] \cap \Sigma_\pm)_\varepsilon, \ g \in A^s(\dot{T}^*(V)). \tag{2.6}$$

PROOF. At every point of $\operatorname{Char} P \cap \dot{T}^*_{x_0}$ we can use the same arguments as in the proof of Theorem 1.3. Therefore we can find a covering $\{\Gamma_j\}_{j=1,\dots,h}$ of $\dot{T}^*(V)$, V being a suitable neighborhood of x_0, and linear operators F_j, $F_j^\pm$ from $A^s(\Gamma_j)$ to $A^s(\Gamma_j)$ such that

$$PF_j = I - F_j^+, \ F_j P = I - F_j^-; \tag{2.7}$$

$$[F_j g] \subset [g], \ [F_j^\pm g] \subset [g] \cap \Sigma_\pm, \ g \in A^s(\Gamma_j) \tag{2.8}$$

and $F_j^\pm \neq 0$ if and only if $\Gamma_j \cap \Sigma_\pm \neq \emptyset$. As in the proof of Proposition 2.1 we can choose the elements of the covering Γ_j in order to have $\Gamma_j \cap \Gamma_j \cap \operatorname{Char} P = \emptyset$ when $i \neq j$. Since the operator P has a unique two-sided inverse in $A^s(\Gamma)$ when $\Gamma \cap \operatorname{Char} P = \emptyset$, it follows that

$$F_i = F_j, \quad F_i^\pm = F_j^\pm = 0 \quad \text{in } A^s(\Gamma_i \cap \Gamma_j), \ i \neq j. \tag{2.9}$$

Now one combines the microlocal constructions with a partition of unity as in the proof of Proposition 2.1 and obtains operators with the desired properties.

REMARK 2.5. In the previous proof the hypothesis $\Omega \subset \mathbb{R}^2$ is used only to obtain the property of *microlocal uniqueness* (2.9). It is possible to perform the same construction of Proposition 2.4 in the case $\Omega \subset \mathbb{R}^n$, $n > 2$, provided that the microlocal operators F_j, $F_i^{\pm}$ satisfy

$$F_i = F_j, \quad F_i^{\pm} = F_j^{\pm} \quad \text{in } A^s(\Gamma_i \cap \Gamma_j). \tag{2.10}$$

In view of Example 1.4 one cannot generally expect property (2.10) to hold when $r \geq 2$ and $n > 2$. In the case of simple characteristics (i.e. $r = 1$) J.J. Duistermaat and J. Sjöstrand [5] prove property (2.10) for every n.

On the other hand, if $\operatorname{Char} P \cap \dot{T}^*_{x_0} = \Sigma_e \cap \dot{T}^*_{x_0}$ then we have (2.7) and (2.8) with $F_j^{\pm} = 0$. Therefore it follows that $F_i = F_j$ in $A^s(\Gamma_i \cap \Gamma_j)$ and the microlocal constructions fit together also in $\mathbb{R}^n$ with $n > 2$, determining a local two-sided parametrix F of P. Thus, in this case, we obtain that the operator P is locally s-solvable at x_0 and s-microhypoelliptic at every $\gamma \in \dot{T}^*_{x_0}$ for $s \in]1, r/(r-1)[$ (see [9] for the local s-solvability of P in $\mathbb{R}^2$, while the s-microhypoellipticity of P in $\mathbb{R}^n$, $n \geq 2$, has been proved in [3]).

REFERENCES

[1] F. CARDOSO, *A necessary condition of Gevrey solvability for differential operators with double characteristics*. Comm. Partial Differential Equations, **14** (1989), 981-1009.

[2] F. CARDOSO - F. TREVES, *A necessary condition of local solvability for pseudodifferential equations with double characteristics*. Ann. Inst. Fourier (Grenoble), **24:1** (1974), 225-292.

[3] L. CATTABRIGA - L. RODINO - L. ZANGHIRATI, *Analytic-Gevrey hypoellipticity for a class of pseudodifferential operators with multiple characteristics*. Comm. Partial Differential Equations, **15** (1989), 81-96.

[4] L. CATTABRIGA - L. ZANGHIRATI, *Fourier integral operators of infinite order on Gevrey spaces. Applications to the Cauchy problem for certain hyperbolic operators*. J. Math. Kyoto Univ., **30** (1990), 149-192.

[5] J.J. DUISTERMAAT - J. SJÖSTRAND, *A global construction for pseudo-differential operators with non-involutive characteristics*. Invent. Math., **20** (1973), 209-225.

[6] Y. EGOROV, *On necessary conditions for local solvability of pseudodifferential equations of principal type*. Trudy Moskov. Mat. Obshch., **24** (1971), 24-42 (Russian); Trans. Moscow Math. Soc., **24** (1971), 632-635.

[7] R. GOLDMAN, *A necessary condition for local solvability of a pseudodifferential equation having multiple characteristics*. J. Differential Equations, **19** (1975), 176-200.

[8] T. GRAMCHEV, *Non solvability for analytic partial differential operators with multiple complex characteristic*. To appear.

[9] T. GRAMCHEV, *Powers of Mizohata type operators in Gevrey classes*. Boll. Un. Mat. Ital. B (7) **1**, 1991.

[10] V.V. GRUŠIN, *On a class of elliptic pseudo differential operators degenerate on a submanifold.* Mat. Sb., **84** (1977), 163-195; Math. USSR-Sb., **13** (1971), 155-185.

[11] N. HANGES, *Almost Mizohata operators.* Trans. Amer. Math. Soc., **293** (1986), 663-675.

[12] L. HÖRMANDER, *The analysis of linear partial differential operators, Voll. I-IV.* Springer-Verlag, Berlin, 1983-85.

[13] H. KOMATSU, *Ultradistributions I, structure theorems and a characterization*, J. Fac. Sci. Univ. Tokyo, Sect. IA Math., **20** (1973), 25-105.

[14] H. NINOMIYA, *Necessary and sufficient conditions for the local solvability of the Mizohata equations.* J. Math. Kyoto Univ., **28:4** (1988), 593-603.

[15] L. NIRENBERG - F. TREVES, *On local solvability of linear partial differential equations, I-Necessary conditions.* Comm. Pure Appl. Math., **23** (1970), 1-38.

[16] P. POPIVANOV, *On the local solvability of a class of partial differential equations with double characteristics.* Trudy Sem. Petrovsk. **1**, (1975), 237-278 (Russian); Amer. Math. Soc. Transl., **118:2** (1982), 51-89.

[17] L. RODINO - A. CORLI, *Gevrey solvability for hyperbolic operators with constant multiplicity.* Recent developments in hyperbolic equations, Proc. of Conference "Hyperbolic Equations" - Pisa 1987, Longman Harlow, 1988.

[18] L. RODINO - L. ZANGHIRATI, *Pseudodifferential operators with multiple characteristics, and Gevrey singularities*, Comm. Partial Differential Equations, **11** (1986), 673-711.

[19] F. TREVES, *Introduction to pseudodifferential and Fourier integral operators, Vol. I*, Plenum Press, New York, 1980.

[20] F. TREVES, *Remarks about certain first order linear PDE in two variables.* Comm. Partial Differential Equations, **5** (1980), 381-425.

[21] L. ZANGHIRATI, *Pseudodifferential operators of infinite order and Gevrey classes*, Ann. Univ. Ferrara, Sez. VII, Sc. Mat., **31** (1985), 197-219.

Dipartimento di Matematica
Università di Ferrara
Via Machiavelli, 35
I-44100, Ferrara
Italy