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Continuity properties of functions from Orlicz-Sobolev spaces and embedding theorems

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1. – Introduction

The classical Sobolev embedding theorem tells us that if G is a sufficiently smooth open subset of $\mathbb{R}^n$, $n \geq 2$, then

$$W^{1,p}(G) \to L^{\frac{np}{n-p}}(G) \tag{1.1}$$

if $p < n$ and

$$W^{1,p}(G) \to C_b(G) \tag{1.2}$$

if $p > n$. Here:

- L^p denotes Lebesgue space;
- $W^{1,p}(G) = \{u \in L^p(G) : u$ is weakly differentiable on G and $|Du| \in L^p(G)\}$, the customary Sobolev space;
- $C_b(G) = L^\infty(G) \cap C(G)$, where $C(G)$ is the space of continuous functions;
- The arrow "$\to$" stands for continuous embedding.

When $p = n$, as long as Lebesgue spaces only are taken into account, what one can say is that

$$W^{1,n}(G) \to L^q(G) \tag{1.3}$$

for every $q \in [n, \infty)$, whereas simple counterexamples show that $W^{1,n}(G) \not\subset L^\infty(G)$.

The embedding (1.3) can be improved if Orlicz spaces are employed. Roughly speaking, the Orlicz space $L^A(G)$ is the Banach space which is obtained when the role played by the function s^p in the definition of $L^p(G)$ is performed by the N-function (or, more generally, Young function) $A(s)$; the

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same replacement in the definition of $W^{1,p}(G)$ yields the Orlicz-Sobolev space $W^{1,A}(G)$ (see Section 2 for precise definitions on this subject).

Indeed, if the Lebesgue measure $m(G)$ of G is finite, then

$$W^{1,n}(G) \to L^B(G), \qquad B(s) = e^{s^{n'}} - 1, \tag{1.4}$$

where $n' = n/(n-1)$, the Hölder conjugate of n ([Tr]; see also [P]). Moreover, this embedding is sharp, in the sense that there is no Orlicz space, strictly contained in $L^B(G)$, $B(s) = e^{s^{n'}} - 1$, into which $W^{1,n}(G)$ is continuously embedded ([HMT]).

The present paper deals with the following optimal embedding problem: given any N-function A, which is the N-function B such that $L^B(G)$ is the smallest Orlicz space into which $W^{1,A}(G)$ is continuously embedded?

Embedding theorems for Orlicz-Sobolev spaces have been studied in [DT] and [Ad2]. In [DT] it is proved that, if G is sufficiently smooth, $m(G) < \infty$ and A is any N-function, then

$$W^{1,A}(G) \to C_b(G) \tag{1.5}$$

if $\int^\infty \frac{A^{-1}(t)}{t^{1+1/n}} dt < \infty$, and

$$W^{1,A}(G) \to L^{A_*}(G) \tag{1.6}$$

if $\int^\infty \frac{A^{-1}(t)}{t^{1+1/n}} dt = \infty$, where A_* is the N-function defined by

$$A_*^{-1}(r) = \int_0^r \frac{A^{-1}(t)}{t^{1+1/n}} dt . \tag{1.7}$$

Observe that such a result reproduces the Sobolev theorem if $A(s) = s^p$ with $p \neq n$. However, it does not solve the optimal embedding problem in general, and the reason it twofold. Firstly, as we shall see below, the convergence of the integral $\int^\infty \frac{A^{-1}(t)}{t^{1+1/n}} dt$ is not a sharp condition for the embedding (1.5) to hold. Secondly, definition (1.5) for a Sobolev conjugate of A fails to give optimal results when $\int^\infty \frac{A^{-1}(t)}{t^{1+1/n}} dt = \infty$ and the asymptotic behaviour of $A(s)$ near infinity is close to that of s^n. For instance, (1.6) yields $W^{1,n}(G) \to L^B(G)$ with $B(s) = e^s - s - 1$, a weaker result than (1.4).

Special situations have been studied by means of ad hoc methods – see, for instance, [EGO] and [FLS] for the case where $A(s)$ behaves like $s^n(\lg(s))^q$ for large s, with $q \geq n-1$ and $q < 0$, respectively.

Our results can be summarized as follows. In [Ta2] it is shown that, if $m(G) < \infty$, then the condition $\int^\infty \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty$ is sufficient for the embedding

$$W_0^{1,A}(G) \to L^\infty(G) \tag{1.8}$$

to hold. Here, $\tilde{A}$ is the Young conjugate of A and $W_0^{1,A}(G)$ is the subspace of $W^{1,A}(G)$ of those functions which vanish (in an appropriate sense) on ∂G. Notice that the convergence of the integral $\int^\infty \frac{\tilde{A}(t)}{t^{1+n'}}dt$ is a weaker condition than the convergence of $\int^\infty \frac{A^{-1}(t)}{t^{1+1/n}}dt$ (see [Ta2]). As a first result, we show that, in fact, the following holds.

THEOREM 1a. *Denote by G an open subset of $\mathbb{R}^n$, $n \geq 2$. Let A be an N-function. A constant C, depending only on A, $m(G)$ and n, exists such that*

$$\|u\|_{C_b(G)} \leq C\|Du\|_{L^A(G)} \tag{1.9}$$

for every G having finite measure and all $u \in W_0^{1,A}(G)$ if and only if

$$\int^\infty \frac{\tilde{A}(t)}{t^{1+n'}}dt < \infty. \tag{1.10}$$

Inequality (1.9) *holds also when $m(G) = \infty$ (and hence with C independent of $m(G)$) if and only if the full integral*

$$\int_0^\infty \frac{\tilde{A}(t)}{t^{1+n'}}dt < \infty. \tag{1.11}$$

An analogous result concerning the embedding (1.5) is proved in Theorem 1b, Section 3. Theorem 3 of the same section deals with the modulus of continuity of functions from $W^{1,A}(G)$. Thus, the existence of embeddings for Orlicz-Sobolev spaces into $C_b(G)$ is characterized.

Assume now that $\int^\infty \frac{\tilde{A}(t)}{t^{1+n'}}dt = \infty$. In this case we prove that, at least when A is sufficiently regular, the optimal embedding problem is solved by the N-function $\bar{A}$ defined as

$$\bar{A}(s) = \int_0^s r^{n'-1}\left(\Phi^{-1}(r^{n'})\right)^{n'} dr\,, \tag{1.12}$$

where

$$\Phi(r) = n' \int_0^r \frac{\tilde{A}(t)}{t^{1+n'}}dt \tag{1.13}$$

and Φ^{-1} is the inverse of Φ.

The following is a prototype of our results in this context.

THEOREM 2. *Denote by G an open subset of $\mathbb{R}^n$, $n \geq 2$. Let A be an N-function such that*

$$\int^\infty \frac{\tilde{A}(t)}{t^{1+n'}} dt = \infty .$$

If

$$\lim_{s\to\infty} \frac{A(\lambda s)}{A(s)} \quad \textit{exists for large} \quad \lambda , \tag{1.14}$$

then a constant C, depending only on A, $m(G)$ and n, exists such that

$$\|u\|_{L^{\bar{A}}(G)} \leq C\|Du\|_{L^A(G)} \tag{1.15}$$

for every G having finite measure and all $u \in W_0^{1,A}(G)$.

Inequality (1.15) *holds also when $m(G) = \infty$ (and hence with C independent of $m(G)$) provided that* (1.14) *is replaced by*

$$\sup_{s>0} \frac{A(\lambda s)}{A(s)} \leq \text{Const.} \inf_{s>0} \frac{A(\lambda s)}{A(s)} \quad \textit{for large} \quad \lambda \tag{1.16}$$

and A satisfies the additional assumption

$$\int_0 \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty .$$

In any case, irrespective of whether (1.14) (*or* (1.16)) *is satisfied or not,* (1.15) *cannot hold if $L^{\bar{A}}(G)$ is replaced by any smaller Orlicz space.*

Theorem 2 is a special case of a more general result which also includes the embedding

$$W^{1,A}(G) \to L^{\bar{A}}(G)$$

(Theorem 5, Section 5). Such a result holds under weaker assumptions than (1.14) (or (1.16)). However, we remark that conditions (1.14) and (1.16), which in a sense amount to requiring that A has a uniform rate of growth, are satisfied by customary N-functions.

Section 4 deals with embeddings of $W^{1,A}(G)$ into Marcinkiewicz spaces, which will be referred to as weak-type embeddings. In this context, an exhaustive picture of the situation is given. Indeed, we show that, for any N-function A and any sufficiently smooth G,

$$W^{1,A}(G) \to M^{\bar{A}}(G) ,$$

where $M^{\bar{A}}(G)$ is the Marcinkiewicz space associated to $\bar{A}$. Furthermore, this is the best possible among all embeddings of weak-type for $W^{1,A}(G)$ (Theorem 4).

2. – Orlicz spaces and rearrangements of functions

In this section we recall a few basic definitions and properties about Orlicz, Orlicz-Sobolev and Marcinkiewicz spaces and about the behaviour of norms in these spaces under rearrangements of functions. For a detailed treatment of Orlicz and Orlicz-Sobolev spaces we refer to [Ad1] and [RR]; Marcinkiewicz spaces are presented e.g. in [BS].

2.1. An N-function A is a convex function from $[0+\infty)$ into $[0,+\infty)$ which vanishes only at 0 and such that $\lim_{s\to 0^+}\frac{A(s)}{s}=0$ and $\lim_{s\to\infty}\frac{A(s)}{s}=\infty$.

Let G be any measurable subset of $\mathbb{R}^n$ and let A be an N-function. For any measurable function u on G, set

$$\|u\|_{L^A(G)} = \inf\left\{\lambda>0:\int_G A\left(\frac{|u(x)|}{\lambda}\right)dx\le 1\right\}. \tag{2.1}$$

The Orlicz space $L^A(G)$ is defined as the set of all (equivalence classes of) measurable functions u on G such that $\|u\|_{L^A(G)}<\infty$. $L^A(G)$, equipped with the norm $\|\cdot\|_{L^A(G)}$, is a Banach space. Clearly, if $A(s)=s^p$, then $L^A(G)=L^p(G)$ and $\|\cdot\|_{L^A(G)}=p^{-1/p}\|\cdot\|_{L^p(G)}$.

The following generalized version of Hölder's inequality holds:

$$\int_G u(x)\,v(x)\,dx\le 2\|u\|_{L^A(G)}\|v\|_{L^{\tilde A}(G)}. \tag{2.2}$$

Moreover,

$$\|u\|_{L^A(G)}\le\sup\left\{\int_G u(x)\,v(x)\,dx/\|v\|_{L^{\tilde A}(G)} : v\in L^{\tilde A}(G)\right\} \tag{2.3}$$

(see e.g. Lemma 8.17 of [Ad1]). Here, A is an N-function and $\tilde A$ is its Young conjugate; namely $\tilde A$ is the N-function defined by

$$\tilde A(s)=\sup\{sr-A(r):0<r<+\infty\}. \tag{2.4}$$

Embeddings between Orlicz spaces defined by different Young functions are characterized in terms of the following partial-ordering relation between functions. A function B is said to dominate a function A globally [respectively near infinity] if a positive constant K exists such that

$$A(s)\le B(Ks)$$

for all $s>0$ [respectively for s greater than some positive number]. Two functions A and B are called equivalent globally [near infinity] if each dominates the other globally [near infinity]. If for every $K>0$ a number $s_0\ge 0$ exists such that the last inequality holds for $s \geq s_0$, then A is said to increase essentially more slowly than B.

The embedding $L^B(G) \to L^A(G)$ holds if and only if either B dominates A globally or B dominates A near infinity and $m(G) < \infty$ ([Ad1], Theorem 8.12).

2.2. Orlicz spaces are rearrangement invariant spaces; namely, $\|u\|_{L^A(G)}$ depends only on the distribution function of u or, equivalently, on the decreasing rearrangement of u. Indeed, the equimeasurabilty of u and of its decreasing rearrangement u^* implies that

$$\int_G A(|u(x)|)dx = \int_0^{m(G)} A(u^*(s))ds \tag{2.5}$$

for any N-function A; hence, by (2.1),

$$\|u\|_{L^A(G)} = \|u^*\|_{L^A(0,m(G))} . \tag{2.6}$$

Recall that u^* is the right-continuous non-increasing function on $[0, m(G))$ which is equimeasurable with u; in formulas,

$$u^*(s) = \sup\{t \geq 0 : \mu_u(t) > s\} , \tag{2.7}$$

where

$$\mu_u(t) = m(\{x \in G : |u(x)| > t\}) , \tag{2.8}$$

the distribution function of u (we refer to [BS] for a detailed treatment of rearrangements and of rearrangement invariant spaces).

The Hardy-Littlewood maximal function of u^*, denoted by u^{**}, is given by

$$u^{**}(s) = \frac{1}{s}\int_o^s u^*(r)dr . \tag{2.9}$$

u^{**} is non-increasing, since u^* is. Clearly,

$$u^*(s) \leq u^{**}(s) \quad \text{for} \quad s \in [0, m(G)] , \tag{2.10}$$

and

$$\operatorname{ess\,sup} |u| = u^*(0) = u^{**}(0) . \tag{2.11}$$

Moreover,

$$(u + v)^{**}(s) \leq u^{**}(s) + v^{**}(s) \tag{2.12}$$

for any couple of measurable functions u and v on G and all $s \in [0, m(G)]$ ([BS], inequality (3.10)).

For any measurable function u on a subset G of $\mathbb{R}^n$ having finite measure, the signed decreasing rearrangement u^0 of u is defined by

$$(2.13)\quad u^0(s) = \sup\left\{t \in \mathbb{R} : m(\{x \in G : u(x) > t\}) > s\right\} \quad \text{for} \quad s \in [0, m(G)].$$

Observe that

$$(2.14)\qquad u^0(s) = u_+^*(s) - u_-^*(m(G) - s) \quad \text{for a.e.} \quad s \in [0, m(G)].$$

Hereafter, $u_+ = \frac{|u|+u}{2}$ and $u_- = \frac{|u|-u}{2}$, the positive and the negative part of u, respectively. Clearly, a property analogous to (2.6) holds for u^0; namely

$$(2.15)\qquad \|u\|_{L^A(G)} = \|u^0\|_{L^A(0,m(G))}.$$

2.3. The definition of Marcinkiewicz spaces is based on the notion of u^{**}. Let $\phi(s)$ be a non-decreasing function from $[0, +\infty)$ into $[0, +\infty]$ which tends to $+\infty$ as s goes to $+\infty$. Denote by ϕ^{-1} the (generalized) inverse of ϕ, i.e.

$$(2.16)\qquad \phi^{-1}(r) = \sup\{t \geq 0 : \phi(t) \leq r\} \quad \text{for} \quad r \geq 0,$$

so that

$$(2.17)\qquad \phi^{-1}(\phi(r)) \geq r \geq \phi(\phi^{-1}(r)).$$

For any measurable subset G of $\mathbb{R}^n$ and any measurable function u on G, set

$$(2.18)\qquad \|u\|_{M^\phi(G)} = \sup\left\{\frac{u^{**}(s)}{\phi^{-1}(1/s)} : 0 < s < m(G)\right\}.$$

The Marcinkiewicz space $M^\phi(G)$ is the collection of all functions u such that $\|u\|_{M^\phi(G)} < \infty$. Thanks to inequality (2.12), $M^\phi(G)$, endowed with $\|\cdot\|_{M^\phi(G)}$, is a (rearrangement invariant) Banach space. If A is an N-function, we shall call $M(A; G)$ the weak Olircz space defined by A. In fact, the embedding

$$(2.19)\qquad L^A(G) \to M^A(G)$$

holds and

$$(2.20)\qquad \|u\|_{M^A(G)} \leq \|u\|_{L^A(G)}$$

for any function u on G (see [O], Lemma 3.1).

Notice that $M^\phi(G) = L(p, \infty)$, the weak Lebesgue space, if $\phi(s) = s^p$ and $M^\phi(G) = L^\infty(G)$ if $\phi(s)$ equals 0 for $0 \leq s < 1$ and takes the value ∞ otherwise.

2.4. In this subsection we consider spaces of weakly differentiable functions. If G is an open subset of $\mathbb{R}^n$, the Orlicz-Sobolev space $W^{1,A}(G)$ is defined as

$$(2.21)\qquad W^{1,A}(G) = \{u \in L^A(G) : u \text{ is weakly differentiable on } G \text{ and } |Du| \in L^A(G)\}.$$

$W^{1,A}(G)$, equipped with the norm $\|u\|_{W^{1,A}(G)} = \|u\|_{L^A(G)} + \|Du\|_{L^A(G)}$, is a Banach space. By $W_0^{1,A}(G)$ we denote the subspace of those functions from $W^{1,A}(G)$ whose continuation by 0 outside G belongs to $W^{1,A}(\mathbb{R}^n)$.

The following generalisation of Pòlya-Szegö principle will play a basic role in our proofs (see e.g. [BZ]). Let u be any function from $W_0^{1,A}(G)$. Then u^* is locally absolutely continuous on $(0, m(G))$ and

$$(2.22)\qquad \|Du\|_{L^A(G)} \geq \left\| n\, C_n^{1/n}\, s^{1/n'} \left(-\frac{du^*}{ds}\right)\right\|_{L^A(0,m(G))}.$$

Here, $C_n = \pi^{n/2}/\Gamma(1+n/2)$, the measure of the unit ball in $\mathbb{R}^n$.

The decreasing rearrangement defined by (2.7) has been shown to be a usefull tool for dealing with Sobolev-type embeddings involving spaces of functions vanishing on the boundary (see [Au] and [Ta1]): indeed the n-dimensional problem is reduced to a 1-dimensional one thanks to (2.6) and (2.22).

To treat spaces of functions which do not necessarily vanish on the boundary, the signed decreasing rearrangement turns out to be appropriate. An inequality analogous to (2.22) for this kind of functions is stated in Lemma 1 below in terms of the isoperimetric function h_G of G. Such function is defined by

$$(2.23)\qquad h_G(s) = \inf\{P(E; G) : E \subseteq G, m(E) = s\},$$

where $P(E; G)$, the perimeter of E relative to G, coincides with the $(n-1)$-dimensional Hausdorff measure of $\partial E \cap G$ provided that E is smooth, and is given by the total variation over G of the gradient of the characteristic function of E otherwise.

Observe that, when $m(G) < \infty$,

$$(2.24)\qquad h_G(s) = h_G(m(G) - s) \quad \text{for} \quad s \in [0, m(G)].$$

The very definition of h_G implies

$$(2.25)\qquad h_G(m(E)) \leq P(E; G)$$

for any measurable subset E of G. (2.25) is called the isoperimetric inequality relative to G. In particular, if positive numbers σ and Q exist such that

$$(2.26)\qquad h_G(s)Q \geq \min{}^{\sigma}\{s, m(G) - s\} \quad \text{for} \quad s \in [0, m(G)],$$

then G is said to satisfy a relative isoperimetric inequality with exponent σ. The smallest number Q which renders (2.26) true is denoted by $Q_\sigma(G)$ and is called the relative isoperimetric constant of G for the exponent σ.

It is easily verified that (2.26) cannot hold with $\sigma < 1/n'$, whatever G is. On the other hand, (2.26) is known to hold with $\sigma = 1/n'$ if G is sufficiently regular, e.g. connected and with the cone property (see [M], Corollary 3.2.1/3 and Lemma 3.2.4). Recall that G has the cone property if there exist a cone Σ such that for any $x \in G$, G contains a cone which is congruent to Σ and whose vertex is x. In case dimension $n = 2$, explicit evaluations and estimates for $Q_\sigma(G)$ are available ([C1]).

A proof of inequality (2.22) is based on the standard isoperimetric inequality in $\mathbb{R}^n$ (see [B] for an alternative approach). Inequality (2.25) is a substitute of the latter for proving (see also [G] and [RT])

LEMMA 1. *Let G be any connected open subset of $\mathbb{R}^n$ having finite measure. Let A be an N-function. Assume that u is a weakly differentiable function on G and $|Du| \in L^A(G)$. Then u^0 is locally absolutely continuous and*

$$\|Du\|_{L^A(G)} \geq \left\| h_G(s)\left(-\frac{du^0}{ds}\right)\right\|_{L^A(0,m(G))}. \tag{2.27}$$

PROOF. The absolute continuity of u^0 is proved in Lemma 6.6 of [CEG]. Moreover, if we set

$$g(s) = h_G(s)\left(-\frac{du^0}{ds}\right),$$

then inequality (6.18) of the same paper tells us that

$$\int_0^s g^*(r)dr \leq \int_0^s |Du|^*(r)dr \quad \text{for} \quad s \in [0, m(G)]. \tag{2.28}$$

Thus, (2.27) follows from Proposition 2.1 of [ALT]. □

3. – Embeddings into $C_b(G)$

Theorem 1a of Section 1 and the results of this section are a completion of the theorem of [Ta2].

Theorem 1a characterizes the existence of the embedding $W_0^{1,A}(G) \to C_b(G)$. The existence of embedding for spaces of functions which do not necessarily vanish on ∂G requires some regularity assumption on G. Classes of sets which are appropriate to deal with such embeddings are introduced in the following definition (see also [M], Chapter 3).

Definition 1. *For $\sigma \geq 1/n'$ we set*

$$\mathcal{I}(\sigma) = \{G \subset \mathbb{R}^n : G \text{ is open and satisfies a relative isoperimetric inequality with exponent } \sigma\} \tag{3.1}$$

and

$$\mathcal{F}(\sigma) = \{G \subset \mathbb{R}^n : G \text{ is the union of a finite number of sets from } \mathcal{I}(\sigma)\}. \tag{3.2}$$

Example 1. Any open set G, with finite measure and having the cone property, belongs to the class $\mathcal{F}(1/n')$. In fact, under these assumptions, G has a finite number of connected components each one having the cone property and thus satisfying a relative isoperimetric inequality with exponent $1/n'$ (see Section 2).

Theorem 1b. *Denote by G an open subset of $\mathbb{R}^n$, $n \geq 2$. Let A be an N-function. The following assertions are equivalent*

i)
$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty .$$

ii) *A constant C, depending only on A, $m(G)$, $Q_{1/n'}(G)$ and n, exists such that*

$$\|u - u_G\|_{C_b(G)} \leq C \|Du\|_{L^A(G)} \tag{3.3}$$

for every $G \in \mathcal{I}(1/n')$ having finite measure and all $u \in W^{1,A}(G)$. Here $u_G = \frac{1}{m(G)} \int_G u(x)dx$, the mean value of u over G.

iii) *The embedding*

$$W^{1,A}(G) \to C_b(G) \tag{3.4}$$

holds for every $G \in \mathcal{F}(1/n')$ having finite measure.

Under the assumption that $\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty$, information about the modulus of continuity of functions from $W^{1,A}(G)$ is available in terms of the function H defined by

$$H(s) = \left(s\, \Theta^{-1}(s^{n'})\right)^{n'}, \tag{3.5}$$

where

$$\Theta(r) = n' \int_r^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt . \tag{3.6}$$

THEOREM 3. *Let G be an open subset of $\mathbb{R}^n$, $n \geq 2$. Let A be an N-function satisfying*

$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty .$$

Then, for every compact subset G' of G, a constant C exists such that

$$|u(x) - u(y)| \leq C \|u\|_{W^{1,A}(G)} H^{-1}(|x-y|^{-n}) \tag{3.7}$$

for any $u \in W^{1,A}(G)$ and a.e. $x, y \in G'$.

Under the additional assumption that G is a bounded strongly Lipschitz domain, (3.7) *holds for a.e. $x, y \in G$.*

Recall that a bounded open set $G \subset \mathbb{R}^n$ is called strongly Lipschitz if, for each $x \in \partial G$, there exist a neighbourhood $\mathcal{U}_x$ of x, a coordinate system $(\xi_1, \ldots, \xi_n)$ centered at x and a Lipschitz continuous function ϕ of $\xi_1, \ldots, \xi_{n-1}$ such that

$$G \cap \mathcal{U}_x = \{(\xi_1, \ldots \xi_n) : \xi_n > \phi(\xi_1, \ldots, \xi_{n-1})\} .$$

Theorems 1a, 1b and 3 yield, via Ascoli-Arzelà theorem, the following compactness result.

COROLLARY 1. *Let G be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. Let A be an N-function satisfying*

$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty .$$

Then the embedding

$$W_0^{1,A}(G) \to C_b(G)$$

is compact. If in addition G has the strong Lipschitz property, then also the embedding

$$W^{1,A}(G) \to C_b(G)$$

is compact.

Proposition 1 below generalises Theorem 1 in case of less smooth domains.

PROPOSITION 1. *Denote by G an open subset of $\mathbb{R}^n$, $n \geq 2$. Let $\sigma \in [1/n', 1)$. Let A be an N-function such that*

$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+1/\sigma}} dt < \infty . \tag{3.8}$$

Then:

i) *A constant C, depending only on A, $m(G)$, $Q_\sigma(G)$ and n, exists such that*

$$\|u - u_G\|_{C_b(G)} \leq C \|Du\|_{L^A(G)}$$

for every $G \in \mathcal{I}(\sigma)$ having finite measure and all $u \in W^{1,A}(G)$.

ii) *The embedding*

$$W^{1,A}(G) \to C_b(G)$$

holds for every $G \in \mathcal{F}(\sigma)$ having finite measure.

A proof of Proposition 1 makes use of the same arguments of the proof of Theorem 1b and will be omitted.

Before proving Theorems 1a, 1b and 3, we establish the following lemma.

LEMMA 2. *Let $n \geq 2$. Let A be an N-function. Then $r^{-1/n'} \in L^{\tilde{A}}(0, s)$ for $s > 0$ if and only if*

$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty .$$

Moreover,

$$\|r^{-1/n'}\|_{L^{\tilde{A}}(0,s)} = H^{-1}(1/s) \tag{3.9}$$

for $s > 0$, where H is the function defined by (3.5).
In particular,

$$\|r^{-1/n'}\|_{L^{\tilde{A}}(0,\infty)} = \left(n' \int_0^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt \right)^{1/n'} . \tag{3.10}$$

PROOF. By definition (2.1),

$$\|r^{-1/n'}\|_{L^{\tilde{A}}(0,s)} = \inf \left\{ \lambda > 0 : \int_0^s \tilde{A}\left(\frac{r^{-1/n'}}{\lambda} \right) dr \leq 1 \right\} . \tag{3.11}$$

A change of variable in the integral on the right-hand side of (3.11) yields

$$\|r^{-1/n'}\|_{L^{\tilde{A}}(0,s)} = \inf \left\{ \lambda > 0 : \frac{n'}{\lambda^{n'}} \int_{\lambda^{-1} s^{-1/n'}}^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt \leq 1 \right\} , \tag{3.12}$$

whence $\|r^{-1/n'}\|_{L^{\tilde{A}}(0,s)} < \infty$ if and only if $\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty$. If this is the case, equation (3.12) easily implies that

$$\|r^{-1/n'}\|_{L^{\tilde{A}}(0,s)} = \frac{s^{-1/n'}}{\Omega^{-1}(1/s)} , \tag{3.13}$$

where (recall (3.6))

$$\Omega(r) = r^{n'} \Theta(r) . \tag{3.14}$$

Equations (3.5) and (3.14) give

$$H^{-1}(s) = \frac{s^{1/n'}}{\Omega^{-1}(s)} . \tag{3.15}$$

Thus, (3.9) follows from (3.13) and (3.15).

Equation (3.10) is a consequence of (3.12). □

PROOF OF THEOREM 1a. Let G be an open subset of $\mathbb{R}^n$. Set $V = m(G)$. Then

$$u^*(0) = \int_0^V -\frac{du^*}{dr}(r)dr \tag{3.16}$$

for all $u \in W_0^{1,A}(G)$. Hence, via (2.2), we get

$$u^*(0) \le \frac{2}{nC_n^{1/n}} \|r^{-1/n'}\|_{L^{\tilde{A}}(0,V)} \left\| n\, C_n^{1/n} r^{1/n'} \left(-\frac{du^*}{dr}\right)\right\|_{L^A(0,V)} . \tag{3.17}$$

This is the point where the main ingredient of the proof, i.e. Pòlya-Szegö principle in the form of inequality (2.22), plays its role. Indeed, using (2.22) and recalling (2.11), one obtains from (3.17)

$$\|u\|_{L^\infty(G)} \le \frac{2}{n\, C_n^{1/n}} \|r^{-1/n'}\|_{L^{\tilde{A}}(0,V)} \|Du\|_{L^A(G)} . \tag{3.18}$$

Thus, by Lemma 2,

$$\|u\|_{L^\infty(G)} \le \frac{2}{n\, C_n^{1/n}} H^{-1}(1/V) \|Du\|_{L^A(G)} \tag{3.19}$$

if $V < \infty$, and

$$\|u\|_{L^\infty(G)} \le \frac{2}{n\, C_n^{1/n}} \left(n' \int_0^\infty \frac{\tilde{A}(t)}{t^{1+n'}} dt \right)^{1/n'} \|Du\|_{L^A(G)} \tag{3.20}$$

if $V = \infty$.

The continuity of u can be established as in the proof of Theorem 1b below. Thus, (1.9) follows from (1.10) when $V < \infty$ and from (1.11) when $V = \infty$.

Conversely, assume that (1.9) is true. Denote by S either a ball of $\mathbb{R}^n$ or the whole of $\mathbb{R}^n$ according to whether (1.9) is assumed to hold for sets having finite measure or for any set. Set $V = m(S)$. Consider radially decreasing functions U from $W_0^{1,A}(S)$. Since

$$\|DU\|_{L^A(S)} = \left\| n\, C_n^{1/n} r^{1/n'} \left(-\frac{dU^*}{dr}\right)\right\|_{L^A(0,V)} , \tag{3.21}$$

inequality (1.9) yields

$$\int_0^V -\frac{dU^*}{dr}(r)dr = U^*(0) \le C \left\| n\, C_n^{1/n} r^{1/n'} \left(-\frac{dU^*}{dr}\right)\right\|_{L^A(0,V)} . \tag{3.22}$$

Hence, thanks to the arbitrariness of U, we infer from (2.3) that

$$\|r^{-1/n'}\|_{L^{\tilde{A}}(0,V)} \leq n\, C_n^{1/n}\, C\,. \tag{3.23}$$

By Lemma 2, (3.23) implies (1.10) or (1.11) according to whether $V < \infty$ or $V = \infty$. □

PROOF OF THEOREM 1b. Let G be a set from $\mathcal{I}(1/n')$ having finite measure. Set $V = m(G)$. Then

$$\operatorname{ess\,sup} u - \operatorname{ess\,inf} u = \int_0^V -\frac{du^0}{dr}(r)dr \tag{3.24}$$

for all $u \in W^{1,A}(G)$.
Applying inequality (2.2) to the right-hand side of (3.24) we get

$$\operatorname{ess\,sup} u - \operatorname{ess\,inf} u \leq 2 \left\| \frac{1}{h_G(r)} \right\|_{L^{\tilde{A}}(0,V)} \left\| h_G(r) \left(-\frac{du^0}{dr}(r) \right) \right\|_{L^A(0,V)} . \tag{3.25}$$

From (2.24), (2.26) with $\sigma = 1/n'$ and (3.9) one infers

$$\left\| \frac{1}{h_G(r)} \right\|_{L^{\tilde{A}}(0,V)} \leq 2Q_{1/n'}(G)H^{-1}(1/V)\,. \tag{3.26}$$

Our basic tool is now the generalized Pòlya-Szegö principle expressed by inequality (2.27). Combining such inequality with (3.25)-(3.26) we get

$$\operatorname{ess\,sup} u - \operatorname{ess\,inf} u \leq 4Q_{1/n'}(G)H^{-1}(1/V)\|Du\|_{L^A(G)}\,, \tag{3.27}$$

whence, in particular,

$$\|u - u_G\|_{L^\infty(G)} \leq 4Q_{1/n'}(G)H^{-1}(1/V)\|Du\|_{L^A(G)}\,. \tag{3.28}$$

Moreover, since

$$\|1\|_{L^{\tilde{A}}(G)} = \frac{1}{\tilde{A}^{-1}(1/m(G))}\,, \tag{3.29}$$

via triangle inequality and inequalities (3.28) and (2.2) one obtains

$$\|u\|_{L^\infty(G)} \leq 4Q_{1/n'}(G)H^{-1}(1/V)\|Du\|_{L^A(G)} + \frac{2}{\tilde{A}^{-1}(1/V)}\|u\|_{L^A(G)}\,. \tag{3.30}$$

Thus ii) and (in the special case where $G \in \mathcal{I}(1/n')$) iii) will follow from (3.28) and (3.30), respectively, if we show that u equals a.e. a continuous function. To this purpose, consider any cube $E(\delta)$ contained in G and having sides of length δ. Call v the restriction of u to $E(\delta)$. Using inequality (3.25) with u replaced by v we get

$$\text{(3.31)}\quad \operatorname{ess\,sup} v - \operatorname{ess\,inf} v \le 2\left\|\frac{1}{h_{E(\delta)}(r)}\right\|_{L^{\tilde{A}}(0,\delta^n)}\left\|h_{E(\delta)}(r)\left(-\frac{dv^0}{dr}(r)\right)\right\|_{L^A(0,\delta^n)}.$$

Since the relative isoperimetric constant $Q_{1/n'}$ is dilation-invariant,

$$\text{(3.32)}\qquad Q_{1/n'}(E(\delta)) = Q_{1/n'}(E(1))$$

for every $\delta > 0$. Owing to (2.26), (3.9) and (3.32), we have

$$\text{(3.33)}\qquad \left\|\frac{1}{h_{E(\delta)}(r)}\right\|_{L^A(0,\delta^n)} \le 2Q_{1/n'}(E(1))H^{-1}(\delta^{-n}).$$

Moreover, by (2.27),

$$\text{(3.34)}\qquad \left\|h_{E(\delta)}(r)\left(-\frac{dv^0}{dr}(r)\right)\right\|_{L^A(0,\delta^n)} \le \|Dv\|_{L^A(E(\delta))} \le \|Du\|_{L^A(G)}.$$

From (3.31), (3.33) and (3.34) one obtains

$$\text{(3.35)}\qquad \operatorname{ess\,sup} v - \operatorname{ess\,sup} v \le 4Q_{1/n'}(E(1))H^{-1}(\delta^{-n})\|Du\|_{L^A(G)}.$$

Notice that $H^{-1}(\delta^{-n})$ tends to 0 as δ goes to 0^+.
Now, for $k \in \mathbb{N}$ set $u_k(x) = u_{E_x(1/k)}$, the mean value of u (extended by 0 outside G) over the cube $E_x(1/k)$ centered at x and having sides of length $1/k$ which are parallel to the coordinate axes. Observe that u_k is a continuous function.
Assume, for instance, that $k < h$; then inequality (3.35) implies that

$$|u_k(x) - u_h(x)| \le 4Q_{1/n'}(E(1))H^{-1}(k^n)\|Du\|_{L^A(G)}$$

whenever $E_x(1/k) \subseteq G$. Thus, $\{u_k\}$ is Cauchy sequence in $C(G')$ for any compact subset G' of G. Therefore, $\{u_k\}$ converges to a continuous function, say $\bar{u}$, in G. By Lebesgue theorem, $\bar{u} = u$ a.e. in G. Hence, the continuity of u follows.
Therefore, ii) is proved and iii) is proved under the additional assumption that $G \in \mathcal{I}(1/n')$. The generalisation of iii) to the case where $G \in \mathcal{F}(1/n')$ is straightforward.

Finally, both ii) and iii) imply i) since $\|u\|_{W^{1,A}(G)}$ and $\|Du\|_{L^A(G)}$ are equivalent norms in $W_0^{1,A}(G)$ by Lemma 3 of [Ta3]. The proof is complete. □

PROOF OF THEOREM 3, SKETCHED. Inequality (3.7) for x, y in compact subsets of G is easily deduced from (3.35).
To prove that (3.7) holds for a.e. $x, y \in G$ when G is a strongly Lipschitz domain one may assume, without loss generality, that G is a cube (see e.g. [DT], Theorem 3.6). In this case, for every $x, y \in G$, a cube $E(|x-y|)$, having sides of length $|x-y|$ which are parallel to those of G, exists such that $x, y \in E(|x-y|)$ and $E(|x-y|) \subseteq G$. Thus, (3.7) follows from (3.35). □

4. – Embeddings into Marcinkiewicz spaces

We have seen that the assumption

$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty$$

is necessary and sufficient for the embedding

$$W^{1,A}(G) \to C_b(G)$$

to hold for any sufficiently smooth open set G having fixed finite measure. Thus, henceforth we shall be concerned with embeddings for $W^{1,A}(G)$ when

$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt = \infty .$$

The following result solves the optimal embedding problem for embeddings into Marcinkiewicz spaces.

THEOREM 4. *Denote by G an open subset of $\mathbb{R}^n$, $n \geq 2$. Let A be an N-function. Assume that*

$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt = \infty . \tag{4.1}$$

Let $\bar{A}$ be the function defined by (1.12). *Then:*

i) *A constant C, depending only on A, $m(G)$ and n, exists such that*

$$\|u\|_{M^{\bar{A}}(G)} \leq C\|Du\|_{L^A(G)} \tag{4.2}$$

for every G having finite measure and all $u \in W_0^{1,A}(G)$.

ii) *A constant C, depending only on A, $m(G)$, $Q_{1/n'}(G)$ and n, exists such that*

$$\|u - u_G\|_{M^{\bar{A}}(G)} \leq C\|Du\|_{L^A(G)} \tag{4.3}$$

for every $G \in \mathcal{I}(1/n')$ having finite measure and all $u \in W^{1,A}(G)$.

iii) *The embedding*

$$W^{1,A}(G) \to M^{\bar{A}}(G) \tag{4.4}$$

holds for every $G \in \mathcal{F}(1/n')$ having finite measure.

Moreover, i) *holds also when $m(G) = \infty$ (and hence C is independent of $m(G)$ in* (4.2)) *provided that A satisfies the additional assumption*

$$\int_0 \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty .$$

In any case, $M^{\bar{A}}(G)$ is the smallest Marcinkiewicz space which renders (4.2)-(4.4) *true.*

REMARK 1. When dealing with sets G having finite measure, one may assume, without loss of generality, that $\int_0 \frac{\tilde{A}(t)}{t^{1+n'}}dt < \infty$. Indeed, if this is not the case, A can be replaced by another N-function which is equivalent to the original one near infinity and makes the relevant integral converge. Such a replacement does not effect the result since the corresponding Orlicz-Sobolev space remains unchanged (see Section 2). In particular, the function $\bar{A}$, given by (1.12), can always be assumed to be well-defined.

EXAMPLE 2. Consider N-functions $A(s)$ which are equivalent to $s^p(\lg(s))^q$ near infinity, where either $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q > 0$. Let G be an open subset of $\mathbb{R}^n$ having finite measure. Then Theorem 1a tells us that the embedding

$$W_0^{1,A}(G) \to C_b(G)$$

holds if and only if either $p > n$, or $p = n$ and $q > n - 1$.

Theorem 4 yields

$$W_0^{1,A}(G) \to M^{\bar{A}}(G)$$

where

$$\text{(4.5)}\quad \bar{A}(s) \text{ is equivalent near infinity to} \begin{cases} s^{\frac{np}{n-p}}(\lg s)^{\frac{nq}{n-p}} & \text{if } 1 \le p < n \\ e^{s^{\frac{n}{n-1-q}}} & \text{if } p = n,\ q < n-1 \\ e^{e^{s^{n'}}} & \text{if } p = n,\ q = n-1\,. \end{cases}$$

The same embeddings are true (and optimal) with $W_0^{1,A}(G)$ replaced by $W^{1,A}(G)$ provided that $G \in \mathcal{F}(1/n')$.

The following version of Theorem 4 (with analogous proof) holds under weaker smoothness assumptions on G. The role played by $\bar{A}$ in Theorem 3 is performed here by the function $\bar{A}_\sigma$ defined by

$$\text{(4.6)}\qquad \bar{A}_\sigma(s) = \int_0^s r^{-1+1/\sigma}(\Phi_\sigma^{-1}(r^{1/\sigma}))^{1/\sigma}dr\,,$$

where

$$\text{(4.7)}\qquad \Phi_\sigma(r) = \frac{1}{\sigma}\int_0^r \frac{\tilde{A}(t)}{t^{1+1/\sigma}}dt\,.$$

Notice that $\bar{A} = \bar{A}_{1/n'}$.

PROPOSITION 2. *Denote by G an open subset of* $\mathbb{R}^n$, $n \ge 2$. *Let* $\sigma \in [1/n', 1)$. *Let A be an N-function such that*

$$\text{(4.8)}\qquad \int^\infty \frac{\tilde{A}(t)}{t^{1+1/\sigma}}dt = \infty\,.$$

Then:

i) *A constant C, depending only on A, m(G), $Q_\sigma(G)$ and n, exists such that*

(4.9)
$$\|u - u_G\|_{M^{\bar{A}_\sigma}(G)} \le C\|Du\|_{L^A(G)}$$

for every $G \in \mathcal{I}(\sigma)$ having finite measure and all $u \in W^{1,A}(G)$.

ii) *The embedding*

(4.10)
$$W^{1,A}(G) \to M^{\bar{A}_\sigma}(G)$$

holds for every $G \in \mathcal{F}(\sigma)$ having finite measure.

The origin of the function $\bar{A}$ is explained by the following lemmas.

LEMMA 3. *Let $n \ge 2$. Let A be an N-function such that*

(4.11)
$$\int_0 \frac{\tilde{A}(t)}{t^{1+n'}}dt < \infty \quad \textit{and} \quad \int^\infty \frac{\tilde{A}(t)}{t^{1+n'}}dt = \infty .$$

Set

(4.12)
$$A_n(s) = \left(s\,\Phi^{-1}(s^{n'})\right)^{n'},$$

where Φ is defined by (1.13). *Then*

(4.13)
$$\|r^{-1/n'}\|_{L^{\tilde{A}}(s,\infty)} = A_n^{-1}(1/s)$$

and

(4.14)
$$\|r^{1/n}\|_{L^{\tilde{A}}(0,s)} \le (n-1)s\,A_n^{-1}(1/s),$$

for $s > 0$.

The function A_n, given by (4.12), is not necessarily convex. However, A_n is equivalent to the N-function $\bar{A}$ defined by (1.12) (Lemma 4 below). Thus, Theorem 4 could have been equivalently stated with $\bar{A}$ replaced by A_n. We have preferred to make use of the former function for homogeneity with Theorems 2 and 5.

LEMMA 4. *Let $n \geq 2$. Let A be an N-function satisfying* (4.11). *Let $\bar{A}$ and A_n be the functions defined by* (1.12) *and* (4.12), *respectively. Then $\bar{A}$ is an N-function and*

$$\bar{A}(s) \leq A_n(s) \leq \bar{A}(2s) \quad \text{for} \quad s \in (0, \infty). \tag{4.15}$$

PROOF OF LEMMA 3. By (2.1)

$$\|r^{-1/n'}\|_{L^{\tilde{A}}(s,\infty)} = \inf\left\{\lambda > 0 : \int_s^\infty \tilde{A}\left(\frac{r^{-1/n'}}{\lambda}\right) dr \leq 1\right\}. \tag{4.16}$$

A change of variable in the latter integral shows that

$$\|r^{-1/n'}\|_{L^{\tilde{A}}(s,\infty)} = \frac{s^{-1/n'}}{\Psi^{-1}(1/s)} \tag{4.17}$$

where

$$\Psi(r) = r^{n'}\Phi(r) \tag{4.18}$$

and Φ is the function defined by (1.13). On the other hand, by (4.12) and (4.18),

$$A_n^{-1}(s) = \frac{s^{1/n'}}{\Psi^{-1}(s)}. \tag{4.19}$$

Equations (4.17) and (4.19) give (4.13).

As far as (4.14) is concerned, we have

$$\begin{aligned} \|r^{1/n}/s\|_{L^{\tilde{A}}(0,s)} &= \inf\left\{\lambda > 0 : \int_0^s \tilde{A}\left(\frac{r^{1/n}}{\lambda s}\right) dr \leq 1\right\} \\ &= \inf\left\{\lambda > 0 : \int_s^\infty (n-1)\tilde{A}\left(\frac{y^{-1/n'}}{\lambda}\right)(s/y)^n dy \leq 1\right\}. \end{aligned} \tag{4.20}$$

Since $\tilde{A}$ is non-decreasing, convex and vanishing at 0,

$$(n-1)\tilde{A}(s) \leq \tilde{A}((n-1)s). \tag{4.21}$$

Combining (4.20) and (4.21) and making use of the fact that $(s/y)^n \leq 1$ for $y \geq s$ yields

$$\|r^{1/n}/s\|_{L^{\tilde{A}}(0,s)} \leq (n-1)\|y^{-1/n'}\|_{L^{\tilde{A}}(s,\infty)}. \tag{4.22}$$

Thus, (4,14) is a consequence of (4.22) and (4.13). □

PROOF OF LEMMA 4. We have

$$\bar{A}(s) = \int_0^s \frac{A_n(r)}{r} dr\,. \tag{4.23}$$

Since $A_n(s)/s$ is increasing, formula (4.23) shows that $\bar{A}$ is an N-function. Moreover, the following chain holds:

$$\begin{aligned} A_n(s/2) = \frac{A_n(s/2)}{s/2} \int_{s/2}^s dr \le \int_{s/2}^s \frac{A_n(r)}{r} dr \le \int_0^s \frac{A_n(r)}{r} dr \\ \le \frac{A_n(s)}{s} \int_0^s dr = A_n(s) \end{aligned}$$

for $s > 0$. Hence, the conclusion follows. □

PROOF OF THEOREM 4. We begin by proving part i). Let G be an open subset of $\mathbb{R}^n$. By Remark 1, we can assume that $\int_0 \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty$ also in case $m(G) < \infty$. Let $u \in W_0^{1,A}(G)$. Extending u by 0 outside G (and still denoting by u the extended function) we have $u \in W^{1,A}(\mathbb{R}^n)$ and

$$u^*(s) = \int_s^\infty -\frac{du^*}{dr}(r) dr \tag{4.24}$$

for $s > 0$. From (4.24), via Fubini's theorem, we obtain

$$u^{**}(s) = \frac{1}{s} \int_0^s u^*(r) dr = \frac{1}{s} \int_0^s -t \frac{du^*}{dt}(t) dt + \int_s^\infty -\frac{du^*}{dt}(t) dt\,. \tag{4.25}$$

By (2.2), equation (4.25) implies

$$\begin{aligned} u^{**}(s) \le \frac{2}{n\, C_n^{1/n}} \Bigg\{ \left\| n\, C_n^{1/n}\, t^{1/n'} \left(-\frac{du^*}{dt} \right) \right\|_{L^A(0,s)} \frac{1}{s} \| t^{1/n} \|_{L^{\tilde{A}}(0,s)} \\ + \left\| n\, C_n^{1/n}\, t^{1/n'} \left(-\frac{du^*}{dt} \right) \right\|_{L^A(s,\infty)} \| t^{-1/n'} \|_{L^{\tilde{A}}(s,\infty)} \Bigg\}\,. \end{aligned} \tag{4.26}$$

Using Pòlya-Szegö principle in the form of (2.22) and recalling (4.13)-(4.14) one gets from (4.26)

$$u^{**}(s) \le \frac{2}{C_n^{1/n}} A_n^{-1}(1/s) \| Du \|_{L^A(G)}\,. \tag{4.27}$$

Thus, (4.2) follows from (4.27) and (4.15).

Conversely, suppose that ϕ is any (admissible) function enjoing the following property: a constant C exists which renders the inequality

$$\| u \|_{M^\phi(G)} \le C \| Du \|_{L^A(G)} \tag{4.28}$$

true for every open subset G of $\mathbb{R}^n$ having a fixed measure V and all $u \in W_0^{1,A}(G)$. Let S be either a ball of $\mathbb{R}^n$ having measure V or the whole of $\mathbb{R}^n$ according to whether $V < \infty$ or $V = \infty$. Fix $s \in (0, V)$. If U is any radially decreasing function from $W_0^{1,A}(S)$ such that $-\frac{dU^*}{dr}$ vanishes outside (s, V), then (4.28) tells us that

$$U^{**}(s) \leq C\phi^{-1}(1/s) \left\| n\, C_n^{1/n}\, r^{1/n'} \left(-\frac{dU^*}{dr}\right) \right\|_{L^A(s;V)} . \tag{4.29}$$

On the other hand, by (2.10),

$$\int_s^V -\frac{dU^*}{dr}(r)dr = U^*(s) \leq U^{**}(s) . \tag{4.30}$$

Owing to (2.3) and to the arbitrariness of U, inequalities (4.29)-(4.30) lead to

$$\|r^{-1/n'}\|_{L^{\tilde{A}}(s,V)} \leq n\, C_n^{1/n} C\, \phi^{-1}(1/s) \quad \text{for} \quad s \in (0, V) . \tag{4.31}$$

Equation (4.13) and inequality (4.31) ensure that a constant K exists such that

$$A_n^{-1}(1/s) \leq K\, \phi^{-1}(1/s) \quad \text{for} \quad s \in (0, V) . \tag{4.32}$$

Inequalities (4.32) and (4.15) imply that $M^{\tilde{A}}(G)$ is continuously embedded into $M^{\phi}(G)$ for any open set G having measure V. Thus, $M^{\tilde{A}}(G)$ is the smallest Marcinkiewicz space which renders (4.2) true.

Consider assertions ii) and iii). Let G be any set from $\mathcal{I}(1/n')$ having finite measure. By Remark 1, we may assume that $\int_0 \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty$. Set $V = m(G)$. Let $u \in W^{1,A}(G)$. Our starting point for the proof of (4.3)-(4.4) are the inequalities

$$(u-u_G)^{**}(s) \leq (u-u^0(V/2))_+^{**}(s)+(u-u^0(V/2))_-^{**}(s)+|u^0(V/2)-u_G| \tag{4.33}$$

and

$$u^{**}(s) \leq (u-u^0(V/2))_+^{**}(s)+(u-u^0(V/2))_-^{**}(s)+|u^0(V/2)-u_G|+|u_G| , \tag{4.34}$$

respectively, which hold for $s \in [0, V]$ and follow, via (2.12), from the equations

$$u = u - u^0(V/2) + u^0(V/2) - u_G + u_G \tag{4.35}$$

and

$$|u - u^0(V/2)| = (u - u^0(V/2))_+ + (u - u^0(V/2))_- . \tag{4.36}$$

Now, first recall (2.14) and deduce that

$$(u - u^0(V/2))_+^*(s) = u^0(s) - u^0(V/2) = \int_s^{V/2} -\frac{du^0}{dr}(r)dr \tag{4.37a}$$

$$(u - u^0(V/2))_-^*(s) = u^0(V/2) - u^0(V-s) = \int_s^{V/2} -\frac{du^0}{dr}(V-r)dr \tag{4.37b}$$

for a.e. $s \in [0, V/2]$, whereas

$$(u - u^0(V/2))_\pm^*(s) = 0 \quad \text{for} \quad s \in (V/2, V]. \tag{4.37c}$$

Starting from equations (4.37), proceeding along the same lines as in the proof of (4.2), making use of (2.27) (instead of (2.22)) and of (2.26) with $\sigma = 1/n'$ one arrives at

$$(u - u^0(V/2))_\pm^{**}(s) \le C\, A_n^{-1}(1/s)\|Du\|_{L^A(G)} \tag{4.38}$$

for some constant C and for $s \in [0, V]$.

Next, it is easily verified that

$$|u^0(V/2) - u_G| \le \frac{1}{V}\|u^0(V/2) - u\|_{L^1(G)}. \tag{4.39}$$

Inequality (4.39), Hölder inequality and Theorem 3.2.3 of [M] tell us that

$$|u^0(V/2) - u_G| \le \frac{Q_{1/n'}(G)}{V^{1/n'}}\|Du\|_{L^1(G)}. \tag{4.40}$$

Hence, by (2.2) and (3.29),

$$|u^0(V/2) - u_G| \le \frac{2Q_{1/n'}(G)}{V^{1/n'}\tilde{A}^{-1}(1/V)}\|Du\|_{L^A(G)}. \tag{4.41}$$

Finally, (2.2) and (3.29) imply

$$|u_G| \le \frac{2}{V\tilde{A}^{-1}(1/V)}\|u\|_{L^A(G)}. \tag{4.42}$$

Thus ii) is a consequence of inequalities (4.33), (4.38), (4.15) and (4.41); iii) follows, in the special case where $G \in \mathcal{I}(1/n')$, from (4.34), (4.38), (4.15), (4.41) and (4.42). Assertion iii) is easily extended to the case where $G \in \mathcal{F}(1/n')$ by means of inequality (2.12).

The fact that $M^{\bar{A}}(G)$ cannot be replaced by any smaller Marcinkiewicz space in (4.3)-(4.4) is a consequence of the same assertion for inequality (4.2), thanks to the equivalence of the norms $\|u\|_{W^{1,A}(G)}$ and $\|Du\|_{L^A(G)}$ in $W_0^{1,A}(G)$ (Lemma 3 of [Ta3]). □

5. – Embeddings into Orlicz spaces

Let A be an N-function such that

$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt = \infty .$$

We already know (Theorem 4) that, if G is sufficiently smooth and has finite measure, then $M^{\bar{A}}(G)$ is the smallest weak Orlicz space into which $W^{1,A}(G)$ is continuously embedded.

On the other hand, Proposition 4 below ensures that any Orlicz space into which $W^{1,A}(G)$ is embedded cannot be smaller than $L^{\bar{A}}(G)$.

These facts suggest that $\bar{A}$ is a good candidate to solve the optimal embedding problem for Orlicz-Sobolev spaces stated in Section 1. We are able to prove that $\bar{A}$ is in fact the solution under some regularity assumption on A. The regularity we need will be expressed in terms of the quantities defined as follows.

For any positive non-decreasing function f on $(0, \infty)$, set

$$\alpha(f) = \lim_{\lambda\to\infty} \frac{\lg\left(\limsup\limits_{s\to\infty} \dfrac{f(\lambda s)}{f(s)}\right)}{\lg \lambda} \qquad \beta(f) = \lim_{\lambda\to\infty} \frac{\lg\left(\liminf\limits_{s\to\infty} \dfrac{f(\lambda s)}{f(s)}\right)}{\lg \lambda}$$

$$\alpha_\infty(f) = \lim_{\lambda\to\infty} \frac{\lg\left(\sup_{s>0} \dfrac{f(\lambda s)}{f(s)}\right)}{\lg \lambda} \qquad \beta_\infty(f) = \lim_{\lambda\to\infty} \frac{\lg\left(\inf_{s>0} \dfrac{f(\lambda s)}{f(s)}\right)}{\lg \lambda} .$$

Notice that the above limits exists by Lemma 1 of [Bo]. Clearly,

$$0 \le \beta_\infty(f) \le \beta(f) \le \alpha(f) \le \alpha_\infty(f) \le \infty$$

for any such f.

Theorem of [Bo] tells us that, if A is an N-function, then the upper and the lower Boyd indices of the Orlicz space $L^A(G)$ agree with $\alpha(A^{-1})$ and $\beta(A^{-1})$, respectively, if $m(G) < \infty$ and with $\alpha_\infty(A^{-1})$ and $\beta_\infty(A^{-1})$ if $m(G) = \infty$. We need not recall the definition of Boyd indices here (see e.g. [BS]); let us only mention that they play a role in the theory of interpolation in rearrangement invariant spaces. The following relations hold for any N-function A:

$$\alpha(A^{-1}) + \beta(\tilde{A}^{-1}) = 1 \qquad \beta(A^{-1}) + \alpha(\tilde{A}^{-1}) = 1 \tag{5.1}$$

(see [Bo]). The same equations are true with α and β replaced by α_∞ and β_∞.

THEOREM 5. *Denote by G an open subset of $\mathbb{R}^n$, $n \leq 2$. Let A be an N-function such that*

$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+n'}} dt = \infty .$$

Assume that

$$\text{(5.2)} \qquad \textit{either} \quad \alpha(A) < n \quad \textit{or} \quad \frac{1}{\beta(A)} - \frac{1}{\alpha(A)} < \frac{1}{n}$$

(in particular, the latter condition is fulfilled if $\lim_{s\to\infty} \frac{A(\lambda s)}{A(s)}$ *exists for large* λ*).*
Let $\bar{A}$ *be the* N*-function defined by* (1.12)*. Then:*
i) *A constant* C*, depending only on* A*,* $m(G)$ *and* n*, exists such that*

$$\text{(5.3)} \qquad \|u\|_{L^{\bar{A}}(G)} \leq C\|Du\|_{L^A(G)}$$

for every G *having finite measure and all* $u \in W_0^{1,A}(G)$*.*
ii) *A constant* C*, depending only on* A*,* $m(G)$*,* $Q_{1/n'}(G)$ *and* n*, exists such that*

$$\text{(5.4)} \qquad \|u - u_G\|_{L^{\bar{A}}(G)} \leq C\|Du\|_{L^A(G)}$$

for every $G \in \mathcal{I}(1/n')$ *having finite measure and all* $u \in W^{1,A}(G)$*.*
iii) *The embedding*

$$\text{(5.5)} \qquad W^{1,A}(G) \to L^{\bar{A}}(G)$$

holds for every $G \in \mathcal{F}(1/n')$ *having finite measure.*

Assertion i) *is true also when* $m(G) = \infty$ *(and hence* C *is independent of* $m(G)$ *in* (5.3)*) provided that* (5.2) *is replaced by*

$$\text{(5.6)} \qquad \textit{either} \quad \alpha_\infty(A) < n \quad \textit{or} \quad \frac{1}{\beta_\infty(A)} - \frac{1}{\alpha_\infty(A)} < \frac{1}{n}$$

(in particular, the latter condition is fulfilled if $\sup_{s>0} \frac{A(\lambda s)}{A(s)} \leq \text{Const.} \inf_{s>0} \frac{A(\lambda s)}{A(s)}$ *for large* λ*) and* A *satisfies the additional assumption*

$$\int_0 \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty .$$

In any case, irrespective of whether (5.2) *(or* (5.6)*) is satisfied or not,* (5.3)-(5.5) *cannot hold if* $L^{\bar{A}}(G)$ *is replaced be any smaller Orlicz space.*

REMARK 3. Let us make a few comments about assumption (5.2) (analogous remarks hold for (5.6)).

Arguing as in the proof of Lemma 2 of [Ta3] shows that, if $\alpha(A) < n$, then $A(s)$ is dominated near infinity by s^p for some $p < n$.
A condition ensuring that $\alpha(A) < n$ is

$$\limsup_{s\to\infty} \frac{s\dfrac{dA}{ds}(s)}{A(s)} < n\,.$$

This assertion can be proved by using the fact that "$\lim_{\lambda\to\infty}$" in the definition of α can be replaced by "$\sup_{\lambda>1}$" (see [Bo], Lemma 1) and arguing as in remark (v) of [Ta3].

The assumption $\int^\infty \frac{\tilde{A}(t)}{t^{1+n'}}dt = \infty$ implies that $n \geq \beta(A)$. Indeed, if $n < \beta(A)$, then $\tilde{A}(s)/\tilde{A}(r) \leq \text{Const.}(s/r)^{n'-\delta}$ for sufficiently small δ when $s \geq r$ and r is large enough (see Lemma 2 of [Ta3]). Consequently, the hypothesis $\frac{1}{\beta(A)} - \frac{1}{\alpha(A)} < \frac{1}{n}$ plays a role in case $\beta(A) \leq n \leq \alpha(A)$. Such an hypothesis requires that $\alpha(A)$ and $\beta(A)$ do not differ "too much", i.e. the rate of growth of A is (any but) uniform enough.

REMARK 4. The same argument as in Theorem 3.7 of [DT] ensures that, whenever the embedding (5.5) is true and G is bounded, then the embedding

$$W^{1,A}(G) \to L^B(G)\,,$$

is compact for any N-function B which increases essentially more slowly than $\bar{A}$ near infinity.

EXAMPLE 3. Let us take into account again N-functions $A(s)$ which are equivalent to $s^p(\lg(s))^q$ near infinity, where either $p > 1$ and $q \in \mathbb{R}$ or $p = 1$ and $q > 0$. Such functions are easily seen to satisfy (1.14). Let G be an open subset of $\mathbb{R}^n$ having finite measure. Then, by Theorem 2,

$$W_0^{1,A}(G) \to L^{\bar{A}}(G)$$

whenever p and q are such that $\int^\infty \frac{\tilde{A}(t)}{t^{1+n'}}dt = \infty$. Furthermore, such embedding is optimal. The asymptotic behaviour of the N-function $\bar{A}$ is described by (4.5).

If $G \in \mathcal{F}(1/n')$, then also the embedding

$$W^{1,A}(G) \to L^{\bar{A}}(G) \tag{5.7}$$

holds and is optimal, by Theorem 5.
Notice that (5.7) includes known results as special cases: when $p < n$ then (5.7) agrees with Sobolev theorem (1.1) if $q = 0$, and overlaps with (1.6) if $q \neq 0$; when $p = n$, (5.7) reproduces the embedding (1.4) for $q = 0$ and gives results of [FLS] if $q < 0$ and of [EGO] if $q = n - 1$.

The following extension of Theorem 5 holds for less smooth domains G.

PROPOSITION 3. *Denote by G an open subset of* $\mathbb{R}^n$, $n \geq 2$. *Let* $\sigma \in [1/n', 1)$. *Let A be an N-function such that*

$$\int^{\infty} \frac{\tilde{A}(t)}{t^{1+1/\sigma}} dt = \infty .$$

Assume that

$$\text{either} \quad \alpha(A) < 1/(1-\sigma) \quad \text{or} \quad \frac{1}{\beta(A)} - \frac{1}{\alpha(A)} < 1 - \sigma .$$

Let $\bar{A}_\sigma$ *be the function defined by* (4.6). *Then:*

i) *A constant C, depending only on A,* $m(G)$, $Q_\sigma(G)$ *and n, exists such that*

$$\|u - u_G\|_{L^{\bar{A}_\sigma}(G)} \leq C \|Du\|_{L^A(G)}$$

for every $G \in \mathcal{I}(\sigma)$ *having finite measure and all* $u \in W^{1,A}(G)$.

ii) *The embedding*

$$W^{1,A}(G) \to L^{\bar{A}_\sigma}(G)$$

holds for every $G \in \mathcal{F}(\sigma)$ *having finite measure.*

A proof of Proposition 3 is analogous to that of Theorem 5 and makes use of versions of Lemma 4 above and Lemmas 5-7 below with $\bar{A}$ replaced by $\bar{A}_\sigma$ and n' replaced by $1/\sigma$. The details will be omitted for brevity.

We establish now some results which will be used in the proof of Theorem 5. We being by a proposition concerning the optimality of our embedding.

PROPOSITION 4. *Let* $n \geq 2$. *Let S be either* $\mathbb{R}^n$ *or a ball of* $\mathbb{R}^n$. *Assume that A and B are N-functions enjoing the following property: a constant C exists such that*

$$\|u\|_{L^B(S)} \leq C \|Du\|_{L^A(S)} \tag{5.8}$$

for all $u \in W_0^{1,A}(S)$. *Then*

$$L^{\bar{A}}(G) \to L^B(G) \tag{5.9}$$

for every open subset G of $\mathbb{R}^n$, *or for every open subset G of* $\mathbb{R}^n$ *having finite measure, according to whether S is the whole of* $\mathbb{R}^n$ *or a ball.*

The embedding (5.9) *is true for every open subset G of* $\mathbb{R}^n$ *having finite measure also if S is a ball and inequality* (5.8) *is replaced by the inequality*

$$\|u - u_G\|_{L^B(S)} \leq C \|Du\|_{L^A(S)}$$

for $u \in W^{1,A}(S)$, *or by the embedding* $W^{1,A}(S) \to L^B(S)$.

PROOF. Set $V = m(S)$. Let U be any radially decreasing function from $W_0^{1,A}(S)$. From the equation in (4.30) and from (2.6), (3.21), (5.8) we get

$$\left\| \int_r^V -\frac{dU^*}{dt}(t)dt \right\|_{L^B(0,V)} \leq C \left\| n\, C_n^{1/n}\, r^{1/n'} \left(-\frac{dU^*}{dr}\right) \right\|_{L^A(0,V)} . \tag{5.10}$$

Fix $s \in (0, V)$. For any U such that $\frac{dU^*}{dr}$ vanishes outside (s, V), one has

$$\left\| n\, C_n^{1/n}\, r^{1/n'} \left(-\frac{dU^*}{dr}\right) \right\|_{L^A(0,V)} = \left\| n\, C_n^{1/n}\, r^{1/n'} \left(-\frac{dU^*}{dr}\right) \right\|_{L^A(s,V)} \tag{5.11}$$

and

$$\begin{aligned} &\left\| \int_r^V -\frac{dU^*}{dt}(t)dt \right\|_{L^B(0,V)} \geq \left\| \int_r^V -\frac{dU^*}{dt}(t)dt \right\|_{L^B(0,s)} \\ &= \|1\|_{L^B(0,s)} \int_s^V -\frac{dU^*}{dt}(t)dt = \frac{1}{B^{-1}(1/s)} \int_s^V -\frac{dU^*}{dt}(t)dt . \end{aligned} \tag{5.12}$$

Let A_n be the function defined by (4.12). From (5.10)-(5.12), thanks to (2.3), (4.13) and to the arbitrariness of U, one infers that

$$B(s) \leq A_n(n\, C_n^{1/n}\, Cs)$$

for $s \in (0, \infty)$, or for s greater than some s_0, according to whether $V = \infty$ or $V < \infty$. Thus, by (4.15), $\bar{A}$ dominates B globally or near infinity, respectively. Hence, the embedding (5.9) is a consequence of Theorem 8.12 of [Ad1].

The second part of the statement follows from the equivalence of the norms $\|u\|_{W^{1,A}(S)}$ and $\|Du\|_{L^A(S)}$ in $W_0^{1,A}(S)$ (Lemma 3 of [Ta3]). □

LEMMA 5. *Let $n \geq 2$. Let A be an N-function satisfying* (4.11). *Let $\bar{A}$ be the N-function defined by* (1.12). *If*

$$\frac{1}{\beta(A)} - \frac{1}{\alpha(A)} < \frac{1}{n} \tag{5.13}$$

then

$$\alpha(\tilde{A}^{-1}) + \alpha(\bar{A}^{-1}) < 1 . \tag{5.14}$$

The same statement is true with α and β replaced by α_∞ and β_∞, respectively.

PROOF. Lemma 4 ensures that $\alpha(\tilde{A}^{-1}) = \alpha(A_n^{-1})$. Moreover, equation (4.19) and the very definition of α and β yield

$$\alpha(A_n^{-1}) = \frac{1}{n'} - \beta(\Psi^{-1}). \tag{5.15}$$

Hence, (5.14) is equivalent to

$$\alpha(\tilde{A}^{-1}) - \beta(\Psi^{-1}) < \frac{1}{n}. \tag{5.16}$$

The following relations hold for any N-function ϕ:

$$\alpha(\phi) = \frac{1}{\beta(\phi^{-1})}, \qquad \beta(\phi) = \frac{1}{\alpha(\phi^{-1})}. \tag{5.17}$$

This is a consequence of the fact that the functions

$$h(\tau) = \limsup_{s\to\infty} \frac{\phi^{-1}(s)}{\phi^{-1}(\tau s)} \tag{5.18}$$

and

$$k(\lambda) = \limsup_{s\to\infty} \frac{\phi(\lambda s)}{\phi(s)} \tag{5.19}$$

are related by

$$k^{-1}(\tau) = \frac{1}{h(\tau)} \tag{5.10}$$

(see the Remark in [Bo]).

Thus, since both $\tilde{A}$ and Ψ are N-function (note that $\frac{d\Psi(s)}{ds}$ is non-decreasing since $\frac{\tilde{A}(s)}{s}$ so is), equations (5.17) imply that (5.16) is in turn equivalent to

$$\frac{1}{\beta(\tilde{A})} - \frac{1}{\alpha(\Psi)} < \frac{1}{n}. \tag{5.21}$$

A change of variable yields

$$\Psi(\lambda s) = n' s^{n'} \int_0^s \frac{\tilde{A}(\lambda t)}{t^{1+n'}} dt \tag{5.22}$$

for $\lambda > 0$. Hence, by l'Hospital rule,

$$\limsup_{s\to\infty} \frac{\Psi(\lambda s)}{\Psi(s)} \le \limsup_{s\to\infty} \frac{\tilde{A}(\lambda s)}{\tilde{A}(s)}. \tag{5.23}$$

From (5.23) we deduce

$$\alpha(\Psi) \le \alpha(\tilde{A}). \tag{5.24}$$

Combining (5.17) and (5.24) we get

$$\frac{1}{\beta(\tilde{A})} - \frac{1}{\alpha(\Psi)} \le \alpha(\tilde{A}^{-1}) - \beta(\tilde{A}^{-1}). \tag{5.25}$$

Hence, via (5.1), (5.17) and (5.13), inequality (5.21) follows.

The proof of the same statement for α_∞ and β_∞ is completely analogous, since (5.18), (5.19) and (5.23) also hold with "$\limsup_{s\to\infty}$" replaced by "$\sup_{s>0}$". □

LEMMA 6. *Let $n \geq 2$. Let A be an N-function satisfying* (4.11). *Let $\bar{A}$ be the function defined by* (1.12). *If* (5.13) *is fulfilled, then for every $a \in (0, \infty)$ a constant C exists such that*

$$\left\| \int_s^a f(r)dr \right\|_{L^{\bar{A}}(0,a)} \leq C \| s^{1/n'} f(s) \|_{L^A(0,a)} \tag{5.26}$$

for every measurable function $f : (0, a) \to [0, \infty)$.

Inequality (5.26) *holds with $a = \infty$ provided that* (5.13) *is satisfied with α and β replaced by α_∞ and β_∞, respectively.*

PROOF. To avoid misunderstandings in the notations, the function $\bar{A}$ will be denoted by B throughout the proof.

A duality argument easily shows, via (2.2) and (2.3), that (5.26) holds if (and only if) a constant C_1 exists such that

$$\left\| s^{-1/n'} \int_0^s g(r)dr \right\|_{L^{\tilde{A}}(0,a)} \leq C_1 \| g(s) \|_{L^{\tilde{B}}(0,a)} \tag{5.27}$$

for all $g \in L^{\tilde{B}}(0, a)$. (5.27) is a weighted Hardy-type inequality in Orlicz spaces. Owing to Theorem 2 of [Be], such inequality will be proved if we show that

(5.28) a number $p > 1$ exists such that $L^{\tilde{A}}(0, a)$ satisfies an upper p-estimate and $L^{\tilde{B}}(0, a)$ satisfies a lower p-estimate

and

$$\sup_{0<s<a} \| 1 \|_{L^B(0,s)} \| r^{-1/n'} \|_{L^{\tilde{A}}(s,a)} < \infty . \tag{5.29}$$

We refer to [LT] for the definition of upper and lower estimates in Banach lattices. Let us only recall that, if a Banach lattice satisfies an upper [respectively lower] p-estimate, then it satisfies an upper [lower] q-estimate for every $q \in [1, p]$ [$q \in [p, \infty]$].

Consider (5.28). By Lemma 5, assumption (5.13) ensures that $\alpha(\tilde{A}^{-1}) < 1 - \alpha(B^{-1})$. Hence, owing to (5.1),

$$\alpha(\tilde{A}^{-1}) < \beta(\tilde{B}^{-1}) .$$

In particular, $\alpha(\tilde{A}^{-1}) < 1$ and $\beta(B^{-1}) > 0$. Proposition 2.b.5 of [LT] tells us that

$$\frac{1}{\alpha(\tilde{A}^{-1})} = \sup\{p : L^{\tilde{A}}(0, a) \text{ satisfies an upper } p\text{-estimate}\}$$

and

$$\frac{1}{\beta(\tilde{B}^{-1})} = \inf\{p : L^{\tilde{B}}(0,a) \text{ satisfies a lower } p\text{-estimate}\}.$$

Thus, there exist numbers p_1 and p_2 such that $1 < p_2 < p_1$, $L^{\tilde{A}}(0,a)$ satisfies an upper p_1-estimate and $L^{\tilde{B}}(0,a)$ satisfies a lower p_2-estimate. Therefore, (5.28) is proved.

As far as (5.29) is concerned, we have

$$1/B^{-1}(1/s) \leq 1/A_n^{-1}(1/s) \tag{5.30}$$

by the former of inequalities (4.15). On the other hand,

$$\|1\|_{L^B(0,s)} = 1/B^{-1}(1/s). \tag{5.31}$$

Thus, by (5.30), (5.31) and (4.13),

$$\sup_{o<s<a} \|1\|_{L^B(0,s)} \|r^{-1/n'}\|_{L^{\tilde{A}}(s,a)} \leq \sup_{0<s<\infty} \|1\|_{L^B(0,s)} \|r^{-1/n'}\|_{L^{\tilde{A}}(s,\infty)} < \infty. \tag{5.32}$$

Hence, (5.29) follows. In case $a = \infty$ the proof is analogous. □

LEMMA 7. *Let $n \geq 2$. Let A be an N-function. If $\alpha_\infty(A) < n$, then a constant K exists such that*

$$\int_0^s \frac{\tilde{A}(t)}{t^{1+n'}} dt \leq \frac{\tilde{A}(s)}{s^{n'}} \tag{5.33}$$

for $s > 0$. If $\alpha(A) < n$ and $\int_0 \frac{\tilde{A}(t)}{t^{1+n'}} dt < \infty$, then a positive number s_0 exists such that (5.33) *is satisfied for $s > s_0$.*

PROOF. Assume that $\alpha_\infty(A) < n$ (the proof of the latter assertion is analogous). Thanks to equations (5.1) and (5.17), our assumption implies that $\beta_\infty(\tilde{A}) > n'$. Let δ be any positive number smaller than $\beta_\infty(\tilde{A}) - n'$. Arguing as in Lemma 2 of [Ta3] we infer that a constant C exists such that

$$\tilde{A}(s)/\tilde{A}(r) \geq C(s/r)^{n'+\delta} \tag{5.34}$$

for $s \geq r > 0$. By (5.34),

$$\int_0^s \frac{\tilde{A}(t)}{\tilde{A}(s)} \frac{dt}{t^{1+n'}} \leq \frac{1}{C} s^{-n'-\delta} \int_0^s t^{-1+\delta} dt = \frac{1}{C\delta} s^{-n'}, \tag{5.35}$$

whence (5.33) follows with $K = \frac{1}{C\delta}$. □

PROOF OF THEOREM 5. Let G be an open subset of $\mathbb{R}^n$ and set $V = m(G)$. Consider assertion i). Assume that $V < \infty$. Therefore, by Remark 1, we way suppose that $\int_0 \frac{\tilde{A}(t)}{t^{1+n'}}dt < \infty$. Our proof of inequality (5.3) proceedes along different lines according to whether the assumption $\alpha(A) < n$ or $\frac{1}{\beta(A)} - \frac{1}{\alpha(A)} < \frac{1}{n}$ is in force.

First, let us take into account the case where $\alpha(A) < n$. Under this assumption, a proof of inequality (5.3) parallels that of the embedding (1.6) in [DT], which in turn is based on the embedding $W_0^{1,1}(G) \to L^{n'}(G)$. An inspection of that proof shows that, thanks to Lemma 4, the whole argument applies to establish (5.3) provided that a positive constant C is known to exist such that

$$\frac{d}{dr}(A_n^{1/n'}(r)) \le C\,\tilde{A}^{-1}(A_n(r)) \quad \text{for} \quad r > 0. \tag{5.36}$$

Straightforward computations show that (5.36) is equivalent to

$$s + \frac{s^{n'+1}}{\tilde{A}(s)} \int_0^s \frac{\tilde{A}(t)}{t^{1+n'}}dt \le C\,\tilde{A}^{-1}\left(s^{n'} \int_0^s \frac{\tilde{A}(t)}{t^{1+n'}}dt\right) \quad \text{for} \quad s > 0, \tag{5.37}$$

where s and r are related by $r^{n'} = \Phi(s)$ (recall (1.13)).

By Lemma 7, positive numbers K and s_0 exist such that

$$\int_0^s \frac{\tilde{A}(t)}{t^{1+n'}}dt \le K\,\frac{\tilde{A}(s)}{s^{n'}} \tag{5.38}$$

for $s > s_0 > 0$. Modifying, if necessary, A near 0, we may assume that (5.38) is fulfilled for all $s \in (0, \infty)$. Thus

$$s + \frac{s^{n'+1}}{\tilde{A}(s)} \int_0^s \frac{\tilde{A}(t)}{t^{1+n'}}dt \le (1+K)s \quad \text{for} \quad s > 0. \tag{5.39}$$

We claim that

$$\tilde{A}\left(\frac{s}{2}\right) \le s^{n'} \int_0^s \frac{\tilde{A}(t)}{t^{1+n'}}dt \quad \text{for} \quad s \ge 0. \tag{5.40}$$

Actually, since $\tilde{A}$ is non-decreasing,

$$\begin{aligned} \tilde{A}\left(\frac{s}{2}\right) &= n'\frac{s^{n'}}{2^{n'}-1}\tilde{A}\left(\frac{s}{2}\right)\int_{s/2}^{s} \frac{1}{t^{1+n'}}dt \\ &\le n'\frac{s^{n'}}{2^{n'}-1}\int_{s/2}^s \frac{\tilde{A}(t)}{t^{1+n'}}dt \le n'\frac{s^{n'}}{2^{n'}-1}\int_0^s \frac{\tilde{A}(t)}{t^{1+n'}}dt. \end{aligned} \tag{5.41}$$

Inasmuch as $n'/(2^{n'}-1)<1$, (5.41) implies (5.40).

Inequality (5.37), with $C=2(1+n'K)$, is a consequence of (5.39)-(5.40). Hence, (5.3) follows.

Assume now that $\frac{1}{\beta(A)}-\frac{1}{\alpha(A)}<\frac{1}{n}$. We have

$$u^*(s)=\int_s^V -\frac{du^*}{dr}(r)dr \tag{5.42}$$

for all $u\in W_0^{1,A}(G)$. Thanks to (5.42), (2.6) and Pòlya-Szegö principle (2.22), inequality (5.3) will follow if we prove that a constant C, independent of u, exists such that

$$\left\|\int_s^V -\frac{du^*}{dr}(r)dr\right\|_{L^{\tilde A}(0,V)}\le C\left\|n\,C_n^{1/n}\,s^{1/n'}\left(-\frac{du^*}{ds}(s)\right)\right\|_{L^{A}(0,V)}. \tag{5.43}$$

Inequality (5.43) follows from Lemma 6.

Assertion i) is fully proved in case $V<\infty$. When $V=\infty$ and $\int_0\frac{\tilde A(t)}{t^{1+n'}}dt<\infty$, the proof of i) proceeds exactly through the same steps as for $V<\infty$: one has simply to make use of Lemmas 6-7 with α_∞ and β_∞ instead of α and β.

Take now into account ii) and iii). Let $G\in\mathcal{I}(1/n')$ and $V<\infty$. Suppose that $\alpha(A)<n$. Our assumption on G ensures that a constant C exists such that

$$\|u-u_G\|_{L^{n'}(G)}\le C\|Du\|_{L^1(G)} \tag{5.44}$$

for all $u\in W^{1,1}(G)$ (see e.g. [C2], where the best constant in (5.44) is characterized). By triangle inequality, (5.44) implies that

$$\|u\|_{L^{n'}(G)}\le C\|Du\|_{L^1(G)}+V^{-1/n}\|u\|_{L^1(G)}. \tag{5.45}$$

Starting from (5.44) and (5.45), inequality (5.4) and embedding (5.5) respectively follow via the same arguments as in the proof on Theorem 3.2 of [DT] and part i) above.

Suppose now that $\frac{1}{\beta(A)}-\frac{1}{\alpha(A)}<\frac{1}{n}$. Let $u\in W^{1,A}(G)$. From equations (4.36), (4.37) and (2.6) one gets

$$\begin{aligned}\|u-u^o(V/2)\|_{L^{\tilde A}(G)}\le&\left\|\int_s^{V/2}-\frac{du^0}{dr}(r)dr\right\|_{L^{\tilde A}(0,V/2)}\\&+\left\|\int_s^{V/2}-\frac{du^0}{dr}(V-r)dr\right\|_{L^{\tilde A}(0,V/2)}.\end{aligned} \tag{5.46}$$

On the other hand, Pòlya-Szegö principle in the form of (2.27) and the isoperimetric inequality (2.26) yield

$$\left\| s^{1/n'} \left(-\frac{du^0}{ds}(s) \right) \right\|_{L^A(0,V/2)} + \left\| s^{1/n'} \left(-\frac{du^0}{ds}(V-s) \right) \right\|_{L^A(0,V/2)} \leq 2Q_{1/n'}(G) \|Du\|_{L^A(G)} . \tag{5.47}$$

Via Lemma 6, inequalities (5.46) and (5.47) give

$$\|u - u^0(V/2)\|_{L^{\bar{A}}(G)} \leq \text{Const. } Q_{1/n'}(G) \|Du\|_{L^A(G)} . \tag{5.48}$$

From (4.41) and (4.42) we obtain

$$\|u^0(V/2) - u_G\|_{L^{\bar{A}}(G)} \leq \frac{2Q_{1/n'}(G)}{V^{1/n'} \tilde{A}^{-1}(1/V) \bar{A}^{-1}(1/V)} \|Du\|_{L^A(G)} \tag{5.49}$$

and

$$\|u_G\|_{L^{\bar{A}}(G)} \leq \frac{2}{V \, \tilde{A}^{-1}(1/V) \bar{A}^{-1}(1/V)} \|u\|_{L^A(G)} , \tag{5.50}$$

respectively.

Assertion ii) follows from (5.48) and (5.49). Moreover, (5.48)-(5.50) prove iii) when $G \in \mathcal{I}(1/n')$; the extension to the case where $G \in \mathcal{F}(1/n')$ is straightforward.

Finally, none of (5.5)-(5.5) can be true for any Orlicz space smaller than $L^{\bar{A}}(G)$ by Proposition 4. The proof is complete. □

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