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Outer ideals and division in $A^\infty(\Omega)$

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1. – Introduction

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$, $\mathbb{T} := \partial\mathbb{D}$ its boundary and $H^p(\mathbb{T})$, $p \geq 1$ the classical Hardy spaces of holomorphic functions in $\mathbb{D}$ with boundary values in $L^p(\mathbb{T})$.

The starting point of this work is the following well known theorem of Beurling, [15]:

THEOREM 1.1. *Let E be a closed subspace of the Hardy space $H^2(\mathbb{T})$, invariant by multiplication by z, then there is a inner function $\Theta \in H^\infty(\mathbb{T})$ such that:*

$$E = \Theta \cdot H^2(\mathbb{T}) .$$

As an easy corollary of it we get the following two facts:

- $g \in \mathcal{A}(\mathbb{D}) := \mathcal{O}(\mathbb{D}) \cap \mathcal{C}^\infty(\overline{\mathbb{D}})$, then $g \cdot H^2(\mathbb{T})$ is closed iff g is a finite Blaschke product (i.e. a smooth inner function) times an invertible function of $H^\infty(\mathbb{T})$;
- $g \in \mathcal{A}(\mathbb{D})$, then $g \cdot H^2(\mathbb{T})$ is dense in $H^2(\mathbb{T})$ iff g is an outer function of $H^\infty(\mathbb{T})$.

These notions of inner and outer functions are very important in one variable and our aim is to define analogous notions in several variables via the invariant subspaces by multiplication by polynomials.

In this paper we shall work in the following context:

- Ω will be a bounded pseudo-convex domain in $\mathbb{C}^n$ smoothly bounded;
- $\mathcal{A}(\Omega) := \mathcal{O}(\Omega) \cap \mathcal{C}^\infty(\overline{\Omega})$;
- $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$;
- μ will be a probability measure on $\overline{\Omega}$;
- and finally $H^p(\mu)$ will denote the closure in $L^p(\mu)$ of $\mathcal{A}(\Omega)$.

The main question treated here will now be: what conditions on the vector valued function $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ imply that $g \cdot H^p(\mu)$ is dense in

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$H^p(\mu)$? If this is the case we may say that the ideal generated by the g_i's in $\mathcal{A}(\Omega)$ is an "outer" ideal for $H^p(\mu)$.

This problem seems to be a purely analytic one but in fact we shall see that, contrary to the one variable case, we have to use powerful tools from the theory of ideals which means that geometry enters strongly in the picture. We introduce a new notion of globally coherent sequence in order to give the same treatment for generators holomorphic in a neighbourood of $\overline{\Omega}$ or generators just smooth up to $\partial\Omega$ but not too far from either a regular sequence or a "weakly" distorted one.

In order to present the main result let me modify the notion of global coherence for a sequence $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ introduced in [1]:

DEFINITION 1.2. *Let $\mathcal{A}$ the sheaf associated to $\mathcal{A}(\Omega)$ over $\overline{\Omega}$, let $\mathcal{W}$ the sheaf associated to $\mathcal{C}^\infty(\overline{\Omega})$ and $\mathcal{F}$ one of these sheaves. We shall say that the (global) sequence $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ is globally coherent of length p over $\mathcal{F}$ if, for any $z \in \overline{\Omega}$, $g_z \cdot \mathcal{F}_z$ has a free resolution of length bigger than $p+1$ over $\mathcal{F}$:*

$$\mathcal{F}_z^{k_p} \longrightarrow \dots \longrightarrow \mathcal{F}_z^{k_1} \longrightarrow \mathcal{F}_z^{k} \xrightarrow{g} g_z \cdot \mathcal{F}_z \longrightarrow 0 \tag{1.1}$$

with $\mathcal{A}_z$ arrows.

A peak set P for $\mathcal{A}(\Omega)$ is a set in $\overline{\Omega}$ such that $\exists \varphi \in \mathcal{A}(\Omega)$ with: $\varphi_{|P} \equiv 1$ and $\forall z \in \overline{\Omega} \setminus P |\varphi(z)| < 1$; this implies by the maximum principle that $P \subset \partial\Omega$. As usual we note $V(g)$ the set of common zeroes of the g_i's in $\overline{\Omega}$.

Now we can state the main result:

THEOREM 1.3. *Let $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ be such that:*

- *$g \cdot \mathcal{C}^\infty(\overline{\Omega})$ is closed in $\mathcal{C}^\infty(\overline{\Omega})$;*
- *g is globally coherent of length $3n+1$ over $\mathcal{W}$;*
- *there is a pic set P for $\mathcal{A}(\Omega)$ such that $V(g) \cap \overline{\Omega} \subset P$;*
- *finally $\mu(P) = 0$.*

Then $g \cdot \mathcal{A}(\Omega)$ is dense in $H^p(\mu)$, $1 \leq p < \infty$.

One of the main tools we use here is the following theorem of Malgrange, [11]:

THEOREM 1.4. *For any $z \in \mathbb{C}^n$, $\mathcal{E}_z$ is a faithfully flat ring extension of $\mathcal{O}_z$.*

Here $\mathcal{E}_z$ is the ring of germs of $\mathcal{C}^\infty$ functions near z and $\mathcal{O}_z$ is the ring of germs of holomorphic functions near z.

This work is organized the following way: In Section 2 we study this notion of global coherence of a sequence $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ introduced in [1] in order to have a kind of "minimal algebraic package" to work in several variables; we also give the main examples of sequences. In Section 3 we study the hypothesis of closedness of $g \cdot \mathcal{C}^\infty(\overline{\Omega})$ in $\mathcal{C}^\infty(\overline{\Omega})$ and see its relations with the geometry of $\partial\Omega$ and $V(g)$. Finally in the last section we prove the main result, state corollaries for the case of strictly pseudo-convex domains and give a simple example which shows that even in the case of the unit ball of $\mathbb{C}^n$ the situation can be not too easy.

Strongly related to this study is the nice paper of M. Putinar [14]. In order to give his main results, let me recall the definition of a A-functional derived from the notion of A-measure by G. Henkin: $\ell \in \mathcal{O}(\overline{\Omega})'$ is a A-functional if, for every sequence $f_n \in \mathcal{O}(\overline{\Omega})$ which is uniformly bounded on $\overline{\Omega}$ and satisfies $\forall z \in \Omega$, $\lim_{n\longrightarrow\infty} f_n(z) = 0$ we have $\lim_{n\longrightarrow\infty} \ell(f_n) = 0$. Now he proved.

THEOREM 1.5. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudo-convex domain with smooth real analytic boundary and let ℓ be a continuous A-functional on $A^\infty(\Omega)$. Let $I \subset \mathcal{O}(\overline{\Omega})$ be an ideal whose zeroes satisfies $V(I) \cap \overline{\Omega} \subset E$, where E is a locally peak set for the algebra $A^\infty(\Omega)$. If $\ell_{|I} = 0$, then $\ell = 0$.*

As an application of this theorem he got:

COROLLARY 1.6. *Let $\Omega \subset \mathbb{C}^n$ be a bounded strictly pseudo-convex domain with smooth real analytic boundary, let $I \subset \mathcal{O}(\overline{\Omega})$ be an ideal whose zeroes satisfies $V(I) \cap \overline{\Omega} \subset E$, where E is a locally peak set for the algebra $A^\infty(\Omega)$, and μ be a positive measure supported by $\overline{\Omega}$ such that $\mu(N) = 0$ for every smooth $(n-1)$ dimensional submanifold N of $\partial\Omega$. Then the ideal I is dense in $H^p(\mu)$, $\forall p \in [1, \infty[$.*

Hence, in particular, strong hypothesis as the real analyticity of the boundary and the fact that it has to be strictly pseudo-convex are removed here.

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2. – Global coherence

Let as usual Ω be a bounded p.c. domain in $\mathbb{C}^n$ with $\mathcal{C}^\infty$ smooth boundary and define the following sheaves: $\mathcal{W}$ is the sheaf of germs of $\mathcal{C}^\infty$ functions on $\overline{\Omega}$, $\mathcal{A}$ is the sheaf of germs of $\mathcal{C}^\infty$ functions on $\overline{\Omega}$ holomorphic in Ω, and for $z \in \overline{\Omega}$, $\mathcal{F}_z$ is the fiber of the sheaf $\mathcal{F}$ over z but usually I shall forget the z index when no confusion may happen. Finally $\mathcal{W}(\overline{\Omega})$ and $\mathcal{A}(\Omega)$ are the spaces of corresponding global sections on $\overline{\Omega}$.

In [1] I proved:

THEOREM 2.1. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ with smooth $\mathcal{C}^\infty$ boundary then:*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{W} \xrightarrow{\bar\partial} \mathcal{W}_{(0,1)} \xrightarrow{\bar\partial} \dots \xrightarrow{\bar\partial} \mathcal{W}_{(0,n)} \xrightarrow{\bar\partial} 0 \tag{2.2}$$

is exact at any point of $\overline{\Omega}$.

This was done using Kohn's $\mathcal{C}^\infty$ estimates for solutions of the $\bar\partial$ equation, [10], and the existence of arbitrary small admissible neighbourhoods.

I introduced too the notion of global coherence of a sequence g over $\mathcal{A}$, but the useful notion for the purpose at hand is a slightly different one: the global coherence of a sequence $g = g_1, \dots, g_k$ over a sheaf $\mathcal{F}$:

DEFINITION 2.2. *The sequence g is globally coherent* (G.C.) *over $\mathcal{F}$ of length p if: $g \cdot \mathcal{F}$ has a free resolution over $\mathcal{F}$ of length at least $p+1$ at any point $z \in \overline{\Omega}$:*

$$\mathcal{F}_z^{k_p} \longrightarrow \dots \longrightarrow \mathcal{F}_z^{k_1} \longrightarrow \mathcal{F}_z^{k} \xrightarrow{g} g_z \cdot \mathcal{F}_z \longrightarrow 0 \tag{2.3}$$

with holomorphic arrows.

Of course if $z \notin V(g) \cap \overline{\Omega}$ one of the g_i's is a unit, we get:

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{F}^k \longrightarrow g \cdot \mathcal{F}$$

with R the module of relations of g:

$$\mathcal{R} := \left\{ f = (f_1, \dots, f_k) / \sum_i f_i g_i = 0 \right\};$$

hence if $g_1(z) \neq 0$ for instance then

$$f \in \mathcal{R} \Longleftrightarrow f_1 = \frac{-1}{g_1} \sum_{i=2}^{n} f_i g_i$$

i.e.:

$$\mathcal{R} \simeq \mathcal{F}^{k-1} \quad \text{and} \quad 0 \longrightarrow \mathcal{F}^{k-1} \longrightarrow \mathcal{F}^k \longrightarrow g \cdot \mathcal{F} \longrightarrow 0$$

is exact, hence the second condition is always valid; the same is true if $z \in \Omega$ for the sheaves we shall deal with, because we have $\mathcal{F}_z = \mathcal{E}_z$ or $\mathcal{F}_z = \mathcal{O}_z$ and: any ideal of $\mathcal{O}_z$ admits a resolution of arbitrary length:

$$\mathcal{O}_z^{k_p} \longrightarrow \mathcal{O}_z^{k_{p-1}} \longrightarrow \dots \longrightarrow \mathcal{O}_z^{k} \xrightarrow{g} \cdot \mathcal{O}_z \longrightarrow 0 \tag{2.4}$$

and then Malgrange's division theorem says that $\mathcal{E}_z$ is faithfully flat on $\mathcal{O}_z$ hence by tensoring the previous sequence we get again the second condition.

Hence this condition is actually a condition for the points $z \in V(g) \cap \partial\Omega$.

The fact that G.C. over $\mathcal{W}$ is stronger than over $\mathcal{A}$ is reflected by the following:

THEOREM 2.3. *If $g \in \mathcal{A}(\Omega)^k$ is* G.C. *over $\mathcal{W}$ of length $n+p$ then g is* G.C. *of length p over $\mathcal{A}$.*

PROOF. First we get that if g is G.C. over $\mathcal{W}$ then obviously g is G.C. over the sheaf of $(0,q)$ forms $\mathcal{W}_{(0,q)}$ with the same $n+p+1$ arrows at each point $z \in \overline{\Omega}$:

$$\mathcal{W}^{k_p}_{(0,q)} \longrightarrow \ldots \longrightarrow \mathcal{W}^{k_1}_{(0,q)} \xrightarrow{g} g \cdot \mathcal{W}_{(0,q)} \longrightarrow 0\,. \tag{2.5}$$

Now the classical diagram chasing gives the answer; let me just recall the method in expliciting the first steps:

If $f \in \mathcal{A}^k$ is such that $g \cdot f = 0$ then, because $\mathcal{A} \subset \mathcal{W}$ we get that:

$$\exists f_1 \in \mathcal{W}^{k_1} \text{ s.t. } r_1 \cdot f_1 = f\,.$$

If f_1 is in $\mathcal{A}^{k_1}$ we are done; if not we notice that $0 = \bar\partial f = r_1 \cdot \bar\partial f_1 \Rightarrow \bar\partial f_1$ is in the kernel of r_1 hence in the image of r_2:

$$\exists f_2 \in \mathcal{W}^{k_2}_{(0,1)} \text{ s.t. } r_2 \cdot f_2 = \bar\partial f_1\,.$$

If $\bar\partial f_2 = 0$ we can solve locally the $\bar\partial$-equation by the exactness of the sequence 2.5:

$$\exists s_2 \in W^{k_2} \text{ s.t. } \bar\partial s_2 = f_2\,.$$

Hence $\bar\partial(r_2 \cdot s_2) = \bar\partial f_1$ and $s_1 := f_1 - r_2 \cdot s_2$ is in $\mathcal{A}^{k_1}$ and $r_1 \cdot s_1 = f$.

If $\bar\partial f_2 \neq 0$ we correct it itself by the same procedure and because any $(0,n)$ form is closed we are sure to end the process after at most n steps. Hence to get the exactness at the first stage for $\mathcal{A}$:

$$\mathcal{A}^{k_1} \longrightarrow \mathcal{A}^k \longrightarrow \mathcal{A}$$

we need g to be G.C. of length n; to get the exactness at the second stage will involve $n+1$ steps etc... □

COROLLARY 2.4. *If g is* G.C. *over $\mathcal{W}$ of length n then we get:*

$$g \cdot \mathcal{A} = g \cdot \mathcal{W} \cap \mathcal{A}$$

PROOF. Take $f \in \mathcal{A} \cap g \cdot \mathcal{W}$ then we have $\exists f_1 \in \mathcal{W}^k / f = g \cdot f_1$ and repeat the previous proof to get a $s_1 \in \mathcal{W}^{k_1}$ such that $s := f_1 - r_1 \cdot s_1$ is in $\mathcal{A}^k$ and $g \cdot s = f$. □

We shall need a result more precise that the one in [1]; with $\mathcal{R}$ the sheaf of relations of g we have:

THEOREM 2.5. *If g is* G.C. *over $\mathcal{A}$ of length greater than $2n$ then $H^p(\overline{\Omega}, g \cdot \mathcal{A}) = 0$, $\forall p \geq 1$. If g is* G.C. *over $\mathcal{A}$ of length greater than $2n+1$ then $H^1(\overline{\Omega}, \mathcal{R}) = 0$.*

PROOF. From the exactness of the Dolbeault's Complex 2.5, we get that:

$$H^p(\overline{\Omega}, \mathcal{A}) \simeq \mathcal{H}^p(\overline{\Omega}, \mathcal{W}.) = 0, p \geq 1$$

because $\mathcal{W}.$ is a thin resolution of $\mathcal{A}$ over $\overline{\Omega}$; hence we have for any l:

$$(*) \qquad H^p(\overline{\Omega}, \mathcal{A}^l) = 0, \quad \forall\, p \geq 1$$

From the exact sequence of sheaves:

$$0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{A}^k \longrightarrow g \cdot \mathcal{A} \longrightarrow 0$$

we get the long exact sequence of cohomology:

$$\begin{aligned} \ldots \longrightarrow H^1(\overline{\Omega}, \mathcal{R}) \longrightarrow H^1(\overline{\Omega}, \mathcal{A}^k) \longrightarrow H^1(\overline{\Omega}, g \cdot \mathcal{A}) \\ \longrightarrow H^2(\overline{\Omega}, \mathcal{R}) \longrightarrow H^2(\overline{\Omega}, \mathcal{A}^k) \longrightarrow \ldots \end{aligned}$$

but $(*)$ implies that

$$0 \longrightarrow H^1(\overline{\Omega}, g \cdot \mathcal{A}) \longrightarrow H^2(\overline{\Omega}, \mathcal{R}) \longrightarrow 0$$

hence $H^1(\overline{\Omega}, g \cdot \mathcal{A}) \simeq H^2(\overline{H}, \mathcal{R})$. Still cutting the resolution of $g \cdot \mathcal{A}$ in short exact sequences, we get again as usual:

$$(**) \qquad 0 \longrightarrow \mathcal{R}_1 \longrightarrow \mathcal{A}^{k-1} \longrightarrow \mathcal{R} \longrightarrow 0$$

which leads to:

$$\begin{aligned} \longrightarrow H^2(\overline{\Omega}, \mathcal{R}_1) \longrightarrow H^2(\overline{\Omega}, \mathcal{A}^{k_1}) \longrightarrow H^2(\overline{\Omega}, \mathcal{R}) \\ \longrightarrow H^3(\overline{\Omega}, \mathcal{R}_1) \longrightarrow H^3(\overline{\Omega}, \mathcal{A}^{k_1}) \longrightarrow . \end{aligned}$$

hence $H^2(\overline{\Omega}, \mathcal{R}) \simeq H^3(\overline{\Omega}, \mathcal{R}_1)$ hence: $H^1(\overline{\Omega}, g \cdot \mathcal{A}) \simeq H^2(\overline{\Omega}, \mathcal{R}) \simeq H^3(\overline{\Omega}, \mathcal{R}_1)$ $\simeq \ldots \simeq H^{2n+1}(\overline{\Omega}, \mathcal{R}_{2n-1}) \simeq 0$.

The last equivalence holds because the homological dimension of $\overline{\Omega}$ is $2n$.

We do exactly the same to get $H^2(\overline{\Omega}, g \cdot A) = 0$ but we need only g to be G.C. of length $2n - 1$, etc... to have $H^p(\overline{\Omega}, g \cdot \mathcal{A}) = 0$.

To get $H^1(\overline{\Omega}, \mathcal{R}) = 0$ we start again with $(**)$ which leads to:

$$\begin{aligned} \longrightarrow H^1(\overline{\Omega}, \mathcal{R}_1) \longrightarrow H^1(\overline{\Omega}, \mathcal{A}^{k_1}) \longrightarrow H^1(\overline{\Omega}, \mathcal{R}) \\ \longrightarrow H^2(\overline{\Omega}, \mathcal{R}_1) \longrightarrow H^1(\overline{\Omega}, \mathcal{A}^{k_1}) \longrightarrow \end{aligned}$$

hence $H^1(\overline{\Omega}, \mathcal{R}) \simeq H^2(\overline{\Omega}, \mathcal{R}_1)$. Playing as before we get:

$$H^1(\overline{\Omega}, \mathcal{R}) \simeq H^2(\overline{\Omega}, \mathcal{R}_1) \simeq H^3(\overline{\Omega}, \mathcal{R}_2) \simeq \ldots \simeq H^{2n+1}(\overline{\Omega}, \mathcal{R}_{2n}) \simeq 0$$

and we need to have a resolution longer by one, i.e. g has to be G.C. of length $2n + 1$ and the theorem is proved. □

We are now in a position to globalize the corollary:

THEOREM 2.6. *Let g be* G.C. *over $\mathcal{W}$ of length $3n+1$. Then: $g \cdot \mathcal{A}(\Omega) = \mathcal{A}(\Omega) \cap g \cdot \mathcal{W}(\overline{\Omega})$.*

PROOF. The short exact sequence: $0 \longrightarrow \mathcal{R} \longrightarrow \mathcal{A}^k \longrightarrow g \cdot \mathcal{A} \longrightarrow 0$ leads to:

$$0 \longrightarrow \Gamma(\overline{\Omega}, \mathcal{R}) \longrightarrow \Gamma(\overline{\Omega}, \mathcal{A}^k) \longrightarrow \Gamma(\overline{\Omega}, g \cdot \mathcal{A}) \longrightarrow H^1(\overline{\Omega}, \mathcal{R}) \longrightarrow \dots$$

but $H^1(\overline{\Omega}, \mathcal{R}) = 0$ because g is G.C. over $\mathcal{W}$ of length $3n+1$ hence by a previous theorem g is G.C. over $\mathcal{A}$ of length $2n+1$, hence $H^1(\overline{\Omega}, \mathcal{R}) = 0$.

Then we have:

$$(***) \qquad 0 \longrightarrow \Gamma(\overline{\Omega}, \mathcal{R}) \longrightarrow \Gamma(\overline{\Omega}, \mathcal{A}^k) \longrightarrow \Gamma(\overline{\Omega}, g \cdot \mathcal{A}) \longrightarrow 0\,.$$

Now take $f \in \mathcal{A}(\Omega) \cap g \cdot \mathcal{W}(\overline{\Omega})$ then we have $\forall z \in \overline{\Omega}$, $f_z \in g_z \cdot \mathcal{A}_z$ by the corollary hence f can be seen as in $\Gamma(\overline{\Omega}, g \cdot \mathcal{A})$ and $(***)$ then gives that f is in the image of $\Gamma(\overline{\Omega}, \mathcal{A}^k)$ i.e. there are functions $f_i \in \mathcal{A}(\Omega)$, $i = 1, \dots, k$ with: $f = \sum_{i=1}^{k} g_i f_i$. □

2.1. – Example of globally coherent sequences

In this subsection we shall give three examples of G.C. sequences.

THEOREM 2.7. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ with smooth C^{∞} boundary; let $g = (g_1, \dots, g_k) \in \mathcal{O}(\overline{\Omega})^k$ then the sequence g is* G.C. *on $\mathcal{W}$ of arbitrary length.*

PROOF. The functions g_i being holomorphic in a neighbourhood $\mathcal{V}$ of $\overline{\Omega}$ we can take the envelope of holomorphy $\overline{\mathcal{V}}$ of $\mathcal{V}$ which exists because $\mathcal{V}$ is a riemannian manifold (!) and then its envelope of holomorphy exists and is Stein by Oka's theorem, [3].

Then $\overline{\Omega}$ can be seen as a compact set in the Stein space $\overline{\mathcal{V}}$ (this remark already appeared in [7]); hence we have a resolution of $g \cdot \mathcal{O}$ of any length on $\overline{\Omega}$:

$$(2.6) \qquad \mathcal{O}^{k_p} \longrightarrow \mathcal{O}^{k_{p-1}} \longrightarrow \dots \longrightarrow \mathcal{O}^k \xrightarrow{g} g \cdot \mathcal{O} \longrightarrow 0\,.$$

Now, because Ω is $\bar{\partial}$-exact at each point of $\partial\Omega$ we may apply Theorem 2.9 in [12]: $\mathcal{W}_z$ is a faithfully flat ring extension of $\mathcal{O}_z$ for any z in $\overline{\Omega}$ hence, tensoring the sequence 2.6 with $\mathcal{W}$, we get:

$$(2.7) \qquad \mathcal{W}^{k_p} \longrightarrow \mathcal{W}^{k_{p-1}} \longrightarrow \dots \longrightarrow \mathcal{W}^k \xrightarrow{g} g \cdot \mathcal{W} \longrightarrow 0$$

which means that g is G.C. of length p over $\mathcal{W}$ and the theorem. □

Of course it means that Nagel's theorem saying that $\mathcal{A}(\Omega) \cap g \cdot \mathcal{W}(\overline{\Omega}) = g \cdot \mathcal{A}(\Omega)$ when $g \in \mathcal{O}(\overline{\Omega})^k$, [12], generalizes to G.C. sequences g.

For stating the other example we need a definition:

DEFINITION 2.8. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ with smooth C^∞ boundary; let $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ we say that g is a strongly regular sequence if: $\forall z \in V(g) \cap \partial\Omega, \exists \mathcal{V} \ni z, \exists G_i \in C^\infty(\mathcal{V})$ with $\mathcal{V}$ a neighbourhood of z in $\mathbb{C}^n$ such that:*

- $G_{i|\mathcal{V}} = g_i, i = 1, \dots, k$
- *$\exists \Phi$ a C^∞ diffeomorphism of $\mathcal{V}$ to an open set $\mathcal{U}$ in $\mathbb{C}^n$ such that: $G_i \circ \Phi^{-1}, i = 1, \dots, k$ make a holomorphic regular sequence in $\mathcal{U}$.*

Of course using Withney's extension theorem the first requirement can always be fulfilled. It can be easily seen that this implies: g is $\mathcal{A}$ regular in the usual sense. In order to show that this definition in not void let us see the:

PROPOSITION 2.9. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ with smooth C^∞ boundary; let $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ be such that on $V(g) \cap \partial\Omega, \partial g_1 \wedge \dots \wedge \partial g_k \neq 0$ then g is strongly regular.*

PROOF. Take $z \in \partial\Omega \cap V(g)$ and extend the g_i's as G_i's by Withney's theorem, then we still have in a neighbourhood $\mathcal{V}$ of z that $\partial G_1 \wedge \dots \wedge \partial G_k \neq 0$, hence a smooth change of variables leads to a diffeomorphism Φ from $\mathcal{V}$ to $\mathcal{U}$ in $\mathbb{C}^n$ such that: $G_i \circ \Phi^{-1} = z_i, i = 1, \dots, k$ on $\mathcal{U}$, and, of course the $z_i, i = 1, \dots, k$ is a holomorphic regular sequence in $\mathcal{U}$. □

Moreover in this case one can easily that $\bar\partial\Phi$ is flat on $\mathcal{V} \cap \overline{\Omega}$.

This will allow us to consider the case of smooth manifold near $\partial\Omega$ not extending holomorphically outside $\partial\Omega$ as in [1].

Now let us give the theorem:

THEOREM 2.10. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ with smooth C^∞ boundary; let $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ be a strongly regular sequence then $I := I(g)$ is* G.C. *on $\mathcal{W}$ of arbitrarily length. In fact we have the resolution of $g \cdot \mathcal{W}$ by the Koszul's complex:*

$$\begin{aligned} 0 \longrightarrow \Lambda^k(\mathcal{W}^k) \longrightarrow \Lambda^{k-1}(\mathcal{W}^k) \longrightarrow \dots \\ \longrightarrow \Lambda^1(\mathcal{W}^k)(\simeq \mathcal{W}^k) \xrightarrow{g} g \cdot \mathcal{W} \longrightarrow 0 \end{aligned} \tag{2.8}$$

where $\Lambda^j(\mathbb{C}^k)$ is the exterior algebra over k generators.

PROOF. Let $z \in \partial\Omega \cap V(g)$ and $\mathcal{V}$ the neighbourhood of z such that the extension G_i and the diffeomorphism Φ are defined and suppose that $\Phi(z) = 0 \in \mathcal{U}$; let $\varphi_i := G_i \circ \Phi^{-1}$ be the holomorphic regular sequence in $\mathcal{U}$, then we know the Koszul's complex is a resolution of the ideal $\varphi \cdot \mathcal{O}$ generated by φ near 0, [16], i.e. the following sequence is exact:

$$0 \longrightarrow \Lambda^k(\mathcal{O}^k) \xrightarrow{\gamma k} \Lambda^{k-1}(\vee^k) \xrightarrow{\gamma k-1} \dots \longrightarrow \Lambda^1(\mathcal{O}^k)(\simeq \mathcal{O}^k) \xrightarrow{\varphi} \varphi \cdot \mathcal{O} \longrightarrow 0 \tag{2.9}$$

with the arrows defined recursively as: let $e_i, i = 1, \dots, k$ be a basis for $\Lambda^1(\mathcal{O}^k)$ then Λ^j is generated by the $e_\alpha := e_{i_1} \wedge \dots \wedge e_{i_j}$, $1 \le i_1 < \dots < i_j \le k$

if $\alpha = (i_1, \dots, i_j)$ and:

$$\gamma_0(e_i) = \varphi(e_i) = \varphi_i$$
$$\gamma_j(e_\alpha \wedge e_i) := \varphi_i e_\alpha - \gamma_{j-1}(e_\alpha) \wedge e_i, \quad \forall\, e_\alpha, |\alpha| = j\,.$$

Tensoring with $\mathcal{E}$ (germs of C^∞ functions near 0) which is flat over $\mathcal{O}_0$ by Malgrange's theorem, we get that:

$$(2.10)\quad 0 \longrightarrow \Lambda^k(\mathcal{E}^k) \xrightarrow{\gamma_k} \Lambda^{k-1}(\mathcal{E}^k) \xrightarrow{\gamma_{k-1}} \dots \longrightarrow \Lambda^1(\mathcal{E}^k)(\simeq \mathcal{E}^k) \xrightarrow{\varphi} \varphi \cdot \mathcal{E} \longrightarrow 0$$

is also exact.

Composing with Φ, we can read this on $\mathcal{V}$ near z and get that the following sequence is exact:

$$(2.11)\ 0 \longrightarrow \Lambda^k(\mathcal{E}^k) \xrightarrow{\gamma_k} \Lambda^{k-1}(\mathcal{E}^k) \xrightarrow{\gamma_{k-1}} 3 \dots \longrightarrow \Lambda^1(\mathcal{E}^k)(\simeq \mathcal{E}^k) \xrightarrow{G} G \cdot \mathcal{E} \longrightarrow 0$$

with γ_j defined as before but with the G_i's in place of the φ_i's.

The idea now is to restrict everything to $\overline{\Omega}$ to get a resolution of $g \cdot \mathcal{W}$ and we want to show that the following sequence is exact:

$$(2.12)\quad 0 \longrightarrow \Lambda^k(\mathcal{W}^k) \xrightarrow{\gamma_k} \Lambda^{k-1}(\mathcal{W}^k) \xrightarrow{\gamma_{k-1}} \dots \longrightarrow \Lambda^1(\mathcal{W}^k)(\simeq \mathcal{W}^k) \xrightarrow{g} g \cdot \mathcal{W} \longrightarrow 0$$

all the arrows restricted to $\overline{\Omega}$.

Let $f \in \Lambda^j(\mathcal{W}^k)$ be such that $\gamma_{j-1}(f) = 0$ we shall extend f to $\tilde{f} \in \Lambda^j(\mathcal{E}^k)$ with the same property $\gamma_{j-1}(f) = 0$ using an analogous idea of [12]: we keep the same extension G_i of g_i as before and also the same γ_j, and if:

$$f = \sum_\alpha f_\alpha e_\alpha \in \Lambda^j(\mathcal{W}^k),\ |\alpha| = j \quad \text{let} \quad F := \sum_\alpha F_\alpha e_\alpha \in \Lambda^j(\mathcal{E}^k),\ |\alpha| = j$$

with F_α any smooth extension of f_α in $\mathcal{V}$; then composing with Φ^{-1} we are back to $\mathcal{U}$ near 0 and, without changing notations for F:

$$H := \gamma_{j-1}(F) \quad \text{is flat on} \quad \mathcal{U} \cap \Phi(\overline{\Omega})$$

hence we have $H = LM$ with L and M also flat on $\mathcal{U} \cap \Phi(\overline{\Omega})$ and $L(\zeta) > 0$ in $\mathcal{U} \setminus \Phi(\overline{\Omega})$, using a result of Tougeron [17], this implies that $M = 0$ on $\mathcal{U} \cap \Phi(\overline{\Omega})$ and $M = \frac{1}{L}\gamma_{j-1}(F)$ on $\mathcal{U} \setminus \Phi(\overline{\Omega})$.

Hence $T_\zeta M \in T_\zeta(\operatorname{Im}\gamma_{j-1})$, $\forall \zeta \in \mathcal{U}$ and by the Withney's spectral theorem, $M \in \overline{\operatorname{Im}\gamma_{j-1}}$ but using again Malgrange's theorem, this module is closed in $\mathcal{E}^p$ because the $\varphi_i = G_i \circ \Phi$ are analytic functions in $\mathcal{U}$, hence $M \in \operatorname{Im}\gamma_{j-1}$ meaning that $\exists \lambda \in \Lambda^j(\mathcal{E}^k)$, $M = \gamma_{j-1}(\lambda)$ but then:

$$H := \gamma_{j-1}(F) = LM = L\gamma_{j-1}(\lambda) = \gamma_{j-1}(L\lambda)$$

hence $F - L\lambda \in \Lambda^j(\mathcal{E}^k)$ is such that: $F - L\lambda_{|\Phi(\overline{\Omega})\cap \mathcal{U}} = F_{|\Phi(\overline{\Omega})\cap \mathcal{U}}$ because L is flat on $\mathcal{U} \cap \Phi(\overline{\Omega})$ and $\gamma_{j-1}(F - L\lambda) = 0$.

Composing with Φ we put $\tilde{f} = (F - L\lambda) \circ \Phi$ to get the desired extension. We now can apply the exactness of sequence 2.11 to conclude to the existence of $\psi \in \Lambda^{j+1}(\mathcal{E}^k) s.t. \tilde{f} = \gamma_j(\psi)$; the restriction of ψ to $\mathcal{V}\cap\overline{\Omega}$ gives the exactness of sequence 2.12 at level j.

The point here is that the arrows γ_j being just matrices with entries linear in the g_i's are still in $\mathcal{A}(\Omega)$ even if Φ is just a diffeomorphism.

In fact in that case we have much more than just the G.C. property: not only the sequence g is generated by global functions but also the module of relations $\mathcal{R}$ of g, the module of relations $\mathcal{R}_1$ of $\mathcal{R}$, etc... □

The last example is a modification of the previous one:

DEFINITION 2.11. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ with smooth $\mathcal{C}^\infty$ boundary; let $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ we say that g is a weakly distorted sequence if: $\forall z \in V(g) \cap \partial\Omega, \exists \mathcal{V} \ni z, \exists G_i \in \mathcal{E}(\mathcal{V})$ with $\mathcal{V}$ a neighbourhood of z in $\mathbb{C}^n$ such that:*

- $G_{i|\mathcal{V}} = g_i, i = 1, \dots, k$
- $\exists\Phi$ *a $\mathcal{C}^\infty$ diffeomorphism of $\mathcal{V}$ to an open set $\mathcal{U}$ in $\mathbb{C}^n$ such that: $G_i \circ \Phi^{-1}$, $i = 1, \dots, k$ are holomorphic functions in $\mathcal{U}$;*
- $\bar{\partial}$ *is flat on $\overline{\Omega}$.*

This time we require just that the $G_i \circ \Phi^{-1}$'s are holomorphic but we paid that $\bar{\partial}\Phi$ has to be flat on $\overline{\Omega}$. The same example as for strongly regularity applies here.

THEOREM 2.12. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ with smooth $\mathcal{C}^\infty$ boundary; let $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ be a weakly distorted sequence then $I := I(g)$ is* G.C. *on $\mathcal{W}$ of arbitrarily length.*

PROOF. We start the same way as before and we have: $\varphi_i := G_i \circ \Phi$ holomorphic in $\mathcal{U}$ near 0. Hence we have a resolution of length p of $\varphi \cdot \mathcal{O}$:

$$\mathcal{O}^k_p \longrightarrow \mathcal{O}^{k_{p-1}} \longrightarrow \dots \longrightarrow \mathcal{O}^k \xrightarrow{\varphi} \varphi \cdot \mathcal{O} \longrightarrow 0 \tag{2.13}$$

and as for the first example we tensor it with $\mathcal{W}$ which is flat by Nagel: $\Phi(\overline{\Omega} \cap \mathcal{V})$ is still a piece of p.c. domain with smooth boundary near 0 because $\bar{\partial}\Phi$ is flat on $\overline{\Omega} \cap \mathcal{V}$, and we know the existence of admissible neighbourhoods, [1] hence:

$$\mathcal{W}^{k_p} \longrightarrow \mathcal{W}^{k_{p-1}} \longrightarrow \dots \longrightarrow \mathcal{W}^k \xrightarrow{\varphi} \varphi \cdot \mathcal{W} \longrightarrow 0\,. \tag{2.14}$$

Now we do exactly the same as before to extend the f_α and we end with smooth matrices γ_j which are holomorphic on $\overline{\Omega} \cap \mathcal{V}$ this time because Φ is biholomorphic on $\overline{\Omega} \cap \mathcal{V}$. □

3. – Closed ideals in $\mathcal{C}^\infty(\overline{\Omega})$

The condition: $g \cdot \mathcal{C}^\infty(\overline{\Omega})$ is closed in $\mathcal{C}^\infty(\overline{\Omega})$, appearing in the main theorem, is a priori a functional condition but this condition was already studied and we have geometric hypothesis which warranty that $g \cdot \mathcal{C}^\infty(\overline{\Omega})$ is closed.

before stating them we need to recall a definition by Lojaciewicz, [17]:

DEFINITION 3.1. *Let X and Y be two closed sets in an open set $U \subset \mathbb{R}^n$; X and Y are regularly situated (R.S. for short) in U if for any point $z_0 \in X \cap Y$ there are a neighbourhood V of z_0 and two constants $C, \alpha > 0$, such that: $\forall\, z \in V, d(z, X) + d(z, Y) \geq C d(z, X \cap Y)^\alpha$.*

This study starts in [1] in the case the functions g_i's define a smooth manifold $V(g)$ near $\partial\Omega$, but does not extend holomorphically across the boundary; in that case we have:

THEOREM 3.2. *Let Ω be a* p.c. *bounded domain in $\mathbb{C}^n$ with a smooth boundary; let $g = (g_1, \ldots, g_k) \in \mathcal{A}(\Omega)^k$ be such that $\forall z \in \partial\Omega, \partial g_1(z) \wedge \ldots \wedge \partial g_k(z) \neq 0$; $g \cdot \mathcal{C}^\infty(\overline{\Omega})$ is closed iff any smooth extension $\widetilde{V}(g)$ of $V(g)$ in a neighbourhood of $\overline{\Omega}$ verifies $\widetilde{V}(g)$ and $\overline{\Omega}$ are* R.S.

This theorem was proved in [1] with a stronger assumption than R.S. but exactly the same proof works with R.S. and it is in [4] that the right condition R.S. was introduced in this setting.

In fact I proved in [2] that if Ω is strictly p.c. and if $V(g)$ is a smooth manifold near $\partial\Omega$ then $g \cdot \mathcal{A}(\Omega)$ is closed iff $g \cdot \mathcal{C}^\infty(\overline{\Omega})$ is closed; if not s.p.c. this question is still open.

The case with analytic singularities was also studied and we have, [3]:

THEOREM 3.3. *Let Ω be a* p.c. *bounded domain in $\mathbb{C}^n$ with a smooth boundary; let $g = (g_1, \ldots, g_k) \in \mathcal{O}(\overline{\Omega})^k$; if the singular filtration of $V(g)$ is* R.S. *with $\overline{\Omega}$ then $g \cdot \mathcal{C}^\infty(\overline{\Omega})$ is closed in $\mathcal{C}^\infty(\overline{\Omega})$.*

Let me recall that the singular filtration of an analytic set X is: $X = S_0 \supset S_1 \ldots \supset S_l$ where S_{i+1} is the singular locus of S_i.

In [5], the authors introduce another filtration of $V(g)$ relying on the Newton diagram and they also get that if this filtration is R.S. with $\overline{\Omega}$ then $g \cdot \mathcal{C}^\infty(\overline{\Omega})$ is closed in $\mathcal{C}^\infty(\overline{\Omega})$. Their theorem is still valid in the real analytic case.

Finally in [8], the author gave a third sufficient condition relying this time on a good desingularization of $V(g)$, and this condition seems to be the nearest of the simplest necessary condition we have namely that $V(g)$ and $\overline{\Omega}$ must be R.S.

The next proposition gives sufficient conditions for $g \cdot \mathcal{C}^\infty(\overline{\Omega})$ to be closed when g is either weakly distorted or strongly regular:

PROPOSITION 3.4. *Let Ω be a p.c. bounded domain in $\mathbb{C}^n$ with a smooth boundary; let $g = (g_1, \ldots, g_k) \in \mathcal{A}(\Omega)^k$ be either weakly distorted or strongly regular at $z \in V(g) \cap \partial\Omega$; if the associated G and Φ are such that $V(G \circ \Phi^{-1})$ verifies the assumptions of one of the previous theorems then: $g_z \cdot \mathcal{W}_z$ is closed in $\mathcal{W}_z$.*

PROOF. Using one of the previous theorems, and the notations of the previous section, we get that the ideal generated by $G_i \circ \Phi^{-1}$, $i = 1, \dots, k$ in $\mathcal{C}^\infty(V \cap \Phi(\overline{\Omega}))$ is closed; composing with the diffeomorphism Φ we get that the ideal generated by the G_i's is still closed in $\mathcal{C}^\infty(U \cap \overline{\Omega})$ but in $U \cap \overline{\Omega}, G = g$. $\square$

And now we have the global version we need in the next section:

COROLLARY 3.5. *Let Ω be a p.c. bounded domain in $\mathbb{C}^n$ with a smooth boundary; let $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ be either weakly distorted or strongly regular at any point $z \in V(g) \cap \partial\Omega$; if the associated G_z and Φ_z are such that $V(G_z \circ \Phi_z^{-1})$ verifies the assumptions of one of the previous theorems at any such point z then: $g \cdot \mathcal{C}^\infty(\overline{\Omega})$ is closed in $\mathcal{C}^\infty(\overline{\Omega})$.*

Because Ω is bounded, $\overline{\Omega}$ is compact hence we can cover it by a finite number of open sets U_i such that: $g \cdot \mathcal{C}^\infty(U_i \cap \overline{\Omega})$ is closed in $\mathcal{C}^\infty(U_i \cap \overline{\Omega})$ by the previous proposition.

Now let $\varphi_n \in \mathcal{C}^\infty(\overline{\Omega})^k$ be such that $g \cdot \varphi_n \longrightarrow f$ in $\mathcal{C}^\infty(\overline{\Omega})$ and let χ_i be a partition of 1 subordinated to the covering U_i then we get: in $U_i, \exists f_i \in \mathcal{C}^\infty(\overline{\Omega})^k s.t. f = g \cdot f_i \Rightarrow \chi_i f = g \cdot \chi_i f_i$ hence $f = \sum_i \chi_i f = \sum_i g \cdot \chi_i f_i = g \cdot \sum_i \chi_i f_i = g \cdot F$ with $F = \sum_i \chi_i f_i \in \mathcal{C}^\infty(\overline{\Omega})^k$. $\square$

4. – Density

As before we shall deal with $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$.

We shall still denote by $V(g)$ the set of common zeroes of the g_i's in $\overline{\Omega}$.

If μ is a probability measure on $\overline{\Omega}$, then we note $H^p(\mu)$ the closure of $\mathcal{A}(\Omega)$ in $L^p(\mu)$. Because we shall no longer use cohomology groups, there will be no confusion with Hardy spaces.

The question here is to give conditions on Ω, g and μ to insure that $g \cdot \mathcal{A}(\Omega)$ is dense in $H^p(\mu)$; of course $V(g) \cap \Omega$ must be empty, at least when dealing with Hardy or Bergman spaces, because any functions in $g \cdot H^p$ have to be zero on it.

But if $V(g) \cap \Omega = \emptyset$ then there are holomorphic functions φ_i in Ω such that: $1 = \sum_i g_i \varphi_i$ and even if any function in $\mathcal{O}(\Omega)$ may be approximated on compacta of Ω by functions in $\mathcal{A}(\Omega)$ this does not imply that $1 \in g \cdot H^2(\mu)$.

This section is devoted to sufficent conditions garantying this.

In all this part, we shall say that g is G.C. if g is G.C. over $\mathcal{W}$ of length $3n + 1$.

THEOREM 4.1. *Let Ω be a bounded pseudo-convex domain in $\mathbb{C}^n$ with smooth $\mathcal{C}^\infty$ boundary; let $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$ be such that:*

- *$g \cdot \mathcal{W}(\overline{\Omega})$ is closed in $\mathcal{W}(\overline{\Omega})$*
- *g is G.C.*
- *there exists a pic set for $\mathcal{A}(\Omega)$, P, containing $V(g) \cap \overline{\Omega}$.*

Let finally μ be a probability measure with no mass on P. Then $g \cdot \mathcal{A}(\Omega)$ is dense in $H^p(\mu)$ for any p with $1 \leq p < \infty$.

PROOF. Because $g \cdot \mathcal{W}(\overline{\Omega})$ is closed in $\mathcal{W}(\overline{\Omega})$, the quotient space $\mathcal{W}(\overline{\Omega})/g \cdot \mathcal{W}(\overline{\Omega})$ is Frechet.

Using Theorem 1.6 we have that $g \cdot \mathcal{A}(\Omega) = \mathcal{A}(\Omega) \cap g \cdot \mathcal{W}(\overline{\Omega})$ hence $g \cdot \mathcal{A}(\Omega)$ is closed as an intersection of closed spaces, so: $\mathcal{A}(\Omega)/g \cdot \mathcal{A}(\Omega)$ is again a Frechet space.

We shall now use the same duality argument as in [13]:
consider the following exact sequences:

$$0 \longrightarrow \mathcal{A}(\Omega) \longrightarrow \mathcal{W}(\overline{\Omega})$$

$$0 \longrightarrow g \cdot \mathcal{A}(\Omega) \longrightarrow g \cdot \mathcal{W}(\overline{\Omega})$$

where the injections are just the canonical ones. This implies that we have an application j: $\mathcal{A}(\Omega)/g \cdot \mathcal{A}(\Omega) \longrightarrow \mathcal{W}(\overline{\Omega})/g \cdot \mathcal{W}(\overline{\Omega})$ and this application is also one to one:
take $F \in \mathcal{A}(\Omega)/g \cdot \mathcal{A}(\Omega)$ such that $j(F) = 0$ then $f \in F \implies f \in g \cdot \mathcal{W}(\overline{\Omega})$ but we still have $f \in \mathcal{A}(\Omega)$ hence $f \in \mathcal{A}(\Omega) \cap g \cdot \mathcal{W}(\overline{\Omega}) = g \cdot \mathcal{A}(\Omega)$ hence we get:

$$0 \longrightarrow \mathcal{A}(\Omega)/g \cdot \mathcal{A}(\Omega) \longrightarrow \mathcal{W}(\overline{\Omega})/g \cdot \mathcal{W}(\overline{\Omega}).$$

Hence $\mathcal{A}(\Omega)/g \cdot \mathcal{A}(\Omega)$, can be seen as a sub Frechet space of $\mathcal{W}(\overline{\Omega})/g \cdot \mathcal{W}(\overline{\Omega})$.

Take a function h in $L^q(\mu)$, $1/p + 1/q = 1$, such that:

$$l(f) := \int f h \, d\mu = 0, \quad \forall f \in g \cdot \mathcal{A}(\Omega)$$

then the functional l is defined on $\mathcal{A}(\Omega)/g \cdot \mathcal{A}(\Omega)$ and, by Hahn-Banach for Frechet spaces [Bourbaki, livre V, pp. 119] it extends to $\mathcal{W}(\overline{\Omega})/g \cdot \mathcal{W}(\overline{\Omega})$ as $\tilde{l}$.

But then $\tilde{l}$ can be seen as a distribution supported by $\overline{\Omega} \cap V(g)$:
$\tilde{l}$ can be lifted as a continuous linear form L on $\mathcal{W}(\overline{\Omega})$ null on $g \cdot \mathcal{W}(\overline{\Omega})$.

Suppose $\varphi \in \mathcal{W}(\overline{\Omega})$, $\operatorname{supp} \varphi \cap \overline{\Omega} \cap V(g) = \emptyset$ then $\forall z \in \overline{\Omega} \cap V(g)$, $0 = T_z\varphi \in T_z(g \cdot \mathcal{W}(\overline{\Omega}))$, where $T_z\varphi$ is the formal power series of φ at z. Hence using the Whitney's spectral theorem we get that $\varphi \in$ closure $(g \cdot \mathcal{W}(\overline{\Omega})) = g \cdot \mathcal{W}(\overline{\Omega})$ because $g \cdot \mathcal{W}(\overline{\Omega})$ is already closed.

Hence $L(\varphi) = 0$ and this means that $\operatorname{supp} L \subset \overline{\Omega} \cap V(g)$.

Then we get that L is of finite order k.

Now let φ the picking function on $P \supset \overline{\Omega} \cap V(g)$ and $f \in \mathcal{A}(\Omega)$; put:

$$f_N := (1 - \varphi^N)^{k+1} \cdot f \in \mathcal{A}(\Omega),$$

then $L(f_N) = 0$ because $f_N = 0$ up to $k + 1$ order on $P \supset \overline{\Omega} \cap V(g)$ hence:

$$0 = \int f_N h \, d\mu \longrightarrow \int f h \, d\mu$$

because $f_N \longrightarrow f$ a.e. and $|f_N| \leq 2^{k+1} \|f\|_\infty$ by Lebesgue's dominated convergence theorem.

Hence any function $h \in L^q(\mu)$ null on $g \cdot \mathcal{A}(\Omega)$ is null on $\mathcal{A}(\Omega)$ and because $\mathcal{A}(\Omega)$ is dense in $H^p(\mu)$ we get the theorem. □

In the strictly p.c. case we have the following:

PROPOSITION 4.2. *Let Ω be a bounded strictly pseudo-convex domain in $\mathbb{C}^n$ with smooth C^∞ boundary; let $g = (g_1, \dots, g_k) \in \mathcal{A}(\Omega)^k$; if $M := V(g) \cap \partial\Omega$ is closed smooth submanifold of $\partial\Omega$ with boundary and $V(g) \cap \Omega = \emptyset$ then M is a local peak set for $A^\infty(\Omega)$.*

PROOF. We need to have: $T_xM \subset T_x^{\mathbb{C}}(\partial\Omega)$ for any interior point of M because $V(g)$ is tangent to $\partial\Omega$ along M and any tangent vector to M must be tangent to the analytic variety $V(g)$ hence its conjugate is tangent to $V(g)$ hence also to $\partial\Omega$.

Now we can use the characterization of local peak set in [6] to conclude M is a local peak set for A^∞. □

The next corollary is a deep generalization of the result in [13]:

COROLLARY 4.3. *Let Ω be a bounded strictly pseudo-convex domain in $\mathbb{C}^n$ with real analytic boundary; let $g = (g_1, \dots, g_k) \in \mathcal{O}(\overline{\Omega})^k$; if $M := V(g) \cap \partial\Omega$ is a closed smooth submanifold of $\partial\Omega$ with boundary, if $V(g) \cap \Omega = \emptyset$ and if $\mu(M) = 0$ then $g \cdot H^2(\mu)$ is dense in $H^2(\mu)$.*

PROOF. Because the g_i's are holomorphic near $\overline{\Omega}$ we have that g is G.C.; adding the fact that $\partial\Omega$ is real analytic implies that $g \cdot \mathcal{W}(\overline{\Omega})$ is closed by Malgrange's theorem; the preceeding proposition allows us to see that the conditions of Theorem 4.1 are fullfilled hence the corollary. □

The fact that even in the unit ball of $\mathbb{C}^n$ this situation may happen with M not reduced to a finite number of points (hence proving that even the ball does not fill the assumption α_4 in [14], Section 2!) is given by the following simple example:

EXAMPLE 4.4. Let $u(z) := z_1^2 + z_2^2 - 1$ then:

- $\{z \in \mathbb{C}^2 / u(z) = 0\} \cap \mathbb{B} = \emptyset$;
- $\{z \in \mathbb{C}^2 / u(z) = 0\} \cap \partial\mathbb{B} = C$ with C the real circle $\{\operatorname{Im} z_1 = \operatorname{Im} z_2 = 0\} \cap \partial\mathbb{B}$

The simple verification is left to the reader. □

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