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A Free Boundary Problem with a Volume Penalization

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0. – Introduction

In this paper we study the problem of minimizing

$$J_\varepsilon(v) := \int_\Omega \Big(\frac{1}{2}|\nabla v|^2 - 2v\Big)dx + f_\varepsilon(|\{x \in \Omega / v(x) > 0\}|)\,, \qquad v \in K^c\,,$$

where

$$f_\varepsilon(s) := \begin{cases} \varepsilon(s-\omega_0) & s \le \omega_0\,, \\ \dfrac{1}{\varepsilon}((s-\omega_0)+(s-\omega_0)^2) & s \ge \omega_0 \end{cases}$$

(we use $|S|$ to denote the Lebesgue measure of a set S). The class K^c consists of all functions v in $H^1(\Omega) \cap L^1(\Omega)$ such that $v = c$ on $\partial\Omega$. Here Ω is an unbounded domain in $\mathbb{R}^n$, more precisely $\Omega := \mathbb{R}^n \setminus \overline{H}$ where H is a bounded domain; c, ε, ω_0 are positive constants.

The present variational problem is motivated by the following optimal design problem: "Among all cylindrical elastic bars, with cross-section of a given area and with a single given hole in it, find the one with the maximum torsional rigidity". This application is studied in the forthcoming paper [10] where we solve this problem for holes belonging to a certain class. Given a hole in that class, we prove —denoting by ω_0 the area of the cross-section and by H the hole— that the optimal cross-section is given by the set $\{u > 0\}$, where u is the solution to the present variational problem for an appropriate pair of constants c and ε (see [10] for a complete discussion).

The aim of this paper is to prove that a solution to our minimization problem exists and to study regularity properties of any solution u and the corresponding free boundary $\Omega \cap \partial\{u > 0\}$: We show that any solution is Lipschitz continuous and that the free boundary is locally analytic except —possibly— for a closed set of $(n-1)$-dimensional Hausdorff measure zero. Moreover, in two dimensions singularities cannot occur, i.e. the free boundary is locally analytic.

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Furthermore, we prove that any solution u satisfies

$$(0.1)\qquad \begin{aligned} &\Delta u = -2 && \text{in} \quad \Omega \cap \{u > 0\}, \\ &u = 0, \quad \partial_{-\nu} u = \lambda_u && \text{on} \quad \Omega \cap \partial\{u > 0\}. \end{aligned}$$

Here λ_u is a positive constant and ν denotes the unit outward normal to the free boundary. The free boundary condition is satisfied globally in a weak sense and therefore in the classical sense along the regular part of $\Omega \cap \partial\{u > 0\}$.

In addition, we show that for small values of the penalization parameter ε the volume of $\{u > 0\}$ automatically adjusts to ω_0 (recall the application described above). The fact that it is not necessary to pass to the limit in ε to adjust the volume of $\{u > 0\}$ to ω_0, allows us —for small values of ε— to get solutions satisfying this property and preserving at the same time the regularity properties mentioned above. This will be a key point in [10].

We also study the dependence of our results on some of the data of the problem, namely on H, c and ε. This analysis is motivated by the nature of the problem studied in [10] and by the stability theory developed in that paper which also includes a stability discussion of our present variational problem as both the constant c and the domain H vary (see Section 5 in [10] for further details).

Many of the ideas we use here were inspired by the papers [1] and [2]. However, our problem presents several new aspects. On one hand, the existence of a solution to our problem is not immediate due to the unboundedness of the domain Ω. We deal with this difficulty by solving first a family of auxiliary minimization problems where the admissible functions have uniformly bounded support.

Another interesting feature appears when studying the regularity of the free boundary. Since we are dealing with weak solutions of (0.1), we cannot directly apply the regularity theory established in [2] (where the right hand side of the equation is zero), nor can we proceed as in the case of the obstacle problem (where the right hand side of the equation is positive). Instead, we combine here the ideas in [2] with a careful rescaling argument which eventually allows us to show that near the free boundary our solutions behave like those in [2].

An outline of the paper is as follows: In Section 1 we formulate the variational problem. In Section 2 we show that the functional has a minimum if we impose that the admissible functions have their supports in a ball $B_R(0)$. We prove preliminary regularity properties of these minima, in particular: $u \geq 0$, u is Lipschitz continuous, $\Delta u = -2$ in $\{u > 0\}$, u has linear growth away from the free boundary $\Omega \cap \partial\{u > 0\}$. We also prove the important fact that for R large enough $\{u > 0\}$ is connected and bounded independently of R.

This allows us to prove the existence of a solution to our original problem, and to show that any solution satisfies analogous properties to the ones just mentioned. In particular, we show that any solution to our original problem has a bounded and connected support (Section 3).

We then obtain preliminary properties of the free boundary in Sections 4 and 5. We show that $\Omega \cap \partial\{u > 0\}$ has finite $(n-1)$-dimensional Hausdorff measure and, in addition, $\Delta u + 2\chi(\{u > 0\})$ is a Radon measure given by a function q_u times the surface measure of $\Omega \cap \partial\{u > 0\}$. Therefore, $\{u > 0\}$ is a set of locally finite perimeter and on the reduced boundary $\partial_{\text{red}}\{u > 0\}$ the normal derivative is well defined. We prove that almost everywhere on the free boundary, the function q_u equals a positive constant λ_u and thus (0.1) is satisfied in a weak sense.

Next, we study the regularity of the free boundary (Sections 6 and 7). We discuss the behaviour of a solution near "flat" free boundary points, and we obtain the analyticity of the free boundary near such points. We also show that singularities cannot occur in two dimensions.

Finally, in Section 8 we show that for ε small enough, the volume of $\{u > 0\}$ automatically adjusts to ω_0. We prove in an Appendix at the end of the paper some results on harmonic functions and on blow-up sequences that are used in different situations throughout the work and cannot be found elsewhere.

The results in this paper extend to the functional

$$J(v) = \int_\Omega (|\nabla v|^2 - gv)dx + f(|\{x \in \Omega / v(x) > 0\}|),$$

for a large family of regular functions g, with $g \geq \gamma$ (γ a positive constant) and for a function f more general than our f_ε. Moreover, the hypotheses on the domain Ω assumed here can be relaxed, as well as the boundary conditions imposed on $\partial\Omega$, and the main results will still hold.

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Notation

- $|S|$ n-dimensional Lebesgue measure of the set S
- $\mathcal{H}^k$ k-dimensional Hausdorff measure
- $B_r(x_0)$ open ball of radius r and center x_0
- $\displaystyle \fint_{B_r(x_0)} u = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u\,dx$
- $\displaystyle \fint_{\partial B_r(x_0)} u = \frac{1}{\mathcal{H}^{n-1}(\partial B_r(x_0))} \int_{\partial B_r(x_0)} u\,d\mathcal{H}^{n-1}$
- $\chi(S)$ characteristics function of the set S
- spt u support of the function u.

1. – Statement of the problem

In this section we give a precise statement of the problem we are going to study.

We assume $n \geq 2$. Suppose we are given

1) a bounded domain H in $\mathbb{R}^n$ with boundary of class C^2 and such that $\mathbb{R}^n \setminus \overline{H}$ is connected,
2) a number $\omega_0 > 0$,
3) a number $c > 0$ and a small number $\varepsilon > 0$.

Our purpose is to study the problem:
$P_\varepsilon^c = P_\varepsilon^c(H)$: minimize

$$J_\varepsilon(v) := \int_\Omega \Big(\frac{1}{2}|\nabla v|^2 - 2v\Big)dx + f_\varepsilon(|\{x \in \Omega / v(x) > 0\}|)$$

among all functions $v \in K^c$, where $\Omega := \mathbb{R}^n \setminus \overline{H}$,

$$f_\varepsilon(s) := \begin{cases} \varepsilon(s - \omega_0) & s \leq \omega_0\,, \\ \dfrac{1}{\varepsilon}((s-\omega_0) + (s-\omega_0)^2) & s \geq \omega_0\,, \end{cases}$$

and

$$K^c = K^c(H) := \{v \in H^1(\Omega) \cap L^1(\Omega)\,,\, v = c \text{ on } \partial\Omega\}\,.$$

We will show that a solution to P_ε^c exists and we will study regularity properties of every solution and the corresponding free boundary, i.e. the boundary of the set where the solution vanishes.

In addition, we will study the dependence of our results on some of the data of the problem, namely on H, c and ε. This analysis is motivated by the problem studied in [10], where all our results are applied. There, a stability theory is developed involving a stability analysis of our problem $P_\varepsilon^c(H)$ as both the constant c and the domain H vary (see Section 5 in [10] for further details).

In order to have a clear understanding of the dependence of our results on the constant c and on the domain H, we fix

(1.1) a number $r^* > 0$, such that H satisfies the uniform interior sphere condition of radius r^*,

(1.2) a number $R_0 > 0$, such that $\{x \in \mathbb{R}^n / \operatorname{dist}(x, H) \leq 1\} \subset B_{R_0}(0)$,

(1.3) a function $u^c = u_H^c \in H_0^1(B_{R_0}(0))$ satisfying $u^c = c$ in H, $u^c \geq 0$, $|\{x \notin H / u^c(x) > 0\}| \leq \omega_0$,

and we define

(1.4) $l = l_H^c := \|\nabla u^c\|^2_{L^2(\mathbb{R}^n)}$.

In this way, the relevant parameters in the paper will be $n, \omega_0, c, \varepsilon, r^*, R_0$ and l.

Since the domain Ω is unbounded, we will not directly solve problem P_ε^c. We will first study an auxiliary version, imposing that the admissible functions have their supports in a ball $B_R(0)$, showing later that for large values of R both problems coincide.

Therefore, for $R \geq R_0$ we state the problem:

$$P_{\varepsilon,R}^c = P_{\varepsilon,R}^c(H): \quad \text{minimize} \quad J_\varepsilon(v)$$

among all functions $v \in K_R^c$, where $\Omega_R := B_R(0) \setminus \overline{H}$ and

$$K_R^c = K_R^c(H) := \{v \in H^1(\Omega_R), v = c \text{ on } \partial H \text{ and } v = 0 \text{ on } \partial B_R(0)\}.$$

We point out that, when necessary, the functions $v \in K_R^c$ will be considered as belonging to $H^1(\mathbb{R}^n)$, by assuming $v = c$ in H and $v = 0$ in $\mathbb{R}^n \setminus B_R(0)$ (analogously with K^c).

Note that we have

$$u^c \in K^c \text{ and } \inf_{v \in K^c} J_\varepsilon(v) \leq l, \tag{1.5}$$

and also, for $R \geq R_0$

$$u^c \in K_R^c \text{ and } \inf_{v \in K_R^c} J_\varepsilon(v) \leq l. \tag{1.6}$$

2. – Existence of a solution to problem $P_{\varepsilon,R}^c$. Basic properties

In this section we show that there exists a solution to the auxiliary problem $P_{\varepsilon,R}^c$ (Theorem 2.3) and we prove some regularity properties that hold for any solution u.

We first show the very basic properties: u is pointwise defined and it satisfies globally $\Delta u \geq -2$ (Lemma 2.4). In addition $u \geq 0$ and $\Delta u = -2$ where positive (Lemmas 2.5 and 2.10). In Theorem 2.6 we get an important bound for the measure of $\{x \in \Omega_R / u(x) > 0\}$. The fact that this bound does not depend on R implies that if R is large enough and u is a solution to $P_{\varepsilon,R}^c$, there exists a set of positive measure in Ω_R where u vanishes and therefore a free boundary $\Omega_R \cap \partial\{u > 0\}$.

We continue our study by proving regularity and nondegeneracy results in the style of [2]: we prove that u grows linearly as we go away from the free boundary, i.e. there are positive constants C_1 and C_2 such that

$$C_1 \operatorname{dist}(x, \Omega_R \cap \partial\{u > 0\}) \leq u(x) \leq C_2 \operatorname{dist}(x, \Omega_R \cap \partial\{u > 0\})$$

near the free boundary, provided $u(x) > 0$ (Lemma 2.13). As a consequence, u is Lipschitz continuous (Lemma 2.14), which is the maximum regularity of u in the whole domain Ω_R. Lemmas 2.9 and 2.16 give a weak version of the linear growth property in terms of the average of u over spheres with center in the free boundary; the first of these lemmas estimates the average from above, and the second one, gives an estimate from below.

We show that the free boundary stays away from ∂H (Theorem 2.11) and we also show that u satisfies the positive density property for $\{u > 0\}$ and for $\{u = 0\}$ at the free boundary (Theorem 2.17).

Throughout the section we carefully study the dependence of our results on the domain H and on the constants of the problem. For this purpose, we define two parameters μ and δ which take into account all these data (see Remarks 2.7 and 2.12), and we get estimates depending on these new parameters.

We finally prove the important fact that for R large enough $\{u > 0\}$ is connected and bounded independently of R (Theorem 2.18).

Let us define for $v \in H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$

$$J(v) = \int_{\mathbb{R}^n} \Big(\frac{1}{2}|\nabla v|^2 - 2v\Big)dx .$$

We will need two preliminary lemmas:

LEMMA 2.1. *If $v \in H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$, then*

$$J(v) \geq -C|\{v > 0\}|^{\frac{n+2}{n}} ,$$

where $C = \dfrac{2}{n(n+2)}|B_1(0)|^{-2/n}$.

PROOF. Since $J(v) \geq J(\max(v, 0))$, we may assume that $v \geq 0$. We will also assume without loss of generality, that $\{v > 0\}$ is bounded. Let B be a ball such that $|B| = |\{v > 0\}|$ and let $v^* \in H_0^1(B)$ be the Schwarz symmetrization of v (see [4] or [8] for definition and properties), which satisfies

$$J(v) \geq J(v^*) . \tag{2.1}$$

By well known results, there exists a unique function $\tilde{v} \in H_0^1(B)$ satisfying

$$J(w) \geq J(\tilde{v}) \text{ for every } w \in H_0^1(B) , \tag{2.2}$$

and explicit calculations show that $\tilde{v} > 0$ in B and

$$J(\tilde{v}) = -C(n)|B|^{\frac{n+2}{n}} , \tag{2.3}$$

where $C(n) := \dfrac{2}{n(n+2)}|B_1(0)|^{-2/n}$. Now the desired inequality follows from (2.1), (2.2), (2.3) and the choice of the ball B. □

LEMMA 2.2. *Let $D \subset \mathbb{R}^n$ be an open set. There is a positive constant C such that if $v \in H_0^1(D)$, then*

$$\|v\|_{H^1(D)} \leq C(1+|D|^{1/n})(J(v)+|D|^{\frac{n+2}{n}})^{1/2}.$$

Here C depends on n only.

PROOF. Without loss of generality we may assume that D is bounded and $v \in C_0^1(D)$. By Poincaré's Inequality ([6, p. 164]), there is a constant $C = C(n)$ such that

$$\left(\int_D v^2\right)^{1/2} \leq C(n)|D|^{1/n}\left(\int_D |\nabla v|^2\right)^{1/2}. \tag{2.4}$$

This, together with Hölder Inequality yields

$$\int_D |\nabla v|^2 = 2J(v) + 4\int_D v \leq 2J(v) + \frac{2}{\alpha}C(n)|D|^{\frac{n+2}{n}} + 2\alpha\int_D |\nabla v|^2$$

for any $\alpha > 0$. Choosing for instance $\alpha = 1/4$ and using again (2.4) we get the desired inequality. □

THEOREM 2.3. *There exists a solution u to problem $P^c_{\varepsilon,R}$.*

PROOF. Using Lemma 2.1, we get $J_\varepsilon(v) \geq -\widetilde{C}$ for all $v \in K_R^c$. We now consider a minimizing sequence u_k, and by Lemma 2.2 we have $\|u_k\|_{H^1(\Omega_R)} \leq \overline{C}$ (here $\widetilde{C}$ and $\overline{C}$ denote positive constants). Hence for some $u \in K_R^c$ and a subsequence

$$\nabla u_k \to \nabla u \text{ weakly in } L^2(\Omega_R),$$

$$u_k \to u \text{ in } L^1(\Omega_R) \text{ and almost everywhere in } \Omega_R.$$

Moreover,

$$|\{x \in \Omega_R / u(x) > 0\}| < \liminf_{k\to\infty} |\{x \in \Omega_R / u_k(x) > 0\}|,$$

$$\int_{\Omega_R} |\nabla u|^2 \leq \liminf_{k\to\infty} \int_{\Omega_R} |\nabla u_k|^2,$$

by the well known semicontinuity properties of these functionals. It follows that u is a solution to $P^c_{\varepsilon,R}$. □

LEMMA 2.4. *If u is a solution to problem $P^c_{\varepsilon,R}$, then $\Delta u \geq -2$ in the distribution sense in Ω_R. Hence we can assume that*

$$u(x) = \lim_{r\downarrow 0} \fint_{B_r(x)} u \quad \forall x \in \Omega,$$

and u is upper semicontinuous.

Proof. For a nonnegative $\eta \in C_0^\infty(\Omega_R)$ and $t > 0$ we have $J_\varepsilon(u - t\eta) \geq J_\varepsilon(u)$ which implies

$$\int_{\Omega_R} (-\nabla u \cdot \nabla \eta + 2\eta) \geq 0,$$

that is, $\Delta u \geq -2$ in the distribution sense. Then, the function $v := u + \frac{|x|^2}{n}$ is subharmonic. It follows that the second assertion of the lemma holds for v and therefore for u. The proof is complete. □

Lemma 2.5. *If u is a solution to $P_{\varepsilon,R}^c$, then $u \geq 0$.*

Proof. Since $J_\varepsilon(v) \leq J_\varepsilon(u)$ for $v := \max(u, 0)$, with strict inequality if $|\{x \in \Omega_R / u(x) < 0\}| > 0$, it follows that $u \geq 0$ in Ω_R. □

In the sequel we shall not indicate the explicit dependence of constants upon ω_0 or R_0.

Theorem 2.6. *There exist positive constants ε_0 and C such that if $\varepsilon \leq \varepsilon_0$ and u is a solution to $P_{\varepsilon,R}^c$, then*

$$|\{x \in \Omega_R / u(x) > 0\}| \leq C.$$

Here ε_0 depends on n only and $C = C(n, |H|, l)$, where l is given by (1.4). *C depends continuously on $|H|$ and l.*

Proof. Let $s(u) := |\{x \in \Omega_R / u(x) > 0\}$. If $s(u) \leq \max(1 - |H|, \omega_0)$, nothing to prove. Otherwise, let $p(s) := -\varepsilon C(n)(s + |H|)^2 + (s - \omega_0)^2 - \varepsilon l$, where $C(n)$ and l denote respectively the constants in Lemma 2.1 and (1.4). From Lemma 2.1 and from (1.6) we find that $p(s(u)) < 0$. It follows that if we choose $\varepsilon \leq \varepsilon_0 := \frac{1}{2C(n)}$ and denote by s_2 the largest zero of the quadratic function p, then

$$s(u) \leq s_2 = s_2(n, \varepsilon, |H|, l) \leq C(n, |H|, l),$$

hence the theorem is established. □

Remark 2.7. In the sequel we shall assume $\varepsilon \leq \varepsilon_0, \varepsilon_0$ given by Theorem 2.6, and we shall fix a positive constant μ, with the following property:

$$|\{x \in \Omega_R / u(x) > 0\}| \leq \mu \text{ for every } R \geq R_0 \text{ and for every solution } u \text{ to } P_{\varepsilon,R}^c, \tag{2.5}$$

noting that such a constant exists by Theorem 2.6.

From the fact that μ does not depend on R, and from Lemma 2.5, we conclude that if R is large enough and u is a solution to $P_{\varepsilon,R}^c$, there exists a set in Ω_R where u vanishes and therefore a free boundary $\Omega_R \cap \partial\{u > 0\}$.

LEMMA 2.8. *There exists a positive constant C such that if u is a solution to $P^c_{\varepsilon,R}$, $D \subset \Omega_R$ is an open set and s is the function satisfying $\Delta s = -2$ in D and $s = u$ on ∂D, then*

$$\int_D |\nabla(u-s)|^2 \leq C(1+\mu+|D|)|\{y \in D/u(y)=0\}|.$$

Here C depends on ε only and μ is given by (2.5).

PROOF. Let us extend s by u into $\Omega_R \backslash D$. Since $u \geq 0$, s is positive in D. Clearly $s \in K^c_R$ and therefore $J_\varepsilon(u) \leq J_\varepsilon(s)$. Then, since

$$f_\varepsilon(|\{s>0\}|) - f_\varepsilon(|\{u>0\}|) \leq C(\varepsilon)(1+\mu+|D|)|\{y \in D/u(y)=0\}|,$$

where μ is given by (2.5), the lemma follows. □

LEMMA 2.9. *There exists a positive constant C, such that if u is a solution to $P^c_{\varepsilon,R}$, $r < 1$ and $B_r(x) \subset \Omega_R$, the following property holds:*

$$\frac{1}{r} \fint_{\partial B_r(x)} u \geq C \text{ implies } u > 0 \text{ in } B_r(x).$$

Here $C = C(n, \varepsilon, \mu)$, and μ is given by (2.5).

PROOF. The idea of the proof is that if the average of u on $\partial B_r(x)$ is large, then replacing u in $B_r(x)$ by the function s satisfying

$$\Delta s = -2 \text{ in } B_r(x), \quad s = u \text{ on } B_r(x),$$

will decrease the functional unless $s = u$.

Since $r < 1$, we have by Lemma 2.8

$$\int_{B_r(x)} |\nabla(u-s)|^2 \leq C(n,\varepsilon,\mu)|\{y \in B_r(x)/u(y)=0\}|. \tag{2.6}$$

We wish to estimate the measure of $B_r(x) \cap \{u = 0\}$ from above by the left hand side of (2.6). Proceeding as in [2, Lemma 3.2] and using that s can be estimated from below by the harmonic function in $B_r(x)$ with boundary values u, we obtain

$$\left(\frac{1}{r}\fint_{\partial B_r(x)} u\right)^2 |\{y \in B_r(x)/u(y)=0\}| \leq C(n) \int_{B_r(x)} |\nabla(u-s)|^2. \tag{2.7}$$

It follows from (2.6) and (2.7) that if

$$\frac{1}{r}\fint_{\partial B_r(x)} u \geq C,$$

and C is sufficiently large depending only on n, ε and μ, then $u > 0$ almost everywhere in $B_r(x)$. From (2.6) we deduce that $u = s$ almost everywhere in $B_r(x)$ and by Lemma 2.4 $u = s > 0$ everywhere in $B_r(x)$. □

LEMMA 2.10. *If u is a solution to $P^c_{\varepsilon,R}$, then $\Delta u = -2$ in the open set $\{x \in \Omega_R / u(x) > 0\}$.*

PROOF. Let $x \in \Omega_R$ such that $u(x) > 0$ and fix α such that $u(x) > \alpha > 0$. Then, by Lemma 2.4

$$\fint_{B_r(x)} u > \alpha$$

for small r. This fact allows us to apply Lemma 2.9 in a small ball $B_\rho(x)$. Hence the set $\{x \in \Omega_R / u(x) > 0\}$ is open, and the result now follows from Lemma 2.8. □

For $0 < \delta < 1$ let us define

$$D_\delta = D_\delta(H) := \{x \in \mathbb{R}^n / 0 < \operatorname{dist}(x, H) < \delta\}. \tag{2.8}$$

Since $R \geq R_0$, we have by (1.2)

$$D_\delta \subset \{x \in \mathbb{R}^n / 0 < \operatorname{dist}(x, H) \leq 1\} \subset \Omega_R .$$

We will now show that the free boundary $\Omega_R \cap \partial\{u > 0\}$ stays away from ∂H.

THEOREM 2.11. *There exists a constant $0 < \delta_0 < 1$, such that if u is a solution to $P^c_{\varepsilon,R}$, then*

$$D_{\delta_0} \subset \{x \in \Omega_R / u(x) > 0\}.$$

Here $\delta_0 = \delta_0(n, \varepsilon, c, r^*, \mu)$, where r^* and μ are given by (1.1) and (2.5) respectively. δ_0 depends continuously on c.

PROOF. The idea of the proof is similar to that of Lemma 2.9.

Let $x_0 \in \partial H, 0 < \delta < \min(1, r^*)$, r^* given by (1.1) and consider B_δ, the interior tangent ball to H in x_0, of radius δ. For convenience let us assume that $B_\delta = B_\delta(0)$. Let s be the function satisfying

$$\Delta s = -2 \text{ in } \Omega_R \cap B_{2\delta}, \quad s = u \text{ on } \partial(\Omega_R \cap B_{2\delta})$$

(here we denote $B_{2\delta} = B_{2\delta}(0)$). Assume, in addition, that $s = u = c$ in $B_{2\delta} \setminus \Omega_R$.

We will first estimate s from below by the harmonic function w in $B_{2\delta} \setminus B_\delta$ such that $w = c$ on ∂B_δ and $w = 0$ on $\partial B_{2\delta}$, using the Maximum Principle. Therefore we get

$$s(x) \geq w(x) \geq c\frac{C(n)}{\delta}(2\delta - |x|) \text{ for } x \in \Omega_R \cap B_{2\delta}. \tag{2.9}$$

On the other hand we have by Lemma 2.8

$$\int_{\Omega_R \cap B_{2\delta}} |\nabla(u - s)|^2 \leq C(n, \varepsilon, \mu)|\{x \in \Omega_R \cap B_{2\delta} / u(x) = 0\}|. \tag{2.10}$$

We wish now to estimate the measure of $B_{2\delta} \cap \{u = 0\}$ from above by the left hand side of (2.10). For any $\xi \in \partial B_1(0)$ we define

$$r_\xi = \inf\{r/\delta \le r \le 2\delta \text{ and } u(r\xi) = 0\},$$

if the set is nonempty and $r_\xi = 2\delta$ otherwise.

Proceeding as in [2, Lemma 3.2], but using (2.9) instead, we get

$$\frac{c^2}{\delta^2}(2\delta - r_\xi) \le C(n) \int_{r_\xi}^{2\delta} |\nabla(u-s)(r\xi)|^2 dr$$

for almost all $\xi \in \partial B_1(0)$, from which we deduce

$$\frac{c^2}{\delta^2}|\{x \in \Omega_R \cap B_{2\delta}/u(x) = 0\}| \le C(n) \int_{\Omega_R \cap B_{2\delta}} |\nabla(u-s)|^2 dx .$$

Hence (2.10) implies that if δ is sufficiently small, depending only on n, ε, μ and c, then $u > 0$ everywhere in $\Omega_R \cap B_{2\delta}$. Thus the desired conclusion is established. □

REMARK 2.12. In the sequel we shall fix a constant $\delta, 0 < \delta < 1$ with the following property:

$$D_\delta \subset \{x \in \Omega_R/u(x) > 0\} \text{ for every } R \ge R_0 \quad \text{and for every solution } u \text{ to } P^c_{\varepsilon,R}, \tag{2.11}$$

noting that such a constant exists by Theorem 2.11.

Given a solution u to $P^c_{\varepsilon,R}$, we will let

$$d(x) = d_u(x) := \operatorname{dist}(x, \Omega_R \cap \partial\{u > 0\})$$

for any $x \in \Omega_R$ such that $u(x) > 0$. In particular we will have $d(x) > 0$ and $B_{d(x)}(x) \cap \Omega_R \subset \{u > 0\}$.

We will now show that u grows linearly away from the free boundary.

LEMMA 2.13. *There exist positive constants C_1 and C_2 such that if u is a solution to $P^c_{\varepsilon,R}$, $x_0 \in \{u > 0\}$, $d(x_0) < 1$ and $B_{d(x_0)}(x_0) \Subset \Omega_R$, then*

$$C_1 d(x_0) \le u(x_0) \le C_2 d(x_0) .$$

Here $C_1 = C_1(n, \varepsilon)$, $C_2 = C_2(n, \varepsilon, \mu)$ and μ is given by (2.5).

PROOF. Let $r := d(x_0)$ and let $B_\rho := B_\rho(x_0)$ for $\rho > 0$. To establish the first inequality, we will assume that

$$u(x_0) < \alpha r \tag{2.12}$$

and derive a lower bound on α. Defining

$$v(x) := u(x) + \frac{|x - x_0|^2}{n}$$

we have $v \geq u \geq 0$ and $\Delta v = 0$ in B_r. Then by Harnack's Inequality and (2.12) we get

$$\sup_{B_{r/2}} u \leq \sup_{B_{r/2}} v \leq C(n) v(x_0) \leq C(n)\alpha r\,. \tag{2.13}$$

Let us consider $\psi \in C^\infty(\mathbb{R}^n)$ satisfying $\psi \equiv 0$ in $|x| \leq 1$, $\psi \equiv 1$ in $|x| \geq 2$, and $\psi > 0$ otherwise. Now define for $x \in B_{r/2}$

$$w(x) := \min\left(u(x), C(n)\alpha r \psi\left(\frac{4x}{r}\right)\right).$$

Extending w by u outside $B_{r/2}$ and recalling (2.13), we see that w is an admissible function, and noting that $w = 0$ in $B_{r/4}$ and $w > 0$ in $B_{r/2} \setminus B_{r/4}$ we get

$$\frac{1}{2}\int_{\Omega_R} |\nabla w|^2 - \frac{1}{2}\int_{\Omega_R} |\nabla u|^2 \leq C\alpha^2 r^n\,,$$

$$-2\int_{\Omega_R} w + 2\int_{\Omega_R} u \leq \alpha C r^{n+1}\,,$$

$$f_\varepsilon(|\{w > 0\}|) - f_\varepsilon(|\{u > 0\}|) \leq -Cr^n\,,$$

with $C = C(n, \varepsilon) > 0$. Since $J_\varepsilon(w) \geq J_\varepsilon(u)$, we deduce that $\alpha \geq C_1$, for a constant $C_1 = C_1(n, \varepsilon) > 0$ and the first inequality follows.

To show the second inequality, we observe that ∂B_r touches $\partial\{u > 0\}$, and by Lemma 2.9 we have

$$\frac{1}{r+\rho} \fint_{\partial B_{r+\rho}} u < C$$

for small $\rho > 0$, hence

$$\frac{1}{r} \fint_{\partial B_r} u \leq C\,.$$

Considering again the function v defined above, we then obtain

$$u(x_0) = \fint_{\partial B_r} v \leq C_2 r\,,$$

where $C_2 = C_2(n, \varepsilon, \mu)$. This completes the proof. □

LEMMA 2.14. *There exists a positive constant L such that if u is a solution to $P^c_{\varepsilon,R}$, $x_0 \in \Omega_R \cap \partial\{u > 0\}$, $0 < r < 1$ and $B_{2r}(x_0) \Subset \Omega_R$, then u is Lipschitz continuous in $B_r(x_0)$ with constant L. Here $L = L(n, \varepsilon, \mu)$ and μ is given by* (2.5). *It follows that $u \in C^{0,1}(\Omega_R)$.*

PROOF. If $x \in B_r(x_0)$ with $u(x) > 0$, we have by Lemma 2.13 that

$$u(x) \leq C(n, \varepsilon, \mu), \tag{2.14}$$

where we have used that $d := d(x) < 1$. Proceeding as in the proof of the same lemma we get

$$\frac{1}{d} \fint_{\partial B_d(x)} u \leq C(n, \varepsilon, \mu). \tag{2.15}$$

From the fact that $u > 0$ and then $\Delta u = -2$ in $B_d(x)$, we see that $\dfrac{\partial u}{\partial x_i}$ is harmonic in $B_d(x)$ for $1 \leq i \leq n$, implying that

$$|\nabla u(x)| \leq \frac{C(n)}{d} \fint_{\partial B_d(x)} u.$$

Therefore, recalling (2.14) and (2.15), and using that $|\nabla u| = 0$ a.e. in $\{u = 0\}$ we get the conclusion. □

LEMMA 2.15. *There exist constants $C > 0$ and $0 < r_0 < 1$ such that if u is a solution to $P^c_{\varepsilon,R}$, $x_0 \in \Omega_R \cap \partial\{u > 0\}$, $0 < r < r_0$ and $B_{2r}(x_0) \Subset \Omega_R$, then*

$$\sup_{B_r(x_0)} u \geq Cr.$$

Here $C = C(n, \varepsilon, \mu)$, $r_0 = r_0(n, \varepsilon, \mu)$ and μ is given by (2.5).

PROOF. Since

$$C_1 d(x) \leq u(x) \leq C_2 d(x) \quad \forall x \in B_r(x_0) \cap \{u > 0\} \tag{2.16}$$

by Lemma 2.13, we will construct a polygonal starting close to x_0, along which u grows linearly.

Step 1. (Essential part of the proof). Let us choose a point x_1 satisfying

$$u(x_1) > 0 \text{ and } B_{d(x_1)}(x_1) \Subset B_r(x_0).$$

Let us call $d = d(x_1)$, let y_1 be a contact point of $B_d(x_1)$ with the free boundary, and consider the function $v(x) = u(x) + \dfrac{|x - x_1|^2}{n}$, which is harmonic in $B_d(x_1)$.

Using (2.16) and Lemma 2.14, and choosing r_0 in the statement small enough (depending only on n, ε, μ), we find a constant $\alpha = \alpha(n, \varepsilon, \mu)$, $0 < \alpha < 1$ such that

$$v(x) < \frac{v(x_1)}{2} \text{ for } x \in \partial B_d(x_1) \text{ with } |x - y_1| < \alpha d\,.$$

This, together with the fact that

$$v(x_1) = \fint_{\partial B_d(x_1)} v$$

allows us to obtain a point $x_2 \in \partial B_d(x_1)$ and a constant $\delta_0 = \delta_0(n, \varepsilon, \mu) > 0$ such that $v(x_2) \geq (1 + 2\delta_0)v(x_1)$. Now using (2.16) (if again r_0 is chosen small enough) we conclude that the point x_2 satisfies

$$u(x_2) \geq (1 + \delta_0)u(x_1)\,, \quad |x_2 - x_1| = d(x_1)\,.$$

Step 2. (Induction argument). Let us assume that a priori x_1 is chosen with $|x_1 - x_0| < r/8$ and thus $|x_2 - x_1| < r/8$.

Now suppose we get points $x_1, \ldots, x_{k+1}$ satisfying

$$\begin{aligned} &u(x_i) > 0,\ B_{d(x_i)}(x_i) \Subset B_r(x_0)\,, \\ &|x_{i+1} - x_i| = d(x_i)\,, u(x_{i+1}) \geq (1 + \delta_0)u(x_i)\,, \end{aligned} \tag{2.17}$$

for $1 \leq i \leq k$, as a result of the iteration of the first step. It is not hard to see that having $\sum_{i=1}^{k} |x_{i+1} - x_i| < r/8$ will allow us to apply the first step to the point x_{k+1}.

However, (2.16) and (2.17) imply that we will be able to perform the iteration only a finite number of times, which means that for some $k_0 > 1$ the points $x_1, x_2, \ldots x_{k_0+1}$ will satisfy

$$\sum_{i=1}^{k_0-1} |x_{i+1} - x_i| < r/8 \text{ and } \sum_{i=1}^{k_0} |x_{i+1} - x_i| \geq r/8\,.$$

This, in conjunction with (2.17) yields

$$u(\overline{x}) \geq Cr, \qquad C = C(n, \varepsilon, \mu) > 0$$

for $\overline{x} = x_{k_0+1}$, and $B_{d(\overline{x})}(\overline{x}) \subset B_r(x_0) \cap \{u > 0\}$. The proof is complete. □

LEMMA 2.16. *For $0 < \kappa < 1$ there exist positive constants C_κ and r_κ, such that for each solution u to $P^c_{\varepsilon,R}$ and for each ball $B_r(x_0) \subset \Omega_R$, with $\operatorname{dist}(x_0, \partial\Omega_R) \geq \delta/2$ and $0 < r < r_\kappa$, the following property holds:*

$$\frac{1}{r} \fint_{\partial B_r(x_0)} u \leq C \text{ implies } u \equiv 0 \text{ in } B_{\kappa r}(x_0)\,.$$

Here $C_\kappa = C(n, \varepsilon, \mu, \delta, \kappa)$, $r_\kappa = r(n, \varepsilon, \mu, \delta, \kappa)$ and, μ and δ are given by (2.5) *and* (2.11) *respectively.*

PROOF.

Step 1. We will first show that under our hypotheses $B_{\kappa r}(x_0) \cap \{u > 0\} \neq \emptyset$ implies

$$\sup_{B_{\sqrt{\kappa} r}(x_0)} u \geq Cr\,, \tag{2.18}$$

for a constant $C = C(n, \varepsilon, \mu, \delta, \kappa)$, provided we choose r_κ in the statement small enough.

Let us first assume that $B_{\kappa r}(x_0) \subset \{u > 0\}$. If $d(x_0) < \delta/2$ we get by Lemma 2.13 that

$$u(x_0) \geq C_1 d(x_0) \geq C_1 \kappa r\,.$$

If $d(x_0) \geq \delta/2$, we consider the function $v(x) = \dfrac{\delta^2}{4n} - \dfrac{|x - x_0|^2}{n}$. Clearly $u - v$ is harmonic in the ball $B_{\delta/2}(x_0)$, nonnegative on its boundary and therefore $u \geq v$ in this ball. Thus

$$u(x_0) \geq v(x_0) \geq \frac{\delta r}{4n}$$

provided we choose $r_\kappa \leq \delta$.

Finally, we will show (2.18) assuming there is a point $x_1 \in B_{\kappa r}(x_0) \cap \partial\{u > 0\}$. In this case, we can apply Lemma 2.15 to $B_{\bar{r}}(x_1)$ for $\bar{r} = (\sqrt{\kappa} - \kappa) r/2$, if again r_κ is small enough. Then, we obtain

$$\sup_{B_{\bar{r}}(x_1)} u \geq Cr\,,$$

thus completing the first step.

Step 2. To establish the lemma, let us consider $w(x) := u(x) + \dfrac{|x - x_0|^2}{n}$ and let h denote the harmonic function in $B_r(x_0)$ with boundary values w. By the Mean Value Theorem we have

$$h(x_0) = \frac{r^2}{n} + \fint_{\partial B_r(x_0)} u\,. \tag{2.19}$$

On the other hand, since $h-w$ is superharmonic in $B_r(x_0)$, with zero boundary values, it follows that $h \geq w \geq u$ in $B_r(x_0)$. Thus, an application of Harnack's Inequality yields

$$\sup_{B_{\sqrt{\kappa}r}(x_0)} u \leq \sup_{B_{\sqrt{\kappa}r}(x_0)} h \leq C(n,\kappa)h(x_0)\,. \tag{2.20}$$

Now suppose that

$$\frac{1}{r} \fint_{\partial B_r(x_0)} u \leq C_\kappa$$

for some constant C_κ. This, together with (2.19) and (2.20) will imply

$$\sup_{B_{\sqrt{\kappa}r}(x_0)} u < Cr\,,$$

with C as in (2.18), provided C_κ and r_κ are small enough. Thus, the first step of the proof establishes our result. □

THEOREM 2.17. *There exist constants* $0 < \lambda_1 \leq \lambda_2 < 1$, $0 < r_0 < 1$, *such that if* u *is a solution to* $P^c_{\varepsilon,R}$, $x_0 \in \Omega_R \cap \partial\{u>0\}$, $0<r<r_0$, $B_{2r}(x_0) \Subset \Omega_R$ *and* $\text{dist}(x_0, \partial\Omega_R) \geq \delta/2$, *then*

$$\lambda_1 \leq \frac{|B_r(x_0) \cap \{u>0\}|}{|B_r(x_0)|} \leq \lambda_2\,.$$

Here $\lambda_i = \lambda_i(n,\varepsilon,\mu,\delta)$, $i=1,2$, $r_0 = r_0(n,\varepsilon,\mu,\delta)$ *and,* μ *and* δ *are given by* (2.5) *and* (2.11) *respectively.*

PROOF. If r_0 in the statement is small enough (depending only on n, ε and μ), we can proceed as in Lemma 2.15 and find a point $\overline{x}$ satisfying

$$u(\overline{x}) \geq Cr,\ B_{d(\overline{x})}(\overline{x}) \subset B_r(x_0) \cap \{u>0\}\,, \tag{2.21}$$

where $C = C(n,\varepsilon,\mu) > 0$. By Lemma 2.13 we have $C_2 d(\overline{x}) \geq u(\overline{x})$, then (2.21) yields

$$|B_r(x_0) \cap \{u>0\}| \geq |B_{d(\overline{x})}(\overline{x})| \geq Cr^n\,,$$

from which the first inequality follows.

To establish the second inequality let s denote the function satisfying

$$\Delta s = -2 \text{ in } B_r(x_0)\,,\ \ s = u \text{ on } \partial B_r(x_0)$$

and consider the function $v(x) := s(x) + \frac{|x-x_0|^2}{n}$, which is harmonic in $B_r(x_0)$. By Lemma 2.8 and Poincare's Inequality we get

$$|\{x \in B_r(x_0)/u(x)=0\}| \geq C \int_{B_r(x_0)} |\nabla(s-u)|^2 \geq \frac{C}{r^2} \int_{B_r(x_0)} (s-u)^2 \tag{2.22}$$

where $C = C(n, \varepsilon, \mu)$. We will now find a lower bound for $(s - u)$ in $B_{\alpha r}(x_0)$, for α small. Given $y \in B_{\alpha r}(x_0)$, the application of the Poisson Integral for v yields

$$v(y) - \frac{|y - x_0|^2}{n} \geq (1 - 2n\alpha) \fint_{\partial B_r(x_0)} u - \frac{(\alpha r)^2}{n} \tag{2.23}$$

and, on the other hand, from Lemma 2.13 we see that

$$u(y) \leq C_2 \alpha r\,. \tag{2.24}$$

We can now apply Lemma 2.16 in $B_r(x_0)$, provided we choose r_0 small enough. Then, from (2.23) and (2.24) it follows that

$$s(y) - u(y) \geq Cr \text{ in } B_{\alpha r}(x_0)\,, C = C(n, \varepsilon, \mu, \delta) > 0$$

for α small enough, which by (2.22) yields the second inequality. □

In the next theorem we will use some results that can be found in the Appendix (Section B) at the end of the paper.

THEOREM 2.18. *There exist positive constants M and R_1 such that if $R \geq R_1$ and u is a solution to $P^c_{\varepsilon,R}$, then $\{x \in \Omega_R / u(x) > 0\}$ is connected and contained in $B_M(0)$. Here $M = M(n, \varepsilon, \mu, \delta)$, $R_1 = R_1(n, \varepsilon, \mu, \delta) \geq R_0$ and, μ and δ are given by* (2.5) *and* (2.11) *respectively.*

PROOF. By Theorem 2.11, the set $\{x \in \Omega_R / u(x) > 0\}$ has a connected nonempty component D such that $\partial H \subset \partial D$.

Step 1. We will start by showing that there exists a positive constant $M = M(n, \varepsilon, \mu, \delta)$ such that $D \subset B_M(0)$.

By (1.2) we have that

$$B_{R_0}(0) \cap D \neq \emptyset, \tag{2.25}$$

and by (2.5)

$$|D| \leq |\{x \in \Omega_R / u(x) > 0\}| \leq \mu\,. \tag{2.26}$$

Thus, if we define, $A_k := \{x / k \leq |x| \leq k + 1/2\}$ for $k \geq R_0$, we find $k_0 = k_0(n, \mu) \geq R_0$ such that

$$A_k \not\subset D \quad \forall k \geq k_0\,. \tag{2.27}$$

Let k_1 be the smallest integer such that $D \subset B_{k_1}(0)$ and suppose $k_1 \geq k_0 + 2$ (if not, there is nothing to prove). Since D is connected, we have by (2.25) that

$$A_k \cap D \neq \emptyset \quad \text{for} \quad k_0 \leq k \leq k_1 - 2\,,$$

and therefore, by (2.27) we can choose points $x_k \in \Omega_R \cap \partial\{u > 0\}$, with $k \le |x_k| \le k + 1/2$, for $k_0 \le k \le k_1 - 2$.

Now set $r_1 = \min\{1/8, r_0/2\}$, for r_0 given by Theorem 2.17. Then, applying that theorem we get

$$|\{u > 0\} \cap B_{r_1}(x_k)| \ge \lambda_1 |B_{r_1}(x_k)| \quad k_0 \le k \le k_1 - 2. \tag{2.28}$$

Therefore, noting that the balls $B_{r_1}(x_k)$ are disjoint and that λ_1 and r_1 depend only on n, ε, μ and δ, we get from (2.26) and (2.28) that $k_1 \le M(n, \varepsilon, \mu, \delta)$, thus completing the first step.

Step 2. We will now suppose that there exists an open set $D' \ne \emptyset$, such that $D \cap D' = \emptyset$ and

$$\{x \in \Omega_R / u(x) > 0\} = D \cup D',$$

and we will find a contradiction, provided $R \ge R_1$ with R_1 a constant depending only on n, ε, μ, and δ.

Let $u_2 \in H^1(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ be the function given by

$$u_2 := \begin{cases} u & \text{in} \quad D' \\ 0 & \text{in} \quad \mathbb{R}^n \setminus D', \end{cases}$$

and let x_0 be a point satisfying

$$x_0 \in \partial D \setminus \partial H, D \subset \{(x - x_0).e_n > 0\}$$

(x_0 exists because D is bounded). Let B^* be a ball such that

$$0 < \operatorname{dist}(x_0, B^*) < 1, B^* \subset \{(x - x_0).e_n < 0\}, \ |B^*| = |\{u_2 > 0\}| \le \mu, \tag{2.29}$$

and let $u^* \in H_0^1(B^*)$ be a function satifying

$$u^* > 0 \text{ in } B^*, \quad J(u^*) \le J(u_2)$$

(we can find such a function proceeding as in Lemma 2.1). Then for $\Omega = \mathbb{R}^n \setminus \overline{H}$ and

$$\overline{u} := \begin{cases} u & \text{in} \quad D \\ u^* & \text{in} \quad B^* \\ 0 & \text{in} \quad \Omega \setminus (D \cup B^*), \end{cases}$$

we have that $\overline{u} \in H^1(\Omega) \cap L^1(\Omega)$, $\overline{u} = c$ on $\partial H = \partial\Omega$, and $J_\varepsilon(u) \ge J_\varepsilon(\overline{u})$.

By the first step, and by (2.29), there exists a constant $R_1 = R_1(n, \varepsilon, \mu, \delta) \ge R_0$, such that $\overline{u}$ has its support in $B_{R_1}(0)$. Therefore, $\overline{u}$ is a solution to $P^c_{\varepsilon,R}$, provided $R \geq R_1$. If in addition $R \geq 2R_1$, we have that $x_0 \in \Omega_R \cap \partial\{\overline{u} > 0\}$ and an application of Theorem 2.17 for small $r > 0$ yields

$$0 < \lambda_1 \leq \frac{|B_r(x_0) \cap D|}{|B_r(x_0)|}. \tag{2.30}$$

On the other hand, we could have proceeded as above in the construction of $\overline{u}$, but allowing in (2.29) that $\operatorname{dist}(x_0, B^*) = 0$. Thus, we find a solution $\overline{u}$ to $P^c_{\varepsilon,R}$, for $R \geq 2R_1$ satisfying in a neighborhood of x_0 the hypotheses of Lemma B.4 (see the Appendix at the end of the paper). Therefore we obtain

$$\frac{|B_r(x_0) \cap D|}{|B_r(x_0)|} \to 0 \quad \text{as} \quad r \to 0,$$

which contradicts (2.30) and gives the desired result. □

3. – Existence of a solution to problem P^c_ε. Basic properties

In this section we show that there is a solution to our original problem P^c_ε, and we derive regularity properties that hold for any solution.

The existence of a solution u to P^c_ε with bounded support is not hard to establish at this stage (Theorem 3.1). It follows from the fact that any solution to a problem $P^c_{\varepsilon,R}$ is supported in a ball centered in the origin with a radius independent of R (recall Theorem 2.18).

As a consequence, this solution u —and any other solution to P^c_ε with bounded support that might exist— will be a solution to $P^c_{\varepsilon,R}$ for R large enough and therefore, the results established in the previous section can be applied. The natural question here is: given a solution to P^c_ε, does it have bounded support and thus the behaviour just described? We will be able to give a positive answer, but not until the end of the section.

We proceed here as follows: We first show that every local result proved in Section 2 for problem $P^c_{\varepsilon,R}$ has an analogous version for problem P^c_ε. Namely, a solution u to P^c_ε is nonnegative and Lipschitz continuous, satisfying $\Delta u \geq -2$ globally and $\Delta u = -2$ where positive (Theorem 3.3). Also we get a bound for $|\{x \in \Omega / u(x) > 0\}|$ (Theorem 3.4) and we show that the free boundary $\Omega \cap \partial\{u > 0\}$ stays away from ∂H (Theorem 3.6). In addition, we see that u has linear growth near the free boundary (Lemmas 3,8, 3.9 and 3.10) and that the positive density property is satisfied for $\{u > 0\}$ and $\{u = 0\}$ (Theorem 3.11).

Next we show that all of these properties imply that any solution u to P^c_ε has bounded and connected support (Theorem 3.13). We eventually get to the desired conclusion which is that problems $P^c_{\varepsilon,R}$ and P^c_ε are equivalent for R large (Corollary 3.14).

As in the previous section, we are concerned with the dependence of our results on the domain H and on the constants of the problem. For this purpose we define two new parameters μ and δ (see Remarks 3.5 and 3.7) which take into account all these data. Let us finally remark that the results in this section that are stated without proof can be obtained by proceeding —with minor modifications— as in Section 2.

THEOREM 3.1. *There exists a solution u to* P_ε^c.

PROOF. Let

$$\gamma := \inf_{v \in K^c} J_\varepsilon(v),$$

and let u_k be a minimizing sequence. Without loss of generality we may assume that for large k, $\operatorname{spt} u_k \subset B_k(0)$ and moreover, u_k is a solution to $P_{\varepsilon,k}^c$. Then, by Theorem 2.18, we get for large k

$$\operatorname{spt} u_k \subset B_M(0),$$

where M is a constant not depending on k. If we choose $R \geq M$ and a solution u to $P_{\varepsilon,R}^c$, we have

$$J_\varepsilon(u) \leq J_\varepsilon(u_k).$$

Now, taking $k \to \infty$, we see that $\gamma > -\infty$ and u is a solution to P_ε^c, thus completing the proof. □

REMARK 3.2. The previous theorem guarantees the existence of a solution of problem P_ε^c, and moreover, from its proof we deduce that any solution to $P_{\varepsilon,R}^c$ is also a solution to P_ε^c, if R is large enough.

THEOREM 3.3. *Let u be a solution to* P_ε^c. *Then* $u \geq 0$, $u \in C^{0,1}(\Omega) \cap C^0(\overline{\Omega})$, $\Delta u \geq -2$ *in the distribution sense in* Ω *and* $\Delta u = -2$ *in* $\{x \in \Omega / u(x) > 0\}$.

THEOREM 3.4. *There exists a positive constant C such that if u is a solution to* P_ε^c, *then*

$$|\{x \in \Omega / u(x) > 0\}| \leq C.$$

Here $C = C(n, |H|, l)$, *where l is given by* (1.4) *and C depends continuously on* $|H|$ *and* l.

REMARK 3.5. In the sequel we shall fix a positive constant μ, with the following property:

$$|\{x \in \Omega / u(x) > 0\}| \leq \mu \text{ for every solution } u \text{ to } P_\varepsilon^c, \tag{3.1}$$

noting that such a constant exists by Theorem 3.4.

The last theorem also implies that if u is a solution to P_ε^c, there exists a set where $u \equiv 0$ and therefore a free boundary $\Omega \cap \partial\{u > 0\}$.

For $0 < \delta < 1$, we define $D_\delta = D_\delta(H)$ as in (2.8) and we have:

THEOREM 3.6. *There exists a constant* $0 < \delta_0 < 1$, *such that if u is a solution to* P_ε^c, *then*

$$D_{\delta_0} \subset \{x \in \Omega / u(x) > 0\}.$$

Here $\delta_0 = \delta_0(n, \varepsilon, c, r^*, \mu)$, *where* r^* *and* μ *are given by* (1.1) *and* (3.1) respectively. δ_0 depends continuously on c.

REMARK 3.7. In the sequel we shall fix a constant $\delta, 0 < \delta < 1$ with the following property:

(3.2) $$D_\delta \subset \{x \in \Omega / u(x) > 0\} \text{ for every solution } u \text{ to } P_\varepsilon^c,$$

noting that such a constant exists by Theorem 3.6.

LEMMA 3.8. *There exists a positive constant C, such that if u is a solution to P_ε^c, $r < 1$ and $B_r(x) \subset \Omega$, the following property holds:*

$$\frac{1}{r} \fint_{\partial B_r(x)} u \geq C \text{ implies } u > 0 \text{ in } B_r(x).$$

Here $C = C(n, \varepsilon, \mu)$, and μ is given by (3.1).

LEMMA 3.9. *For $0 < \kappa < 1$ there exist positive constants C_κ and r_κ, such that for each solution u to P_ε^c and for each ball $B_r(x_0) \subset \Omega$, with $\operatorname{dist}(x_0, \partial\Omega) \geq \delta/2$ and $0 < r < r_\kappa$, the following property holds:*

$$\frac{1}{r} \fint_{\partial B_r(x_0)} u \leq C_\kappa \text{ implies } u \equiv 0 \text{ in } B_{\kappa r}(x_0).$$

Here $C_\kappa = C(n, \varepsilon, \mu, \delta, \kappa)$, $r_\kappa = r(n, \varepsilon, \mu, \delta, \kappa)$ and, μ and δ are given by (3.1) *and* (3.2) *respectively.*

Given a solution u to P_ε^c, we will let

$$d(x) = d_u(x) := \operatorname{dist}(x, \Omega \cap \partial\{u > 0\}),$$

for any $x \in \Omega$ such that $u(x) > 0$.

LEMMA 3.10. *There exist positive constants C_1 and C_2 such that if u is a solution to P_ε^c, $x_0 \in \{u > 0\}$, $d(x_0) < 1$ and $B_{d(x_0)}(x_0) \Subset \Omega$, then*

$$C_1 d(x_0) \leq u(x_0) \leq C_2 d(x_0).$$

Here $C_1 = C_1(n, \varepsilon)$, $C_2 = C_2(n, \varepsilon, \mu)$ and μ is given by (3.1).

THEOREM 3.11. *There exist constants $0 < \lambda_1 \leq \lambda_2 < 1$, $0 < r_0 < 1$, such that if u is a solution to P_ε^c, $x_0 \in \Omega \cap \partial\{u > 0\}$ and $0 < r < r_0$, then*

$$\lambda_1 \leq \frac{|B_r(x_0) \cap \{u > 0\}|}{|B_r(x_0)|} \leq \lambda_2.$$

Here $\lambda_i = \lambda_i(n, \varepsilon, \mu, \delta)$, $i = 1, 2$, $r_0 = r_0(n, \varepsilon, \mu, \delta)$ and μ and δ are given by (3.1) *and* (3.2) *respectively.*

REMARK 3.12. Theorem 3.11 implies that $\Omega \cap \partial\{u > 0\}$ has Lebesgue measure zero, for any solution u to P_ε^c.

THEOREM 3.13. *There exists a positive constant M such that if u is a solution to P^c_ε, then*

$$\{x \in \Omega / u(x) > 0\}$$

is connected and contained in $B_M(0)$. Here $M = M(n, \varepsilon, \mu, \delta)$ and μ and δ are given by (3.1) *and* (3.2) *respectively.*

We now have, as a direct consequence of the last theorem:

COROLLARY 3.14. *There exists a constant R_1 such that*

$$u \text{ is a solution to } P^c_\varepsilon \iff u \text{ is a solution to } P^c_{\varepsilon,R} \quad \forall R \geq R_1 .$$

Here $R_1 = R_1(n, \varepsilon, \mu, \delta) \geq R_0$, and μ and δ are given by (3.1) *and* (3.2) *respectively.*

The next lemma provides bounds that will be useful later.

LEMMA 3.15. *Let u be a solution to P^c_ε.*

1) *If x_0 satisfies*

$$x_0 \in \Omega \cap \partial\{u > 0\}, \ 0 < r < 1 \text{ and } B_{2r}(x_0) \Subset \Omega ,$$

then

$$\|\nabla u\|_{L^\infty(B_r(x_0))} \leq L .$$

Here $L = L(n, \varepsilon, \mu) > 0$, and μ is given by (3.1).

2) *Let D be a domain satisfying*

$$D \Subset \Omega, D \cap \partial\{u > 0\} \neq \emptyset .$$

Then

$$\|u\|_{L^\infty(D)} + \|\nabla u\|_{L^\infty(D)} \leq C .$$

Here $C = C(n, \varepsilon, \mu, s, D) > 0$, s is any number such that $0 < s \leq \operatorname{dist}(D, \partial\Omega)$ and μ is given by (3.1).

PROOF. To prove part 1) we can proceed as in Lemma 2.14. To see 2), let us define $D' = \{x \in \mathbb{R}^n / \operatorname{dist}(x, D) < s/2\}$.

Step 1. Let $r_0 := \min\{s/4, 1\}$ and choose a point $z \in D \cap \partial\{u > 0\}$. Since D' is connected, for $x \in D'$ there exist points $x_0, \ldots, x_k$ in D' (k depending only on D and s), such that

$$x_0 = x , \quad x_k = z , \quad |x_{i+1} - x_i| < r_0/2 , \quad 1 \leq i \leq k ,$$

and there exists $0 \leq j \leq k$ such that $B_{r_0}(x_i) \subset \{u > 0\}$ for $0 \leq i \leq j - 1$ and $B_{r_0}(x_j) \not\subset \{u > 0\}$.

For $0 \leq i \leq j - 1$ we consider in $B_{r_0}(x_i)$ the harmonic function

$$v_i(x) = u(x) + \frac{|x - x_{i+1}|^2}{n} ,$$

which satisfies

$$u(x_i) \le \sup_{B_{r_0/2}(x_i)} v_i(x) \le C(n)u(x_{i+1}) \tag{3.3}$$

by Harnack's Inequality. Since from Lemma 3.10 we get $u(x_j) \le Cr_0$, $C = C(n, \varepsilon, \mu)$, we obtain, applying inductively (3.3), that $u(x) = u(x_0) \le Cr_0$. That is,

$$\|u\|_{L^\infty(D)} \le \|u\|_{L^\infty(D')} \le C\,, \qquad C = C(n, \varepsilon, \mu, s, D)\,. \tag{3.4}$$

Step 2. Let $r_1 := \min\{s/10, 1/4\}$ and let $x \in D \cap \{u > 0\}$. If $d(x) > r_1$, then

$$|\nabla u(x)| \le \frac{C(n)}{r_1} \fint_{\partial B_{r_1}(x)} u \le \frac{C(n)}{r_1} \|u\|_{L^\infty(D')}\,. \tag{3.5}$$

On the other hand, if $d(x) \le r_1$, we choose $y \in \Omega \cap \partial\{u > 0\}$ such that $|x - y| = d(x)$, and part 1) implies

$$|\nabla u(x)| \le \|\nabla u\|_{L^\infty(B_{2r_1}(y))} \le L\,, \qquad L = L(n, \varepsilon, \mu)\,,$$

which together with (3.4) and (3.5) establishes the result. □

4. – The measure λ_u and the function q_u

In this section we prove preliminary regularity properties of the free boundary.

We first prove that $\lambda_u := \Delta u + 2\chi(\{u > 0\})$ is a positive Radon measure supported in the free boundary (Lemma 4.1). Next, the linear growth condition proved in the previous section implies a density property on the free boundary (Theorem 4.2).

As a consequence we get a representation theorem, which says that $\Omega \cap \partial\{u > 0\}$ has finite $\mathcal{H}^{n-1}$ measure and that the measure λ_u is given by a function q_u times the surface measure of $\Omega \cap \partial\{u > 0\}$ (Theorem 4.3). Therefore $\{u > 0\}$ is a set of locally finite perimeter.

Our next step is the study of the behaviour of u near the reduced boundary $\partial_{\text{red}}\{u > 0\}$, that is, near the points in $\Omega \cap \partial\{u > 0\}$ where the unit outward normal with respect to $\{u > 0\}$ exists. We prove that the normal derivative of u is well defined for certain points x_0 on the reduced boundary: there, u behaves like the positive part of a linear function with slope $q_u(x_0)$ (Theorem 4.7). We finally show that this behaviour holds $\mathcal{H}^{n-1}$ almost everywhere on the free boundary (Remark 4.8).

In this section we use the notions of blow-up sequences and limits. To this effect we refer to definitions and results given separately in the Appendix at the end of the paper.

LEMMA 4.1. *If u is a solution to P_ε^c then $\lambda_u := \Delta u + 2\chi(\{u > 0\})$ is a positive Radon measure with support in $\Omega \cap \partial\{u > 0\}$.*

PROOF. Let $f(s) = \max(\min(2 - s, 1), 0)$. Since $\Delta u = -2$ in $\{u > 0\}$, we have for nonnegative functions $\eta \in C_0^\infty(\Omega)$ and $k \in \mathbb{N}$

$$-\int_\Omega \nabla u \nabla \eta + 2 \int_{\{u>0\}} \eta \geq -\int_\Omega \nabla u \nabla(\eta f(ku)) \geq -\int_{0<u<2/k} |\nabla u \nabla \eta| .$$

Letting $k \to \infty$ we conclude that $\Delta u + 2\chi(\{u > 0\}) \geq 0$ in the sense of distributions. Consequently, a measure λ_u with the desired properties exists. □

THEOREM 4.2. *Let u be a solution to P_ε^c. For any $D \Subset \Omega$, there exist constants $0 < \bar{c} \leq \overline{C}$ and $r_0 > 0$ such that for any ball $B_r \subset D$ with center in $\Omega \cap \partial\{u > 0\}$ and $r \leq r_0$*

$$\bar{c} r^{n-1} \leq \int_{B_r} d\lambda_u \leq \overline{C} r^{n-1} .$$

PROOF. Let

$$c_1 := \|\nabla u\|_{L^\infty(D)} , \tag{4.1}$$

let $\Lambda := \Delta u$, and choose $x_0 \in D \cap \partial\{u > 0\}$. For $\eta \in C_0^\infty(\Omega)$ we have

$$\int_\Omega \eta d\Lambda = -\int_\Omega \nabla \eta \nabla u .$$

Approximating $\chi(B_r(x_0))$ from below by suitable test functions η and using (4.1) we get for almost all $r < 1$ with $B_r(x_0) \subset D$

$$\int_{B_r(x_0)} d\lambda_u = \int_{\partial B_r(x_0)} \nabla u \cdot \nu d\mathcal{H}^{n-1} + 2 \int_{B_r(x_0)} \chi(\{u > 0\}) \leq \overline{C} r^{n-1} ,$$

which proves the second inequality. To prove the first one, let us fix $0 < \kappa < 1$ and let C_κ and r_κ be the constants in Lemma 3.9. Now choose $r < r_\kappa$ such that $B_r(x_0) \subset D$. By Lemma 3.9, given $0 < \alpha < 1$ there is a point $y \in \partial B_{\alpha r}(x_0)$ with $u(y) > C_\kappa \alpha r > 0$. Thus (4.1) implies

$$u(y) \leq c_1 |y - x_0| = c_1 \alpha r \text{ and } u > 0 \text{ in } B_{c(\alpha) r}(y) , \tag{4.2}$$

where $c(\alpha) := \dfrac{\alpha C_\kappa}{\alpha c_1}$. Let G_y be the positive Green function for the Laplacian in $B_r(x_0)$ with pole y. Let s be the function satisfying $\Delta s = -2$ in $B_r(x_0)$ and

$s = u$ on $\partial B_r(x_0)$. Now set $\widetilde{\Lambda} := \Delta(u-s)$. It follows from (4.2) that $d\widetilde{\Lambda} = 0$ in $B_{c(\alpha)r}(y)$, thus we can write

$$(u-s)(y) = -\int_{B_r(x_0)} G_y d\widetilde{\Lambda}\,,$$

which implies

$$\int_{B_r(x_0)} G_y d\Lambda = -u(y) + \int_{\partial B_r(x_0)} u\partial_{-\nu} G_y d\mathcal{H}^{n-1}\,.$$

Then, using (4.2) and Lemma 3.9 we have

$$\int_{B_r(x_0)} G_y d\lambda_u \geq -u(y) + \int_{\partial B_r(x_0)} u\partial_{-\nu} G_y d\mathcal{H}^{n-1}$$

$$\geq -c_1\alpha r + (1-2n\alpha)C_\kappa r \geq \tilde{c}r\,,$$

for α small enough. On the other hand, by Lemma 4.1 and (4.2)

$$\int_{B_r(x_0)} G_y d\lambda_u \leq \sup_{B_r(x_0)\setminus B_{c(\alpha)r}(y)} G_y \int_{B_r(x_0)} d\lambda_u \leq \widetilde{C}r^{2-n}\int_{B_r(x_0)} d\lambda_u\,,$$

and this establishes the desired inequality. □

We shall denote by $\mathcal{H}^{n-1}\lfloor\partial\{u>0\}$ the measure $\mathcal{H}^{n-1}$ restricted to the set $\partial\{u>0\}$.

THEOREM 4.3. *Let u be a solution to P_ε^c. Then,*

1) $\mathcal{H}^{n-1}(\Omega\cap\partial\{u>0\}) < \infty$.
2) *There is a Borel function q_u such that*

$$\Delta u + 2\chi(\{u>0\}) = q_u\mathcal{H}^{n-1}\lfloor\partial\{u>0\}\,,$$

that is, for every $\varphi \in C_0^\infty(\Omega)$ we have

$$-\int_\Omega \nabla u\nabla\varphi + 2\int_{\{u>0\}}\varphi = \int_{\Omega\cap\partial\{u>0\}} \varphi q_u d\mathcal{H}^{n-1}\,.$$

3) *For any $D \Subset \Omega$ there exist constants $0 < c^* \leq C^*$ and $r_0 > 0$ such that for every ball $B_r(x) \subset D$ with $r \leq r_0$ and $x \in \Omega\cap\partial\{u>0\}$,*

$$c^* \leq q_u(x) \leq C^*\,, c^*r^{n-1} \leq \mathcal{H}^{n-1}(B_r(x)\cap\partial\{u>0\}) \leq C^*r^{n-1}\,.$$

PROOF. It follows from Theorem 4.2 precisely as in [2, Theorem 4.5], if we us that, by Theorems 3.6 and 3.13, there exists a set $E \Subset \Omega$ such that $\Omega\cap\partial\{u>0\} \subset E$. □

REMARK 4.4. It is not hard to see that given $D \Subset \Omega$, the constants c^* and C^* in Theorem 4.3 depend only on n and on the constants $\bar{c}$ and $\overline{C}$ that we get for D in Theorem 4.2.

Let u be a solution to P_ε^c. Fix $\bar{x}$ and R satisfying

$$\bar{x} \in \Omega \cap \partial\{u > 0\}, B_{2R}(\bar{x}) \Subset \Omega\,, \quad 0 < R < 1\,,$$

and now set $D = B_R(\bar{x})$ in Theorem 4.2. It follows from its proof (and from Lemmas 3.9 and 3.15) that in this case the constants $\bar{c}$ and $\overline{C}$ (and therefore the constants c^* and C^* in Theorem 4.3) will depend only on n, ε, μ and δ (μ and δ the constants in (3.1) and (3.2) respectively).

REMARK 4.5. Let u be a solution to P_ε^c. From Theorem 4.3, 1) it follows ([5, Theorem 4.5.11]) that the set $A = \{u > 0\}$ has finite perimeter locally in Ω, that is $-\nabla\chi(A)$ is a Borel measure. We denote by $\partial_{\text{red}} A$ the reduced boundary of A, that is,

$$\partial_{\text{red}} A := \{x \in \Omega / |\nu_u(x)| = 1\}\,,$$

where $\nu_u(x)$ is the unique unit vector such that

$$\int_{B_r(x)} |\chi(A) - \chi(\{y/(y-x).\nu_u(x) < 0\})| = o(r^n) \tag{4.3}$$

for $r \to 0$, if such a vector exists, and $\nu_u(x) = 0$ otherwise; see [5], [7].

REMARK 4.6. Given a solution u to P_ε^c and a sequence of points $x_k \in \Omega$ such that $x_k \to x_0 \in \Omega$ and $u(x_k) = 0$, there exists a neighborhood $D \subset \Omega$ of x_0, where u satisfies the hypotheses B.1 in the Appendix (see Lemma 3.9 and Theorem 3.11). Therefore, we can define blow-up sequences and limits as in B.2, and make use of the results in the Appendix (Section B).

In the next statement we use the following notation (see [5, 2.10.19, 3.1.21]): For any set E and $x_0 \in E$ we denote by $\text{Tan}(E, x_0)$ the tangent cone of E at x_0, i.e.,

$$\text{Tan}(E, x_0) = \{v/v = \lim_{m\to\infty} r_m v_m, r_m > 0, x_0 + v_m \in E, v_m \to 0\}\,.$$

Given a measure μ on $\mathbb{R}^n$ and $x_0 \in \mathbb{R}^n$ we denote by $\theta^{*n-1}(\mu, x_0)$ the $(n-1)$-upper density of μ at x_0, i.e.

$$\theta^{*n-1}(\mu, x_0) = \frac{\limsup_{r\to 0} \mu(B_r(x_0))}{\alpha_{n-1} r^{n-1}}\,,$$

where α_{n-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.

THEOREM 4.7. *Let u be a solution to P_ε^c and let $x_0 \in \partial_{\text{red}}\{u > 0\}$. Then*

$$\operatorname{Tan}(\partial\{u > 0\}, x_0) = \{x / x.\nu_u(x_0) = 0\}.$$

If, in addition

$$\theta^{*n-1}(\mathcal{H}^{n-1} \lfloor \partial\{u > 0\}, x_0) \leq 1$$

and

$$\int_{B_r(x_0)\cap\partial\{u>0\}} |q_u - q_u(x_0)| d\mathcal{H}^{n-1} = o(r^{n-1}) \text{ as } r \to 0,$$

then for $x \to 0$

$$u(x_0 + x) = q_u(x_0)\max(-x.\nu_u(x_0), 0) + o(|x|).$$

PROOF. Take for simplicity $\nu_u(x_0) = e_n$. Let u_k be a blow-up sequence with respect to balls $B_{\rho_k}(x_0)$, with blow-up limit u_0. Then $\chi(\{u_k > 0\})$ converges in $L^1_{\text{loc}}(\mathbb{R}^n)$ to $\chi(\{u_0 > 0\})$ by (B.6) and to $\chi(\{x_n < 0\})$ by (4.3). It follows that $u_0 = 0$ in $\{x_n \geq 0\}$ and $u_0 > 0$ a.e. in $\{x_n < 0\}$. From (B.10) we deduce that $u_0 > 0$ in $\{x_n < 0\}$ and therefore, $\{x_n = 0\}$ is the (topological) tangent space of $\partial\{u > 0\}$ at x_0.

Now define

$$\xi(x) = \min(2(1 - |x_n|), 1)\eta(x_1, \dots, x_{n-1})$$

for $|x_n| \leq 1$ and zero otherwise, where $\eta \in C_0^\infty(B_r')(B'$ is an $(n-1)$-dimensional ball with radius r). By Theorem 4.3, we have for large k

$$-\int_{\mathbb{R}^n} \nabla u_k \nabla \xi + 2\rho_k \int_{\{u_k>0\}} \xi = \int_{\partial\{u_k>0\}} \xi(x) q_u(x_0 + \rho_k x) d\mathcal{H}_x^{n-1}.$$

Then, using (B.2), (B.3), (B.5) and (B.7), and proceeding as in [2, p. 121] we get

$$-\int_{\mathbb{R}^n} \nabla u_0 \nabla \xi = \int_{\mathbb{R}^{n-1}} q_u(x_0)\eta d\mathcal{H}^{n-1}.$$

Recalling that $\Delta u_0 = 0$ in $\{x_n < 0\}$ and $u_0 = 0$ in $\{x_n > 0\}$, we obtain

$$u_0(x) = -q_u(x_0)x_n \text{ for } x_n < 0.$$

Finally, since the blow-up sequence was arbitrary whereas the limit is unique, the last statement of the theorem follows. □

REMARK 4.8. If u is a solution to P_ε^c, then by [5, Theorem 4.5.6(2)] and [5, Theorem 2.9.9] the conclusion of Theorem 4.7 holds for $\mathcal{H}^{n-1}$ almost all x_0 in $\partial_{\text{red}}\{u > 0\}$. We observe, in addition, that Theorem 3.11 together with [5, Theorem 4.5.6(3)] imply that $\mathcal{H}^{n-1}(\Omega \cap \partial\{u > 0\} \setminus \partial_{\text{red}}\{u > 0\}) = 0$.

5. – Estimates on $|\nabla u|$ and q_u

In this section we prove estimates on $|\nabla u|$ and related ones near the free boundary, that will be needed later for the regularity theory.

Our first result is Theorem 5.1, where we prove that there exists a positive constant λ_u such that $q_u(x_0) = \lambda_u$ for $\mathcal{H}^{n-1}$ almost all x_0 in $\Omega \cap \partial\{u > 0\}$. This is an important step in proving that the free boundary condition

$$u = 0\,, \quad \partial_{-\nu} u = \lambda_u \text{ on } \Omega \cap \partial\{u > 0\}$$

is satisfied (notice that this is already known is the weak sense given by Theorems 4.3 and 4.7). The fact that almost everywhere the function q_u equals a constant —which is a regular function— will imply later the smoothness of the free boundary.

On the other hand, we observe that the free boundary condition can be written as $|\nabla u| = \partial_{-\nu} u = \lambda_u$, which evidences that the regularity of the free boundary is related to the behaviour of $|\nabla u|$ near the free boundary. In Theorem 5.1 (see (5.1)) we give an estimate from above for $|\nabla u|$ at every point in the free boundary. Later in the section we prove a strengthened version of this estimate (Theorem 5.4).

We also study the behaviour of u locally if a ball contained in $\{u=0\}$ touches the free boundary (Lemma 5.2). Finally, Lemma 5.3 provides estimates for the constant λ_u, depending on the paramenters of the problem.

In this section we will use some results that can be found in the Appendix at the end of the paper.

THEOREM 5.1. *If u is a solution to P^c_ε, then there exists a positive constant λ_u such that*

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} |\nabla u(x)| = \lambda_u \textit{ for all } x_0 \textit{ in } \Omega \cap \partial\{u > 0\}\,, \tag{5.1}$$

$$q_u(x_0) = \lambda_u \quad \textit{for } \mathcal{H}^{n-1} \textit{ almost all } x_0 \quad \textit{in} \quad \Omega \cap \partial\{u > 0\}\,. \tag{5.2}$$

PROOF.

Step 1. Suppose we are given two points $x_0, x_1 \in \Omega \cap \partial\{u > 0\}$ and ρ_k a sequence with $\rho_k > 0$, $\rho_k \to 0$. Suppose, in addition, that for $i = 0, 1$, there exists a sequence of balls $B_{\rho_k}(x_{ik}) \subset \Omega$ with $x_{ik} \to x_i$ and $u(x_{ik}) = 0$, such that the blow-up sequence with respect to $B_{\rho_k}(x_{ik})$

$$u_{ik}(x) := \frac{u(x_{ik} + \rho_k x)}{\rho_k}$$

has limit $u_i(x) = \lambda_i \max(-x.\nu_i, 0)$, with $0 < \lambda_i < \infty$ and ν_i a unit vector. We will prove that $\lambda_0 = \lambda_1$.

Assume that $\lambda_1 < \lambda_0$. We will arrive to a contradiction by constructing suitable admissible functions. For this, choose ϕ a nonnegative C_0^∞ function, $\phi \not\equiv 0$, supported in the unit interval, and let t be a small positive constant. For $k \in \mathbb{N}$ define

$$\tau_k(x) := \begin{cases} x + t\rho_k\phi\Big(\dfrac{|x - x_{0k}|}{\rho_k}\Big)\nu_0 & \text{for } x \in B\rho_k(x_{0k}), \\ x - t\rho_k\phi\Big(\dfrac{|x - x_{1k}|}{\rho_k}\Big)\nu_1 & \text{for } x \in B\rho_k(x_{1k}), \\ x & \text{elsewhere}, \end{cases}$$

which is a diffeomorphism if t is small enough (depending only on ϕ). It follows that the functions

$$v_k(x) := u(\tau_k^{-1}(x))$$

are admissible for P_ε^c, if k is large, and by (B.6) we have

$$\rho_k^{-n}|\{v_k > 0\} \cap B_{\rho k}(x_{ik})| = \int\limits_{B_1(0)\cap\{u_{ik}>0\}} \det D\tau_k(x_{ik} + \rho_k y)dy$$

$$\to \int\limits_{B_1(0)\cap\{y.\nu_i<0\}} \Big(1 + (-1)^i t\phi'(|y|)\frac{y}{|y|}.\nu_i\Big)dy,$$

for $i = 0, 1$ as $k \to \infty$. This shows that

$$\rho_k^{-n}(|\{v_k > 0\}| - |\{u > 0\}|) \to 0,$$

and since $|\{u > 0\}|$ and $|\{v_k > 0\}$ are bounded (independently of k), and f_ε is Lipschitz continuous in bounded intervals we conclude

$$f_\varepsilon(|\{v_k > 0\}|) - f_\varepsilon(|\{u > 0\}|) = o(\rho_k^n). \tag{5.3}$$

On the other hand, using (B.1) and (B.9), and proceeding as in [1, p.194] we obtain

$$\begin{aligned} &\int\limits_\Omega |\nabla v_k|^2 - \int\limits_\Omega |\nabla u|^2 \\ &= \rho_k^n\Big(t(\lambda_1^2 - \lambda_0^2) \int\limits_{B_1(0)\cap\{y_n=0\}} \phi(|y|)d\mathcal{H}_y^{n-1} + o(t)\Big) + o(\rho_k^n). \end{aligned} \tag{5.4}$$

We now notice that by Lemma 3.10 there exists a constant C (independent of k) such that for $y \in B_{\rho k}(x_{ik})$, $i = 0, 1$, we have

$$v_k(y) = C\rho_k, \quad u(y) \le C\rho_k,$$

which yields

$$(5.5)\qquad -\int_\Omega v_k + \int_\Omega u = o(\rho_k^n)\,.$$

Since we have supposed that $\lambda_1 < \lambda_0$, the main term in (5.4) becomes negative if we choose t small enough (independent of k). Then from (5.3), (5.4) and (5.5) we have that $J_\varepsilon(v_k) < J_\varepsilon(u)$ for large k, which is a contradiction and completes the first step.

Step 2. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and let

$$(5.6)\qquad \lambda = \lambda(x_0) := \limsup_{\substack{x \to x_0 \\ u(x)>0}} |\nabla u(x)|\,.$$

We will prove that $0 < \lambda < \infty$, and we will find a sequence of balls $B_{d_k}(y_k) \subset \Omega$ with $d_k > 0$, $d_k \to 0$, $y_k \to x_0$ and $y_k \in \Omega \cap \partial\{u > 0\}$, such that the blow-up sequence with respect to $B_{d_k}(y_k)$ has limit $u_0(x) = \lambda \max(-x \cdot v, 0)$, with $v = v(x_0)$ a unit vector.

By (5.6), there exists a sequence $z_k \to x_0$ such that

$$u(z_k) > 0\,, \quad |\nabla u(z_k)| \to \lambda\,.$$

Let y_k be the nearest point to z_k on $\Omega \cap \partial\{u > 0\}$ and let $d_k = |z_k - y_k|$. Consider a blow-up sequence with respect to $B_{d_k}(y_k)$ with limit u_0, such that there exists

$$v := \lim_{k\to\infty} \frac{y_k - z_k}{d_k}\,,$$

and suppose for simplicity $v = e_n$.

Using that u_0 and thus its directional derivatives are harmonic in $\{u_0 > 0\}$, and applying the results in Lemma B.3, we can proceed as in [3, p. 22] to prove that $0 < \lambda < \infty$ and

$$u_0(x) = -\lambda x_n \text{ in } \{x_n \leq 0\}\,.$$

Finally, by (B.11) we have that $0 \in \partial\{u_0 > 0\}$ and then, using (B.10) we see that u_0 satisfies the hypotheses of Theorem A.1 (see Appendix). Therefore $u_0 = 0$ in $\{x_n > 0\}$, and this completes Step 2.

Step 3. Now comes actual proof of the theorem. Choose $x_1 \in \partial_{\text{red}}\{u > 0\}$ for which the conclusion of Theorem 4.7 holds. Given $x_0 \in \Omega \cap \partial\{u > 0\}$, set

$$\lambda(x_0) = \limsup_{\substack{x \to x_0 \\ u(x)>0}} |\nabla u(x)|\,,$$

and apply Step 2 to x_0, finding in this way a sequence of balls $B_{d_k}(y_k) \subset \Omega$ and a unit vector $v(x_0)$, such that the blow-up sequence with respect to $B_{d_k}(y_k)$ has limit

$$u_0(x) = \lambda(x_0)\max(-x.v(x_0), 0)\,.$$

By Theorem 4.7 the blow-up sequence with respect to $B_{d_k}(x_1)$ has limit

$$u_1(x) = q_u(x_1)\max(-x \cdot \nu_u(x_1), 0)\,.$$

Hence, an application of Step 1 gives $\lambda(x_0) = q_u(x_1)$. If we now set $\lambda_u := q_u(x_1)$ and notice that x_0 was any point in $\Omega \cap \partial\{u > 0\}$, we obtain (5.1). The result (5.2) is now obtained from Remark 4.8.

LEMMA 5.2. *Let u be a solution to P_ε^c. If B is a ball in $\{u = 0\}$ touching $\Omega \cap \partial\{u > 0\}$ at x_0, then*

$$\limsup_{\substack{x \to x_0 \\ u(x) > 0}} \frac{u(x)}{\operatorname{dist}(x, B)} = \lambda_u\,.$$

PROOF. Denote the left-hand side by γ and let $z_k \to x_0$, $u(z_k) > 0$,

$$\frac{u(z_k)}{d_k} \to \gamma\,, \quad d_k := \operatorname{dist}(z_k, B)\,.$$

Consider the blow-up sequence u_k with respect to $B_{d_k}(y_k)$, where $y_k \in \partial B$ are points with $|y_k - x_k| = d_k$, and choose a subsequence with a blow-up limit u_0, such that

$$v := \lim_{k \to \infty} \frac{y_k - z_k}{d_k}$$

exists. Therefore, by construction we have $u_0(-v) = \gamma$,

$$u_0(x) = 0 \text{ for } x \cdot v \geq 0\,, \quad u_0(x) \leq -\gamma x \cdot v \text{ for } x \cdot v \leq 0\,,$$

and then by (B.7) $0 < \gamma < \infty$. Consequently, by the Strong Maximum Principle and by analytic continuation

$$u_0(x) = \gamma \max(-x \cdot v, 0)\,.$$

To finish the proof, we can proceed as in the last step of Theorem 5.1, but working instead with the blow-up sequence we have just constructed. Choosing a point $x_1 \in \partial_{\mathrm{red}}\{u > 0\}$ for which the conclusion of Theorem 4.7 holds, we get $\gamma = q_u(x_1) = \lambda_u$, thus establishing the desired result. □

LEMMA 5.3. *There exist positive constants $c_{\min}$, $C_{\max}$ such that if u is a solution to P_ε^c, then*

$$c_{\min} \leq \lambda_u \leq C_{\max}\,.$$

Here the constants $c_{\min}$, $C_{\max}$ depend only on n, ε, μ and δ and, μ and δ are given by (3.1) *and* (3.2) *respectively.*

PROOF. Choose $x_0 \in \Omega \cap \partial\{u > 0\}$ and $0 < R < 1$ with $B_{2R}(x_0) \Subset \Omega$ and set $D := B_R(x_0)$. By Theorem 4.3 and Remark 4.4, there exist positive constants c^* and C^* (depending only on n, ε, μ and δ) such that for every ball $B_r(x) \subset D$ with r small and $x \in \Omega \cap \partial\{u > 0\}$

$$c^* r^{n-1} \leq \mathcal{H}^{n-1}(B_r(x) \cap \partial\{u > 0\}) \leq C^* r^{n-1}, \tag{5.7}$$

and in addition

$$c^* \leq q_u \leq C^* \qquad \mathcal{H}^{n-1} \text{ almost everywhere in } D \cap \partial\{u > 0\}. \tag{5.8}$$

Therefore, since $\mathcal{H}^{n-1}(D \cap \partial\{u > 0\}) > 0$ by (5.7), the result follows from the combination of (5.2) and (5.8). □

THEOREM 5.4. *There exist positive constants C and r_0 such that if u is a solution to P_ε^c, $x_0 \in \Omega \cap \partial\{u > 0\}$ and $r \leq r_0$, then*

$$\sup_{x \in B_r(x_0)} |\nabla u(x)| \leq \lambda_u(1 + Cr).$$

Here $r_0 = r_0(\delta)$, $C = C(n, \varepsilon, \mu, \delta)$ and μ and δ are given by (3.1) *and* (3.2) *respectively.*

PROOF. Since the functions $\dfrac{\partial u}{\partial x_i}$ are harmonic in $\{u > 0\}$ for $1 \leq i \leq n$, we have that the function $|\nabla u|$ is subharmonic in $\{u > 0\}$, and so is the function

$$U_k := (|\nabla u| - \lambda_u - 1/k)^+$$

for $k \in \mathbb{N}$. From Theorem 5.1 it follows that U_k vanishes in a neighborhood of the free boundary and therefore, U_k is continuous and subharmonic in the entire domain Ω.

Let $r_0 = \delta/3$ with δ as in (3.2), and let $x_0 \in \Omega \cap \partial\{u > 0\}$. By Lemma 3.15 there exists a constant $L = L(n, \varepsilon, \mu) > 0$ such that

$$U_k \leq L \text{ in } B_{r_0}(x_0) \cap \{u > 0\}.$$

Now consider the function $v(x) = u(x) + \dfrac{|x - x_0|^2}{n}$, which is harmonic in $B_{r_0}(x_0) \cap \{u > 0\}$. Setting $C = \dfrac{nL}{r_0^2}$ we have

$$U_k \leq Cv \text{ on } \partial(B_{r_0}(x_0) \cap \{u > 0\})$$

and therefore, the inequality also holds in $B_{r_0}(x_0) \cap \{u > 0\}$. Taking $k \to \infty$ we get

$$|\nabla u| \leq \lambda_u + Cv \text{ in } B_{r_0}(x_0) \cap \{u > 0\}. \tag{5.9}$$

From Lemma 3.10 we have that $v \leq Cr$ in $B_r(x_0) \cap \{u > 0\}$, for $r \leq r_0$ and now the result follows as an immediate consequence of (5.9) and Theorem 5.3. □

6. – Flat free boundary points

This section, together with Section 7, is devoted to the study of the regularity of the free boundary. We will prove that if the free boundary is "flat" enough at some point, then it is regular near that point.

Our approach is inspired in [2], which in turn is oriented by the regularity theory for minimal surfaces. We combine here the ideas in [2] with a careful rescaling argument which eventually allows us to show that near the free boundary our solutions behave like those in [2].

We start the section by giving a precise definition of the "flatness condition" at a free boundary point (Definition 6.1). We next proceed in the following way: We perform a non-homogeneous scaling (or blow-up) in the "flat" direction (Theorem 6.4), obtaining the graph of a subharmonic function f (Lemma 6.5). We show further regularity properties of this function f (Lemma 6.6 to Lemma 6.8), that allows us to prove Lemmas 6.9 and 6.10, which are our main results in the section. These last two lemmas roughly say that if the free boundary is "flat" enough near some point, it is still "flatter" in a smaller neighborhood after and adequate change of coordinates. The regularity study of the free boundary is completed in Section 7.

DEFINITION 6.1. Let $0 < \sigma_+, \sigma_- \le 1$ and $\tau > 0$. We say that u belongs to class $\mathcal{F}(\sigma_+, \sigma_- ; \tau)$ in $B_\rho = B_\rho(0)$ if u is a solution to P_ε^c, $B_\rho \subset \Omega$, $0 \in \partial\{u > 0\}$ and

$$u(x) = 0 \text{ for } x_n \ge \sigma_+\rho, x \in B_\rho\,, \tag{6.1}$$

$$u(x) \ge -\lambda_u(x_n + \sigma_-\rho) \text{ for } x_n \le -\sigma_-\rho, x \in B_\rho\,, \tag{6.2}$$

$$|\nabla u| \le \lambda_u(1+\tau) \text{ in } B_\rho\,, \tag{6.3}$$

where λ_u is the constant in Theorem 5.1. If the origin is replaced by x_0 and the direction of flatness e_n is replaced by a unit vector ν, then we say that u belongs to the class $\mathcal{F}(\sigma_+, \sigma_- ; \tau)$ in $B_\rho(x_0)$ in direction ν.

REMARK 6.2. Let u be a solution to P_ε^c and let λ_u be the constant in Theorem 5.1. By Lemma 5.3 there exists a constant $c_{\min}$ (not depending on u) such that

$$0 < c_{\min} \le \lambda_u\,. \tag{6.4}$$

Then, if $B_\rho(0) \subset \Omega$, we can consider the function $v \in C^{0,1}(B_1(0))$ given by

$$v(x) = \frac{u(\rho x)}{\lambda_u \rho}\,,$$

which will clearly satisfy

$$v \ge 0\,, \Delta v \ge \frac{-2\rho}{\lambda_u} \text{ in } B_1(0)$$

and

$$\Delta v = \frac{-2\rho}{\lambda_u} \text{ in } \{v > 0\}.$$

In this section we shall denote balls with center in the origen by B_ρ.

THEOREM 6.3. *There exist positive constants $C = C(n)$ and $\sigma_0 = \sigma_0(n)$ such that if $u \in \mathcal{F}(\sigma, 1; \sigma)$ in B_ρ with $\sigma \le \sigma_0$ and $\rho \le c_{\min}\sigma$ then $u \in \mathcal{F}(2\sigma, C\sigma; \sigma)$ in $B_{\rho/2}$. Here $c_{\min}$ is the constant in* (6.4).

PROOF. Let $\lambda = \lambda_u$ and define $v(x) := \dfrac{u(\rho x)}{\lambda\rho}$ for $x \in B_1$. Let

$$\eta(y) := \exp\Big(\frac{-9|y|^2}{1-9|y|^2}\Big) \quad \text{for } |y| < 1/3,$$
$$:= 0 \qquad\qquad \text{for } |y| \ge 1/3,$$

and choose $s \ge 0$ maximal such that

$$B_1 \cap \{v > 0\} \subset D = D_\sigma := \{x \in B_1 / x_n < \sigma - s\eta(x')\}$$

where $x = (x', x_n)$. Thus there exists a point $z \in B_{1/2} \cap \partial D \cap \partial\{v > 0\}$. Also $s \le \sigma$ since $0 \in \partial\{v > 0\}$.

We consider the function w_1 satisfying

$$\Delta w_1 = \frac{-2\rho}{\lambda} \text{ in } D,$$
$$w_1 = 0 \text{ on } \partial D \cap B_1,$$
$$w_1 = (1 + 2\sigma)(\sigma - x_n) \text{ on } \partial D \setminus B_1.$$

By well known estimates we have

$$\text{(6.5)} \qquad \partial_{-\nu} w_1(z) \le 1 + c(n)\rho/\lambda + c(n)\sigma \le 1 + C(n)\sigma,$$

where we have used that $\rho \le c_{\min}\sigma$. On the other hand, since $v - w_1$ is subharmonic in D and $v - w_1 \le 0$ on ∂D by construction of D and by (6.3), we infer $v \le w_1$ in D.

We will prove an estimate for v from below for points $\xi \in \partial B_{3/4}$, $\xi_n < -1/2$. For that, let w_2 be the harmonic function in $D \setminus \overline{B}_{1/10}(\xi)$ such that

$$w_2 = 0 \text{ on } \partial D,$$
$$w_2 = -x_n \text{ on } \partial B_{1/10}(\xi).$$

Then, if σ is small (depending only on n) we have by the Hopf Principle

$$\text{(6.6)} \qquad \partial_{-\nu} w_2(z) \ge c(n) > 0.$$

If we suppose that $v(x) \leq w_1(x) + d\sigma x_n$ for $x \in B_{1/10}(\xi)$, for a positive constant d, we conclude as above that $v \leq w_1 - d\sigma w_2$ in $D \setminus B_{1/10}(\xi)$. Then, from (6.5), (6.6) and Lemma 5.2 it follows that

$$1 \leq \limsup_{x \to z} \frac{v(x)}{\operatorname{dist}(x, \partial D)} \leq 1 + C(n)\sigma - d\sigma c(n),$$

a contradiction, if d is large. Thus

$$v(x_\xi) \geq w_1(x_\xi) + C\sigma x_{\xi n},$$

for some $x_\xi \in B_{1/10}(\xi)$ and $C = C(n)$. Now, using similar arguments to those in [2, Lemma 7.2], we get for $\alpha > 0$

$$v(\xi + \alpha e_n) \geq -(\xi_n + \alpha) - C(n)\sigma,$$

which says that $u \in \mathcal{F}(2\sigma, C\sigma; \sigma)$ in $B_{\rho/2}$. □

We shall denote points in $\mathbb{R}^n$ by $x = (y, h)$ with $y \in \mathbb{R}^{n-1}$, balls in $\mathbb{R}^{n-1}$ by $B'_\rho(y)$; and $B'_\rho(0)$ by B'_ρ.

LEMMA 6.4. *Let u_k be a sequence of class $\mathcal{F}(\sigma_k, \sigma_k; \tau_k)$ in $B_{\rho k}$ with $\sigma_k \to 0$, $\tau_k \sigma_k^{-2} \to 0$ and $\rho_k = O(\tau_k)$. Let $\lambda_k := \lambda_{u_k}$ and for $x \in B_1$ define*

$$v_k(x) := \frac{u_k(\rho_k x)}{\lambda_k \rho_k}.$$

If for $y \in B'_1$

$$f_k^+(y) := \sup\{h/(y, \sigma_k h) \in \partial\{v_k > 0\}\},$$

$$f_k^-(y) := \inf\{h/(y, \sigma_k h) \in \partial\{v_k > 0\}\},$$

then, for a subsequence

$$f(y) := \limsup_{\substack{z \to y \\ k \to \infty}} f_k^+(z) = \liminf_{\substack{z \to y \\ k \to \infty}} f_k^-(z) \text{ for all } y \in B'_1.$$

Further, f is continuous with $f(0) = 0$, and locally $f_k^+ \to f$ and $f_k^- \to f$ uniformly.

PROOF. We can prove this result proceeding as in [2, Lemma 7.3], but using Theorem 6.3 instead. We observe that this is possible since by our hypotheses we have $\sigma_k \to 0$, $\tau_k = o(\sigma_k)$ and $\rho_k = o(\sigma_k)$. □

LEMMA 6.5. *f is subharmonic.*

PROOF. If the assertion is not true then there is a ball $B'_\rho(y_0) \Subset B'_1$ and a harmonic function g in a neighborhood of this ball such that

$$g > f \text{ on } \partial B'_\rho(y_0) \text{ and } f(y_0) > g(y_0).$$

We proceed as in [2], setting

$$Z := B'_\rho(y_0) \times \mathbb{R}, \quad Z^+(\phi) := \{(y,h) \in Z / h > \phi(y)\},$$

and similarly $Z^0(\phi)$ and $Z^-(\phi)$. Letting $d_\delta(Z^+(\sigma_k g))$ be a function which converges as $\delta \to 0$ to the characteristic function of $Z^+(\sigma_k g)$, we have by Theorems 4.3 and 5.1, and Remark 4.8.

$$-\int_{\{v_k>0\}} \nabla v_k \nabla d_\delta(Z^+(\sigma_k g)) + \frac{2\rho_k}{\lambda_k} \int_{\{v_k>0\}} d_\delta(Z^+(\sigma_k g))$$
$$= \int_{\partial_{\text{red}}\{v_k>0\}} d_\delta(Z^+(\sigma_k g)) d\mathcal{H}^{n-1}.$$

Taking $\delta \to 0$ (and assuming that $Z^0(\sigma_k g) \cap \partial\{v_k > 0\}$ has $\mathcal{H}^{n-1}$ measure zero; otherwise we replace g by $g + c$ for some suitable small c), we get

$$\int_{\partial(Z^+(\sigma_k g))\cap\{v_k>0\}} \nabla v_k \cdot \nu d\mathcal{H}^{n-1} + \frac{2\rho_k}{\lambda_k} |Z^+(\sigma_k g) \cap \{v_k > 0\}|$$
$$= \mathcal{H}^{n-1}(Z^+(\sigma_k g)) \cap \partial_{\text{red}}\{v_k > 0\}).$$

Using that by (6.3) $|\nabla v_k| \le 1 + \tau_k$, we deduce

$$\text{(6.7)} \quad \mathcal{H}^{n-1}(Z^+(\sigma_k g) \cap \partial_{\text{red}}\{v_k > 0\}) \le \frac{C\rho_k}{\lambda_k} + (1+\tau_k)\mathcal{H}^{n-1}(Z^0(\sigma_k g) \cap \{v_k > 0\}).$$

The set

$$E_k := \{v_k > 0\} \cup Z^-(\sigma_k g)$$

has finite perimeter in the cylinder Z, with

$$\text{(6.8)} \quad \begin{aligned} \mathcal{H}^{n-1}(Z \cap \partial_{\text{red}} E_k) &\le \mathcal{H}^{n-1}(Z^+(\sigma_k g) \cap \partial_{\text{red}}\{v_k > 0\}) \\ &\quad + \mathcal{H}^{n-1}(Z^0(\sigma_k g) \cap \{v_k = 0\}). \end{aligned}$$

If we show the estimate

$$\text{(6.9)} \quad \mathcal{H}^{n-1}(Z \cap \partial_{\text{red}} E_k) \ge \mathcal{H}^{n-1}(Z^0(\sigma_k g)) + C\sigma_k^2,$$

for large k, and we substitute this into (6.8) and use (6.7) and (6.4), we get a contradiction to the relations $\tau_k = o(\sigma_k^2)$, $\rho_k = O(\tau_k)$ which were assumed in Lemma 6.4. Now to get (6.9), we proceed as in [2, p. 136] and the lemma follows. □

We will let $B_\rho^- := B_\rho \cap \{h < 0\}$ for $\rho > 0$.

LEMMA 6.6. *Let*

$$w_k(y,h) := \frac{v_k(y,h) + h}{\sigma_k}.$$

Then, there exists a constant $C = C(n)$ such that

$$|w_k| \leq C \text{ in } B_1^- \tag{6.10}$$

for k large, and for a subsequence,

$$w := \lim_{k\to\infty} w_k \text{ exists everywhere in } B_1^-, \tag{6.11}$$

where the convergence is uniform in compact subsets of B_1^-. In addition, w satisfies:

$$|w| \leq C, \tag{6.12}$$

$$\Delta w = 0 \text{ in } B_1^-, \tag{6.13}$$

$$w(y,0) = f(y) \tag{6.14}$$

in the sense that $\lim_{h\uparrow 0} w(y,h) = f(y)$ and

$$w(y,h) - w(y,0) \leq 0 \quad \text{for} \quad (y,h) \in B_1^-. \tag{6.15}$$

PROOF. By definition of w_k and v_k and by the flatness assumption in Lemma 6.4, we get (6.10) and also

$$\Delta w_k = \frac{-2\rho_k}{\lambda_k \sigma_k} \quad \text{in} \quad B_1^- \cap \{h < -\sigma_k\}. \tag{6.16}$$

hence, the functions

$$\tilde{w}_k(x) := w_k(x) + \frac{\rho_k}{\lambda_k\sigma_k}\frac{|x|^2}{n}$$

are harmonic in $B_1^- \cap \{h < -\sigma_k\}$. Since $\frac{\rho_k}{\lambda_k\sigma_k} \to 0$, by (6.10) we have that $|\nabla \tilde{w}_k|$ are bounded (independently of k) in compact subsets of B_1^- and then, for a subsequence $\tilde{w}_k \to w$ uniformly in compact subsets of B_1^-. Therefore (6.11), (6.12) and (6.13) follow.

On the other hand, since

$$-\frac{\partial w_k}{\partial h} = -\frac{1}{\sigma_k}\Big(\frac{\partial v_k}{\partial h} + 1\Big) \leq \frac{|\nabla v_k| - 1}{\sigma_k} \leq \frac{\tau_k}{\sigma_k},$$

we have for $(y,h) \in B_1^-$ and $h < h' < 0$

$$w_k(y,h) - w_k(y,h') \le |h-h'|\frac{\tau_k}{\sigma_k},$$

and letting $k \to \infty$

$$w(y,h) - w(y,h') \le 0. \tag{6.17}$$

If we suppose that (6.14) holds, we can take $h' \to 0$ in (6.17) and (6.15) is proved. It remains to establish (6.14).

We will need to use that, for any small $\alpha > 0$ and any large constant K

$$w_k(y, h\sigma_k) \to f(y) \text{ uniformly for } y \in B'_{1-\alpha},\ -K \le h \le -1. \tag{6.18}$$

We can prove this by proceeding a in [3, Lemma 5.7], but applying Theorem 6.3 and Lemma 6.4 instead. This will be possible since we have $\sigma_k \to 0$, $\frac{\tau_k}{\sigma_k^2} \to 0$ and $\rho_k = O(\tau_k)$.

Now using (6.18), (6.16) and proceeding again as in [3, Lemma 5.7] we deduce (6.14), thus establishing the result. □

LEMMA 6.7. *There exists a positive contant $C = C(n)$ such that, for any $y \in B'_{1/2}$*

$$\int_0^{1/4} \frac{1}{r^2}\Big(\frac{1}{\mathcal{H}^{n-2}(\partial B'_r(y))} \int_{\partial B'_r(y)} (f - f(y))\,d\mathcal{H}^{n-2}\Big)dr \le C. \tag{6.19}$$

PROOF. Let $\bar{y} \in B'_{1/2}$ and

$$w^*(y,h) := w\Big(\frac{1}{2}y + \bar{y}, \frac{h}{2}\Big) - w(\bar{y},0) \text{ for } (y,h) \in B_1^-.$$

Setting

$$g(y) := f\Big(\frac{1}{2}y + \bar{y}\Big) - f(\bar{y}) \text{ for } y \in B'_1,$$

we have by Lemma 6.6, 6.4 and 6.5 that [3, Lemma 5.5] can be applied to the function w^*, which satisfies

$$\lim_{h\uparrow 0} w^*(y,h) = g(y) \text{ for } y \in B'_1,$$

where g subharmonic and continuous in B'_1, with $g(0) = 0$. Therefore (6.19) holds. □

Using now Lemma 6.4, 6.5 and 6.7 we obtain as in [2, Lemma 7.8] the following lemma:

LEMMA 6.8. *There is a large constant $C = C(n) > 0$ and for $0 < \theta < 1$ a small constant $c_\theta = C(\theta, n) > 0$, such that we find a ball B'_r and a vector $l \in \mathbb{R}^{n-1}$ with*

$$c_\theta \le r \le \theta, |l| \le C, \text{ and } f(y) \le l.y + \frac{\theta}{2} r \text{ for } |y| \le r.$$

LEMMA 6.9. *Let $C^* > 0, 0 < \theta < 1$ and C, c_θ as in Lemma 6.8. Then, there is a constant $\sigma_\theta > 0$, such that*

$$u \in \mathcal{F}(\sigma, \sigma; \tau) \text{ in } B_\rho \text{ in direction } \nu$$

with $\sigma \le \sigma_\theta$, $\tau \le \sigma_\theta \sigma^2$ and $C^*\rho \le \tau$ implies

$$u \in \mathcal{F}(\theta\sigma, 1; \tau) \text{ in } B_{\overline{\rho}} \text{ in direction } \overline{\nu}$$

for some $\overline{\rho}$ and $\overline{\nu}$ with $c_\theta \rho \le \overline{\rho} \le \theta\rho$ and $|\overline{\nu} - \nu| \le C\sigma$. Here $\sigma_\theta = \sigma(\theta, n, C^*, c_{\min})$ and $c_{\min}$ as in (6.4).

PROOF. If the result were not true, then for each positive integer k there would exist

$$u_k \in \mathcal{F}(\sigma_k, \sigma_k; \tau_k) \text{ in } B_{\rho_k}$$

with $\sigma_k \le 1/k$, $\tau_k \le \frac{1}{k}\sigma_k^2$, $C^*\rho_k \le \tau_k$ and such that the conclusion of the lemma does not hold. Clearly, u_k is a sequence as in Lemma 6.4 and therefore, Lemma 6.8 will yield a contradiction if we proceed as in [2, Lemma 7.9]. □

LEMMA 6.10. *Let $0 < \theta < 1$ and $C^* > 0$, then there exist positive constants σ_θ, c_θ and C such that if*

$$u \in \mathcal{F}(\sigma, 1; \tau) \quad \text{in } B_\rho \text{ in direction } \nu$$

with $\sigma \le \sigma_\theta$, $\tau \le \sigma_\theta \sigma^2$ and $C^*\rho \le \tau$, then

$$u \in \mathcal{F}(\theta\sigma, \theta\sigma; \theta^2\tau) \quad \text{in } B_{\bar{\rho}} \text{ in direction } \bar{\nu}$$

for some $\bar{\rho}$ and $\bar{\nu}$ with $c_\theta \rho \le \bar{\rho} \le \frac{1}{4}\rho$ and $|\bar{\nu} - \nu| \le C\sigma$. Here $c_\theta = c(\theta, n)$, $C = C(n)$, $\sigma_\theta = \sigma(\theta, n, C^*, c_{\min})$ and $c_{\min}$ as in (6.4).

PROOF. The idea is similar to that of [2, Lemma 7.10]. Let $0 < \theta_1 \le 1/2$. If σ_θ in the statement is small enough we obtain by Theorem 6.3 and Lemma 6.9 that

$$u \in \mathcal{F}(\theta_1 C\sigma, 1; \tau) \quad \text{in } B_{r_1\rho} \text{ in direction } \nu_1,$$

for some r_1, ν_1 with $c_{\theta_1} \le 2r_1 \le \theta_1$ and $|\nu_1 - \nu| \le C\sigma$.

For $k \in \mathbb{N}$, we consider the function

$$U_k := (|\nabla u| - \lambda_u - 1/k)^+,$$

which is subharmonic and continuous (see Theorem 5.4). Then, there is a constant $0 < c(n) < 1$ such that

$$U_k \leq (1 - c(n))\lambda_u \tau \quad \text{in} \quad B_{r_1}\rho \,.$$

If we let $k \to \infty$, it follows that

$$u \in \mathcal{F}(\theta_0\sigma, 1; \theta_0^2\tau) \quad \text{in } B_{r_1}\rho \text{ in direction } \nu_1 \,,$$

where $\theta_0 := \sqrt{1 - c(n)}$, provided θ_1 is chosen small enough (depending only on n). Then repeating this argument a suitable number of times and using Theorem 6.3 again, we finish the proof. □

7. – Smoothness of the free boundary

In this section we complete the study of the regularity of the free boundary started in Section 6. Here we prove that the reduced free boundary is locally analytic and that in two dimensions singularities cannot occur.

In Section 6 we showed that if the free boundary is flat enough near some point, it is still flatter in a smaller neighborhood. This fact, together with the estimates in Section 5, is used in Theorem 7.1 to prove —by means of an iteration argument— that the free boundary is a $C^{1,\alpha}$ surface near flat enough free boundary points. Moreover, we show that the free boundary is locally analytic except possibly for a closed set of $\mathcal{H}^{n-1}$ measure zero (Theorem 7.2). Now Theorem 4.3 implies that along the regular part of the free boundary, the condition

$$\partial_{-\nu} u = \lambda_u$$

is satisfied in the classical sense. We complete the section with additional results valid for the two dimensional case. Theorem 7.4 gives an estimate from below for $|\nabla u|$ near any free boundary point (compare with Theorem 5.4). This important estimate allows us to prove that in two dimensions singular points of the free boundary cannot occur (Theorem 7.6).

In this section we will use some results that can be found in the Appendix at the end of the paper.

THEOREM 7.1. *Let u be a solution to P_ε^c. Then there exist positive constants α, σ_0 and τ_0 such that*

$$u \in \mathcal{F}(\sigma, 1; \infty) \quad \text{in} \quad B_\rho(x_0) \quad \text{in direction} \quad \nu$$

with $\sigma \leq \sigma_0$ and $\rho \leq \tau_0\sigma^2$ implies that

$$B_{\rho/8}(x_0) \cap \partial\{u > 0\} \quad \text{is a} \quad C^{1,\alpha} \quad \text{surface} \,,$$

more precisely, it is a graph in the direction ν of a $C^{1,\alpha}$ function. Moreover, for any x_1, x_2 on this surface,

$$|\nu(x_1) - \nu(x_2)| \leq C\sigma \left|\frac{x_1 - x_2}{\rho}\right|^{\alpha}.$$

The constants depend only on n, ε, μ and δ, μ and δ given by (3.1) *and* (3.2) *respectively.*

PROOF. We will proceed as in [2, Theorem 8.1]. If σ_0 and τ_0 are chosen small enough, we have by Theorem 5.4 that

$$\sup_{B_\rho(x_0)} |\nabla u(x)| \leq \lambda_u(1 + \widetilde{C}\rho).$$

Let $\tau := \widetilde{C}\rho$ and $x_1 \in B_{\rho/2}(x_0) \cap \partial\{u > 0\}$, then using Theorem 6.3 we get

$$u \in \mathcal{F}(C\sigma, C\sigma; \tau) \quad \text{in} \quad B_{\rho/4}(x_1) \quad \text{in direction} \quad \nu.$$

We want to apply Lemma 6.10 in $B_{\rho/4}(x_1)$ for some $0 < \theta < 1$ and $C^* = \widetilde{C}$. Indeed, if we choose

$$\sigma_0 \leq \frac{\sigma_\theta}{C} \quad \text{and} \quad \tau_0 \leq \frac{\sigma_\theta C^2}{\widetilde{C}},$$

we can apply this lemma. If in addition, we choose $\theta = 1/2$, it will be possible to apply Lemma 6.10 inductively, following the idea of [2, Lemma 8.1]. Finally, observing that the constants in our proof depend only on those of Theorems 5.4 and 6.3, and Lemma 6.10, we complete the theorem. □

THEOREM 7.2. *Let u be a solution to P_ε^c. Then $\partial_{\text{red}}\{u > 0\}$ is a $C^{1,\alpha}$ surface locally in Ω, and the remainder of $\Omega \cap \partial\{u > 0\}$ has $\mathcal{H}^{n-1}$ measure zero. It follows that $\partial_{red}\{u > 0\}$ is locally analytic.*

PROOF. Let $x_0 \in \partial_{\text{red}}\{u > 0\}$ and u_k be a blow-up sequence with respect to balls $B_{\rho_k}(x_0)$ with limit u_0. By the first part of Theorem 4.7 and (B.7) there is a sequence $\sigma_k \to 0$ such that

$$u \in \mathcal{F}(\sigma_k, 1; \infty) \quad \text{in} \quad B_{\rho_k}(x_0) \quad \text{in direction} \quad \nu_u(x_0),$$

hence for large k Theorem 7.1 can be applied. Analyticity now follows from [9, Theorem 2], if we use that by Theorems 4.3 and 5.1, $|\nabla u| = \lambda_u$ on any smooth portion of $\Omega \cap \partial\{u > 0\}$. □

REMARK 7.3 Let u be a solution of P_ε^c and $\bar{x}$ a free boundary point such that $B_r(\bar{x}) \cap \partial\{u > 0\}$ is smooth for some small $r > 0$. Near $\bar{x}$ we make a smooth pertubation of the set $\{u > 0\}$, increasing its volume by $\alpha > 0$ (α small) and in the perturbed set D_α, where

$$B_r(\bar{x}) \cap \{u > 0\} \subset D_\alpha \subset B_r(\bar{x}),$$

we consider the function v_α satisfying

$$\begin{aligned} \Delta v_\alpha &= -2 && \text{in} \quad D_\alpha, \\ v_\alpha &= u && \text{on} \quad \partial D_\alpha \cap \partial B_r(\bar{x}), \\ v_\alpha &= 0 && \text{in} \quad B_r(\bar{x}) \setminus D_\alpha. \end{aligned}$$

We have that $v_\alpha \to u$ in $H^1(B_r(\bar{x}))$ as $\alpha \to 0$, and since $\partial_{-\nu} u = \lambda_u$ on $B_r(\bar{x}) \cap \partial\{u > 0\}$ it follows that

$$\int_{B_r(\bar{x})} |\nabla v_\alpha - \nabla u|^2 = \lambda_u^2 \alpha + o(\alpha).$$

We get the same estimate if we decrease by $\alpha > 0$ the volume of $\{u > 0\}$ and define v'_α analogously in the perturbed set.

THEOREM 7.4. *Let $n = 2$ and let u be a solution to P_ε^c. If $x_0 \in \Omega \cap \partial\{u > 0\}$ then*

$$\frac{1}{r^2} \int_{B_r(x_0) \cap \{u>0\}} \max(\lambda_u^2 - |\nabla u|^2, 0) \to 0 \quad \textit{for} \quad r \to 0. \tag{7.1}$$

PROOF. Let $0 < r < \rho$ small and

$$\eta(x) := \begin{cases} \dfrac{\log(\rho/|x - x_0|)}{\log(\rho/r)} & \text{for } x \in B_\rho(x_0) \setminus B_r(x_0), \\ 1 & \text{for } x \in B_r(x_0). \end{cases}$$

Let $t > 0$ and define in $B_\rho(x_0)$

$$v_0 := \max(u - t\eta, 0) = u - \min(u, t\eta).$$

Consider now a point $x_1 \in \Omega \cap \partial\{u > 0\}$ away from x_0 such that $B_{r_1}(x_1) \cap \partial\{u > 0\}$ is smooth for some small $r_1 > 0$. We make a smooth pertubation of the set $\{u > 0\}$ near x_1 increasing its volume by α where

$$\alpha = \alpha_{\rho,t} := |\{x \in B_\rho(x_0) / 0 < u \le t\eta\}|. \tag{7.2}$$

Then, letting v_α be the function defined in the perturbed set as in Remark 7.3, we get that the function

$$v := \begin{cases} v_0 & \text{in } B_\rho(x_0)\,, \\ v_\alpha & \text{in } B_{r_1}(x_1)\,, \\ u & \text{elsewhere}\,, \end{cases}$$

is admisible for P_ε^c and satisfies $|\{v > 0\}| = |\{u > 0\}|$. We thus obtain by construction that

$$0 \leq J_\varepsilon(v) - J_\varepsilon(u) = \frac{1}{2} \int_{B_\rho(x_0)} |\nabla \min(u, t\eta)|^2 - \frac{\lambda^2}{2}\alpha + o(\alpha)\,,$$

where $\lambda = \lambda_u$ and this estimate, together with (7.2) implies

$$\int_{B_\rho(x_0)\cap\{0<u\leq t\eta\}} (\lambda^2 - |\nabla u|^2) \leq t^2 \int_{B_\rho(x_0)\cap\{u>t\eta\}} |\nabla \eta|^2 + o(\alpha)\,.$$

By Lemma 3.10 we have $u \leq Cr$ in $B_r(x_0)$ and if we choose $t = Cr$ we get

$$\int_{B_r(x_0)\cap\{u>0\}} \max(\lambda^2 - |\nabla u|^2, 0) \leq \int_{B_\rho(x_0)} \max(|\nabla u|^2 - \lambda^2, 0)$$

$$+ \frac{Cr^2}{\log(\rho/r)} + o(\delta_\rho)\,,$$

where $\delta_\rho := |\{x \in B_\rho(x_0)/0 < u \leq C\rho\}|$. Using Theorem 5.4 we obtain

$$\frac{1}{r^2} \int_{B_r(x_0)\cap\{u>0\}} \max(\lambda^2 - |\nabla u|^2, 0) \leq \frac{1}{r^2}(C\rho^3 + o(\delta_\rho)) + \frac{C}{\log(\rho/r)}\,.$$

Finally, if we observe that $\delta_\rho \leq C\rho^2$ by Theorem 3.11 and we choose $r = \rho f(\rho)^{1/4}$ with $f(\rho) := \max\left(\rho, \frac{o(\delta_\rho)}{\delta_\rho}\right)$, we get (7.1). □

COROLLARY 7.5. *Let $n = 2$ and let u be a solution to P_ε^c. If $x_0 \in \Omega \cap \partial\{u > 0\}$ and u_k is a blow-up sequence with respect to balls $B_{\rho_k}(x_0)$ with limit u_0, then*

$$u_0(x) = \lambda_u \max(-x.e, 0)\,.$$

Here $e = e(x_0)$ is a unit vector.

PROOF. By Lemma B.3 and Theorems 5.1 and 7.4, we see that

$$|\nabla u_0| = \lambda_u \quad \text{in} \quad B_1(0) \cap \{u_0 > 0\}. \tag{7.3}$$

Since $0 \in \partial\{u_0 > 0\}$, there exists a connected component D of $\{u_0 > 0\}$ such that $0 \in \partial D$. Let $y_1 \in D \cap B_1(0)$ and $e := \dfrac{-\nabla u_0(y_1)}{|\nabla u_0(y_1)|}$. From (7.3) and from the fact that u_0 is harmonic in $\{u_0 > 0\}$ we deduce that $\dfrac{\partial u_0}{\partial e}$ is harmonic in $D \cap B_1(0)$ and achieves its minimum in y_1. Then,

$$u_0(y) = -\lambda_u y.e \quad \text{for} \quad y.e \leq 0.$$

The result now follows if we apply Theorem A.1 to show that $u_0 \equiv 0$ in $y.e > 0$. □

THEOREM 7.6. *Let $n = 2$ and let u be a solution to P_ε^c. Then $\partial\{u > 0\}$ is a $C^{1,\alpha}$ curve locally in Ω. It follows that $\partial\{u > 0\}$ is locally analytic.*

PROOF. Let $x_0 \in \Omega \cap \partial\{u > 0\}$ and let u_k be a blow-up sequence with respect to balls $B_{\rho_k}(x_0)$ with limit u_0. From Corollary 7.5 and from (B.7) there is a unit vector e and a sequence $\sigma_k \to 0$ such that

$$u \in \mathcal{F}(\sigma_k, 1; \infty) \quad \text{in} \quad B_{\rho k}(x_0) \quad \text{in direction} \quad e,$$

hence for large k Theorem 7.1 can be applied. Analyticity now follows as in Theorem 7.2. □

8. – Behaviour of the solutions for small ε

The aim of this section is to show that the volume of $\{u > 0\}$ automatically adjusts to ω_0, for small values of ε.

In the previous section, we proved that the free boundary is locally analytic (except possibly for a closed subset of $\mathcal{H}^{n-1}$ measure zero), and thus $\partial_{-\nu} u = \lambda_u$ along the regular part of it. This allows us to use a comparison argument in Lemma 8.2 to prove that if ε is small and u is a solution to P_ε^c, then λ_u stays away from

$$0 < C_1 \leq \lambda_u \leq C_2 < \infty,$$

where C_1 and C_2 are constants not depending on ε (compare with Lemma 5.3). As a consequence, we prove in Theorem 8.3 that $|\{x \in \Omega/u(x) > 0\}| = \omega_0$ (when ε is small). This result is derived from Lemma 8.2 and from the fact that $f_\varepsilon'(s)$ jumps from a small to a large number at $s = \omega_0$ (recall that f_ε is the function appearing in the penalized functional J_ε). The fact that it is not necessary to pass to the limit in ε to adjust the volume of $\{u > 0\}$ to ω_0, allows us —for small values of ε— to get solutions satisfying this property and preserving at the same time the properties proved before. This will be a key point in [10].

As in the previous sections, we analize the dependence of our results on the data of the problem.

LEMMA 8.1. *There exists a positive constant* $C = C(\Omega) = C(H)$ *such that if* $v \in H^1(\Omega) \cap L^1(\Omega)$ *and* $v \geq 0$ *then*

$$\left(\int_{\partial\Omega} v d\mathcal{H}^{n-1}\right)^2 \leq C(H)|\{x \in \Omega / v(x) > 0\}|. \int_{\Omega} (v^2 + |\nabla v|^2). \tag{8.1}$$

PROOF. The result follows if we integrate along lines □

LEMMA 8.2. *There exists positive constants* C_1 *and* C_2 *such that if* u *is a solution to* P_ε^c *then*

$$C_1 \leq \lambda_u \leq C_2 .$$

Here the constants C_i *depend only on* n, $|H|$, c, r^*, l *and* $C(H)$ (r^*, l *and* $C(H)$ *as in* (1.1), (1.4) *and* (8.1) *respectively). Moreover,* C_i *depend continuously on* $|H|$, c *and* l.

PROOF.

Step 1. (First inequality) Since $H \subset B_{R_0}(0)$ and $\overline{H} \cup \{u > 0\}$ is connected, satisfying by Theorem 3.4 that

$$|\overline{H} \cup \{u > 0\}| \leq C(n, |H|, l), \tag{8.2}$$

there is a constant R_1 and a point y_1 such that

$$y_1 \in \Omega \cap \partial\{u > 0\}, |y_1| \leq R_1 \quad \text{and} \quad R_1 = R_1(n, |H|, l) > R_0 .$$

Let $r_1 := \text{dist}(y_1, \partial H)$ and r^* as in (1.1). Then, there are points $y_0 \in \partial H$ and $y_2 \in H$ such that $y_0 \in \overline{B}_{r_1}(y_1) \cap \overline{H}$ and $\overline{B}_{r^*}(y_2) \cap \overline{\Omega} = \{y_0\}$. Now assume that e_n is the outward normal to H at y_0 and for $0 \leq t \leq 1$ consider

$$D_t := \{x \in \mathbb{R}^n / \text{dist}(x, I_t) < r^*\},$$

where

$$I_t := \{x \in \mathbb{R}^n / x = y_2 + s e_n, 0 \leq s \leq 2R_1 t\},$$

that is D_t is a smooth family of smooth domains such that $D_t \subset D_{t'}$ if $t < t'$. Moreover, $D_0 = B_{r^*}(y_2)$, $D_t \cap \Omega \neq \emptyset$ for $t > 0$ and $y_1 \in D_1$. Let t be the first value for which D_t touches a free boundary point $x_0 \in \partial D_t \cap \Omega \cap \partial\{u > 0\}$. We have that $0 < t < 1$ and

$$D_t \cap \Omega \subset \{u > 0\}. \tag{8.3}$$

Consider the function w satisfying

$$\Delta w = 0 \quad \text{in} \quad D_t \setminus \overline{B_{r^*/2}}(y_2)\,,$$
$$w = 1 \quad \text{on} \quad \partial B_{r^*/2}(y_2)\,,$$
$$w = 0 \quad \text{in} \quad \mathbb{R}^n \setminus D_t\,.$$

Then, by the Hopf Principle, there is a constant γ such that

$$\partial_{-\nu} w(x_0) \geq \gamma > 0, \quad \gamma = \gamma(n\,, R_1\,, r^*)\,, \tag{8.4}$$

where ν is the unit outward normal to D_t in x_0. Recalling (8.3) we have $\Delta(u - cw) = -2$ in $D_t \cap \Omega$, $u - cw \geq 0$ in $\partial(D_t \cap \Omega)$ and therefore, from the Maximum Principle and the estimate (8.4) we get for small $r > 0$

$$\frac{1}{r} \fint_{\partial B_r(x_0)} u \geq \frac{c}{r} \fint_{\partial B_r(x_0)} w \geq c\alpha \tag{8.5}$$

where $\alpha = \alpha(n, |H|, r^*, l)$.

Now consider the function v_0 satisfying $\Delta v_0 = -2$ in $B_r(x_0)$, $v_0 = u$ on $\partial B_r(x_0)$. Reasoning as in (2.7) and using (8.5) we get

$$\begin{aligned}\int_{B_r(x_0)} |\nabla(v_0 - u)|^2 &\geq C(n)|B_r(x_0) \cap \{u = 0\}| \left(\frac{1}{r} \fint_{\partial B_r(x_0)} u\right)^2 \\ &\geq (c\alpha)^2 C(n) |B_r(x_0) \cap \{u = 0\}|\,.\end{aligned} \tag{8.6}$$

Next choose a point $x_1 \in \Omega \cap \partial\{u > 0\}$ away from x_0 such that $B_\rho(x_1) \cap \partial\{u > 0\}$ is smooth for some small $\rho > 0$. Near x_1 we make a smooth pertubation of the set $\{u > 0\}$ decreasing its volume by δ_r, where $\delta_r := |B_r(x_0) \cap \{u = 0\}|$ and we consider the function $v_1 = v'_{\delta_r}$ defined in $B_\rho(x_1)$ as in Remark 7.3.

Then the function

$$v := \begin{cases} v_0 & \text{in } B_r(x_0)\,, \\ v_1 & \text{in } B_\rho(x_1)\,, \\ u & \text{elsewhere}\,, \end{cases}$$

is admissible to P_ε^c and satisfies $|\{v > 0\}| = |\{u > 0\}|$. Consequently, from (8.6) and Remark 7.3 we deduce that

$$0 \leq J_\varepsilon(v) - J_\varepsilon(u) \leq -(c\alpha)^2 \frac{C(n)}{2} \delta_r + \frac{\lambda_u^2}{2} \delta_r + o(\delta_r)\,,$$

that is, $c\alpha C(n) \leq \lambda_u$.

Step 2. (Second inequality) We first see, using (8.2), Lemma 2.2 and (1.5), that

$$\|u\|_{H^1(\Omega)} \leq C(n, |H|, l) \tag{8.7}$$

and therefore, denoting by ν the unit outward normal to Ω, we have

$$c \int_{\partial\Omega} \partial_\nu u d\mathcal{H}^{n-1} = \int_{\{u>0\}} u\Delta u + \int_{\{u>0\}} |\nabla u|^2 \leq C(n, |H|, l) \tag{8.8}$$

(note that we have used that $\Delta u = -2$ in $\{u > 0\}$). On the other hand, we have the Isoperimetric Inequality

$$\mathcal{H}^{n-1}(\Omega \cap \partial\{u > 0\}) \geq C(n)|\{u > 0\}|^{\frac{n-1}{n}} . \tag{8.9}$$

In addition, from Lemma 8.1 (applied to u) , (8.7) and again the Isoperimetric Inequality, it follows

$$C(H).C(n, |H|, l)|\{u > 0\}| \geq c^2(\mathcal{H}^{n-1}(\partial\Omega))^2 \geq c^2 C(n, |H|) . \tag{8.10}$$

Now, recalling Theorem 5.1, 2) and applying Theorem 4.3, 2) to suitable functions we deduce

$$\int_{\partial\Omega} \partial_\nu u d\mathcal{H}^{n-1} + 2|\{u > 0\}| = \lambda_u \mathcal{H}^{n-1}(\Omega \cap \partial\{u > 0\}) . \tag{8.11}$$

Finally, we combine (8.2) and (8.8) to estimate the left hand side of (8.11) and then, (8.9) and (8.10) to estimate its right hand side. Thus, we conclude that $C_2 \geq \lambda_u$. □

THEOREM 8.3. *There exists a positive constant ε_1 such that if $\varepsilon \leq \varepsilon_1$ and u is a solution to P_ε^c, then*

$$|\{x \in \Omega / u(x) > 0\}| = \omega_0 .$$

Here ε_1 depends only on n, $|H|$, c, r^*, l and $C(H)$ (r^*, l and $C(H)$ as in (1.1), (1.4) *and* (8.1) *respectively). Moreover, ε_1 depends continuously on $|H|$, c and l.*

PROOF. Suppose $|\{u > 0\}| > \omega_0$. Choose a regular point in $\Omega \cap \partial\{u > 0\}$ and in a neighborhood of it make a smooth inward pertubation of $\{u > 0\}$ decreasing its volume by $\delta v > 0$. Let v_1 be the admissible function that equals u everywhere except in the perturbed region, where it is defined as in Remark 7.3. If δv is small enough, we will have $|\{v_1 > 0\}| > \omega_0$ and since $f_\varepsilon'(s) \geq \dfrac{1}{\varepsilon}$ for $s > \omega_0$, we get

$$J_\varepsilon(v_1) - J_\varepsilon(u) \leq \frac{1}{2} \int_\Omega |\nabla(u - v_1)|^2 - \frac{1}{\varepsilon}\delta v = \Big(\frac{1}{2}\lambda_u^2 - \frac{1}{\varepsilon}\Big)\delta v + o(\delta v) .$$

Now using Lemma 8.2 we find a constant $\bar{\varepsilon}$ (depending only on C_2) such that if $\varepsilon \leq \bar{\varepsilon}$ then $J_\varepsilon(v_1) < J_\varepsilon(u)$, provided δv is small, a contradiction.

If $|\{u > 0\}| < \omega_0$ we argue similarly, using now the first inequality in Lemma 8.2 and the fact that $f_\varepsilon(s) = \varepsilon(s - \omega_0)$ for $s < \omega_0$. □

Appendix

In this Appendix we prove some results on harmonic functions and on blow-up sequences that are used in different situations throughout the paper.

A. – A result on harmonic functions with linear growth

THEOREM A.1. *Let u be a Lipschitz continuous function on $\mathbb{R}^n$ with constant L such that*

1) $u \geq 0$ *in* $\mathbb{R}^n$, $\Delta u = 0$ *in* $\{u > 0\}$,
2) $\{x_n < 0\} \subset \{u > 0\}$, $u = 0$ *on* $\{x_n = 0\}$,
3) *There exists* $0 < \lambda < 1$ *such that* $\dfrac{|\{u = 0\} \cap B_R(0)|}{|B_R(0)|} > \lambda, \quad \forall R > 0$.

Then

$$u \equiv 0 \quad in \quad \{x_n > 0\}. \tag{A.1}$$

PROOF.
Step 1. (Essential part of the proof) Let $R > 0$ and

$$w(x) := \frac{r(Rx)}{R}.$$

Suppose that there is $0 \leq \alpha \leq L$ such that

$$w(x) \leq \alpha x_n \quad \text{in} \quad B_1(0) \cap \{x_n > 0\}. \tag{A.2}$$

We will show that there are constants $c_\alpha = c_\alpha(n, \lambda, L, \alpha), \gamma = \gamma(n, \lambda)$ such that

$$w(x) \leq (\alpha - c_\alpha) x_n \quad \text{in} \quad B_\gamma(0) \cap \{x_n > 0\},$$

with $0 < \gamma < 1, 0 \leq c_\alpha \leq \alpha$ and $c_\alpha > 0$ when $\alpha > 0$.

To prove this, suppose $\alpha > 0$ and consider the superharmonic function

$$v(x) := \alpha x_n - w(x).$$

Setting $\beta := \dfrac{\lambda |B_1(0)|}{2^n}$ we deduce from 2) and 3) that there is a point z such that

$$z \in B_1(0), \quad z_n > \beta \quad \text{and} \quad w(z) = 0.$$

Let $r := |z|$. We consider the harmonic function h in $B_r(0) \cap \{x_n > 0\}$ with boundary values v and extend it to a harmonic function in the entire ball by setting $h(x_1, \dots, x_n) = h(x_1, \dots, -x_n)$ for $x_n < 0$.

Using the Maximum Principle and the Poisson Integral we get for all y such that $|y| < \beta/2$ and $y_n > 0$

$$v(y) \geq h(y) = \frac{r^2 - |y|^2}{n w_n r} \int_{\substack{|x|=r \\ x_n>0}} v(x)\Big(\frac{1}{|x-y|^n} - \frac{1}{|\bar{x}-y|^n}\Big) d\mathcal{H}_x^{n-1}\,, \tag{A.3}$$

where for $x = (x_1, \dots, x_n)$ we have denoted $\bar{x} = (x_1, \dots, -x_n)$. On the other hand, using that w is Lipschitz continuous with constant L, we get

$$v(x) > \frac{\alpha\beta}{2} \quad \text{and} \quad x_n > \frac{3\beta}{4} \quad \text{for} \quad |x - z| < \frac{\alpha\beta}{4L}\,.$$

Therefore (A.2) and (A.3) yield

$$v(y) \geq \alpha C y_n\,,$$

where $C = C(n, \lambda, L, \alpha) > 0$ and if we now set

$$c_\alpha := \alpha C, \qquad \gamma := \beta/2\,,$$

the first step is establised.

Step 2. (Introduction argument) Let $r_0 > 0$ and

$$w_0(x) := \frac{1}{r_0} u(r_0 x)\,.$$

By 2), using that u is Lipschitz continuous, we have for $\alpha_0 = L$

$$w_0(x) \leq \alpha_0 x_n \quad \text{in} \quad B_1(0) \cap \{x_n > 0\}\,,$$

and therefore, an application of the first step yields

$$w_0(x) \leq (\alpha_0 - c_{\alpha_0}) x_n \quad \text{in} \quad B_\gamma(0) \cap \{x_n > 0\}\,.$$

We can normalize the situation to the unit ball and apply the first step inductively. Then, if we denote

$$w_i(x) := \frac{1}{r_i} u(r_i x), \quad r_{i+1} := \gamma^{i+1} r_0, \quad \alpha_{i+1} := \alpha_i - c_{\alpha_i} \tag{A.4}$$

for $i \geq 0$, we get

$$0 \leq \alpha_{i+1} \leq \alpha_i \leq L, \quad w_i(x) \leq \alpha_i x_n \quad \text{in} \quad B_1(0) \cap \{x_n > 0\}$$

and consequently

$$u(x) \leq \alpha_i x_n \quad \text{in} \quad B_{r_i}(0) \cap \{x_n > 0\} \quad \text{for} \quad i \geq 0\,. \tag{A.5}$$

In oder to prove (A.1), now suppose there exists a point ξ such that

$$\xi_n > 0 \quad \text{and} \quad u(\xi) > 0\,.$$

Observing that the sequence α_i, which does not depend on r_0, tends to zero as $i \to \infty$, we have that

$$u(\xi) > \alpha_k \xi_n \tag{A.6}$$

for some $k \in \mathbb{N}$. Then, if we start in (A.4) with $r_0 > \frac{|\xi|}{\gamma^k}$, we will have by (A.5) that $u(\xi) \leq \alpha_k \xi_n$, which contradicts (A.6). The result is established. □

B. – Blow-up limits

HYPOTHESES B.1. D is a domain in $\mathbb{R}^n$ and u a function satisfying

1) u is nonnegative and Lipschitz continuous with constant L in D and $\Delta u = -2$ in $D \cap \{u > 0\}$,
2) For $0 < \kappa < 1$ there exist positive constants C_κ and r_κ, such that for balls $B_r(x) \subset D$ with $0 < r < r_\kappa$

$$\frac{1}{r} \fint_{\partial B_r(x)} u \leq C_\kappa \quad \text{implies} \quad u \equiv 0 \quad \text{in} \quad B_{\kappa r}(x)\,,$$

3) There exist positive constants r_0 and $\lambda_1 \leq \lambda_2 < 1$, such that for balls $B_r(x) \subset D$ with $x \in \partial\{u > 0\}$ and $0 < r < r_0$

$$\lambda_1 \leq \frac{|B_r(x) \cap \{u > 0\}|}{|B_r(x)|} \leq \lambda_2\,.$$

DEFINITION B.2. Let u be a function as in B.1, and let $B_{\rho_k}(x_k) \subset D$ be a sequence of balls with $\rho_k \to 0, x_k \to x_0 \in D, u(x_k) = 0$. We call the sequence of functions defined by

$$u_k(x) := \frac{1}{\rho_k} u(x_k + \rho_k x)$$

the blow-up sequence with respect to $B_{\rho_k}(x_k)$. Since

$$|\nabla u_k| \leq L \quad \text{in every compact set of} \quad \mathbb{R}^n\,, \tag{B.1}$$

if k is large enough, and since $u_k(0) = 0$, there exists a blow-up limit $u_0 : \mathbb{R}^n \to \mathbb{R}$, such that for a subsequence

$$u_k \to u_0 \quad \text{in} \quad C^{0,\alpha}_{\text{loc}}(\mathbb{R}^n) \quad \text{for every} \quad 0 < \alpha < 1\,, \tag{B.2}$$

$$\nabla u_k \to \nabla u_0 \quad \text{weakly star in} \quad L^\infty_{\text{loc}}(\mathbb{R}^n)\,. \tag{B.3}$$

LEMMA B.3. *Let u_k and u_0 in B.2. Then, the following properties hold:*

(B.4) u_0 *is nonnegative in* $\mathbb{R}^n$ *and harmonic in* $\{u_0 > 0\}$,

(B.5) $\partial\{u_k > 0\} \to \partial\{u_0 > 0\}$ *locally in the Hausdorff distance*,

(B.6) $\chi(\{u_k > 0\}) \to \chi(\{u_0 > 0\})$ *in* $L^1_{\text{loc}}(\mathbb{R}^n)$, (B.7) *If* $K \Subset \operatorname{int}\{u_0 = 0\}$, *then* $u_k \equiv 0$ *in* K *for large* k,

(B.8) *If* $K \Subset \{u_0 > 0\} \cup \operatorname{int}\{u_0 = 0\}$, *then* $\nabla u_k \to \nabla u_0$ *uniformly in* K,

(B.9) $\nabla u_k \to \nabla u_0$ *a.e.*,

(B.10) *There is* $0 < \lambda < 1$ *such that* $\dfrac{|B_R(y) \cap \{u_0 = 0\}|}{|B_R(y)|} > \lambda$, *for all* $R > 0$, *and* $y \in \partial\{u_0 > 0\}$

(B.11) *If* $x_k \in \partial\{u > 0\}$ *then* $0 \in \partial\{u_0 > 0\}$.

PROOF. Let $B \Subset \{u_0 > 0\}$. Then, the functions

$$v_k(x) := u_k(x) + \rho_k \frac{|x|^2}{n}$$

are harmonic in B if k is large enough, and $v_k \to u_0$ uniformly in B. Therefore (B.4) follows.

The proof of (B.5) now follows from (B.2) and B.1,2). Using B.1,3) and (B.5), and arguing as in [2, p. 120] we obtain

$$|\partial\{u_0 > 0\}| = 0 \tag{B.12}$$

and the assertion (B.6) follows.

The proof of (B.7) follows from B.1,2). To prove (B.8) we need (B.7) when $u_0 = 0$ and when $u_0 > 0$ we use the functions v_k as we did above. Combining (B.8) with (B.12) we show (B.9).

By (B.5), for $y \in \partial\{u_0 > 0\}$ there are points $y_k \in \partial\{u_k > 0\}$ converging to y. By B.1,3) we have for $R > 0$ and k large

$$\lambda_1 \le \frac{|B_R(y_k) \cap \{u_k > 0\}|}{|B_R(y_k)|} \le \lambda_2 .$$

Recalling (B.6), we see that (B.10) holds. Finally, combining (B.2) with B.1,2) we obtain (B.11). □

LEMMA B.4. *Let u be a function as in B.1 satisfying*

1) *There is* $x_0 \in D \cap \partial\{u > 0\}$ *such that* $u(x) = 0$ *in* $\{x \in D/(x - x_0).e_n = 0\}$,
2) *There is a ball* B *such that* $x_0 \in \partial B$, $B \subset \{x \in D/(x - x_0).e_n < 0\}$ *and* $u(x) > 0$ *for* $x \in B$.

Then,

$$\lim_{r \to 0} \frac{|\{x \in B_r(x_0)/(x - x_0).e_n > 0\} \cap \{u > 0\}|}{|B_r(x_0)|} = 0 .$$

PROOF. Let u_k be a blow-up sequence with respect to balls $B_{\rho k}(x_0)$ with limit u_0. By 1), 2) and Lemma B.3, we can apply Theorem A.1 to u_0 and this fact in conjunction with (B.6) yields the desired result. □

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