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New *CR* invariants and their application to the *CR* equivalence problem

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New CR Invariants and their Application to the CR Equivalence Problem

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1. - Introduction

Let M be a strictly pseudoconvex CR manifold (of hypersurface type) of CR dimension $n-1$. Let $K(M) = \Omega^{n,0}(M)$ be its canonical bundle and $K^0(M) = K(M) - \{\ zero\ section\ \}$. Let $C(M) = K^0(M)/\mathbb{R}_+$. Then $C(M)$ is a principal circle bundle over M and, by work of C.L. Fefferman [4], with each fixed pseudohermitian structure θ on M one may associate a Lorentz metric g on $C(M)$. This is the *Fefferman metric* of (M, θ). Its properties are closely tied to those of the base CR manifold. For instance, if M is a real hypersurface in $\mathbb{C}^n$ then the null geodesics of the Fefferman metric project on biholomorphic invariant curves (known as the *chains* of M, cf. S.S. Chern & J. Moser [1]). Although not fully understood as yet, the Fefferman metric proved useful in a number of situations, e.g. provided a simpler proof (cf. L.K. Koch [9]) of the striking result of H. Jacobowitz (cf. [6]) that two nearby points of a strictly pseudoconvex CR manifold are joined by a chain. See also C.R. Graham [5], for a characterization of Fefferman metrics among all Lorentz metrics on $C(M)$.

By classical work of S.S. Chern & J. Simons [2], the Pontrjagin forms of a riemannian manifold are conformal invariants. On the other hand, the restricted conformal class of the Fefferman metric is known (cf. J.M. Lee [10]) to be a CR invariant. This led us to investigate whether the result by S.S. Chern & J. Simons may carry over to Lorentz geometry. We find (cf. Theorem 2) that the Pontrjagin forms $P(\Omega^\ell)$ of the Fefferman metric are CR invariants of M. Also, whenever $P(\Omega^\ell) = 0$, the De Rham cohomology class of the corresponding transgression form is a CR invariant, as well. As an application, we show that a necessary condition for M to be globally CR equivalent to a sphere S^{2n-1} is that $P_1(\Omega^2) = 0$ (i.e. the first Pontrjagin form of $(C(M), g)$ must vanish) and the corresponding transgression form gives an integral cohomology class (cf. Theorem 3).

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2. – The Fefferman metric

Let $(M, T_{1,0}(M))$ be an orientable CR manifold (of hypersurface type) of CR dimension $n-1$, where $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$ denotes its CR structure. Its Levi distribution $H(M) = Re\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ carries the complex structure $J : H(M) \to H(M)$ given by $J(Z + \overline{Z}) = i(Z - \overline{Z})$ for any $Z \in T_{1,0}(M)$. Here $T_{0,1}(M) = \overline{T_{1,0}(M)}$. Overbars denote complex conjugation and $i = \sqrt{-1}$. The annihilator $E \subset T^*(M)$ of $H(M)$ is a trivial line bundle, hence it admits global nowhere vanishing cross sections $\theta \in \Gamma^\infty(E)$, each of which is referred to as a *pseudohermitian structure*. The Levi form L_θ is given by $L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W})$ for any $Z, W \in T_{1,0}(M)$. Two pseudohermitian structures θ, $\hat{\theta}$ are related by $\hat{\theta} = e^{2u}\theta$ for some C^∞ function $u : M \to \mathbb{R}$ and the corresponding Levi forms satisfy $L_{\hat{\theta}} = e^{2u} L_\theta$. This accounts for the (already highly exploited, cf. e.g. D. Jerison & J.M. Lee [7], and references therein) analogy between CR and conformal geometry. If L_θ is nondegenerate for some choice of θ (and thus for all) then $(M, T_{1,0}(M))$ is a *nondegenerate* CR manifold. Any nondegenerate CR manifold, on which a pseudohermitian structure θ has been fixed, admits a unique linear connection ∇ (the *Tanaka-Webster connection*) parallelizing both the Levi form and the complex structure (in the Levi distribution). Cf. also [3] for an axiomatic description of the Tanaka-Webster connection.

A complex valued p-form ω on M is a *$(p, 0)$-form* if $T_{0,1}(M) \rfloor \omega = 0$. Let $\Omega^{p,0}(M)$ be the bundle of all $(p, 0)$-forms on M. Set $K(M) = \Omega^{n,0}(M)$. There is a natural action of $\mathbb{R}_+ = (0, \infty)$ on $K^0(M) = K(M) - \{0\}$ and the quotient space $C(M) = K^0(M)/\mathbb{R}_+$ is a principle S^1-bundle over M. Let $\pi : C(M) \to M$ be the projection. A local frame $\{\theta^\alpha\}$ of $T_{1,0}(M)^*$ on $U \subseteq M$ induces the trivialization chart:

$$\pi^{-1}(U) \to U \times S^1, \qquad [\omega] \mapsto \left(x, \frac{\lambda}{|\lambda|}\right)$$

where $\omega \in K^0(M)$, $\pi([\omega]) = x$ and $\omega = \lambda \left(\theta \wedge \theta^1 \wedge \cdots \wedge \theta^{n-1}\right)_x$ with $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Define $\gamma : \pi^{-1}(U) \to [0, 2\pi)$ by $\gamma([\omega]) = \arg(\lambda)$. Moreover, consider the (globally defined) 1-form σ on $C(M)$ given by:

$$\sigma = \frac{1}{n+1}\left(d\gamma + \pi^*\left(i\omega_\alpha{}^\alpha - \frac{i}{2}h^{\alpha\overline{\beta}} dh_{\alpha\overline{\beta}} - \frac{R}{2n}\theta\right)\right).$$

Here $h_{\alpha\overline{\beta}}$, $\omega_\alpha{}^\beta$ and $R = h^{\alpha\overline{\beta}} R_{\alpha\overline{\beta}}$ are respectively the (local) components of the Levi form, the connection 1-forms (of the Tanaka-Webster connection) and the pseudohermitian scalar curvature (cf. e.g. (2.17) in [12], p. 34).

Let us extend the Hermitian form $\langle Z, W\rangle_\theta = L_\theta(Z, \overline{W})$ to the whole of $T(M) \otimes \mathbb{C}$ by requesting that $\langle Z, \overline{W}\rangle_\theta = 0$, $\langle \overline{Z}, \overline{W}\rangle_\theta = \overline{\langle Z, W\rangle_\theta}$ and $\langle T, V\rangle_\theta = 0$ for any $Z, W \in T_{1,0}(M)$, $V \in T(M) \otimes \mathbb{C}$. Then:

$$g = \pi^*\langle\ ,\ \rangle_\theta + 2(\pi^*\theta) \odot \sigma \tag{1}$$

is a semi-riemannian metric on $C(M)$. Assume from now on that M is strictly pseudoconvex and choose θ so that L_θ is positive definite. Then g is a Lorentz metric on $C(M)$, known as the *Fefferman metric* of (M, θ). By a result of J.M. Lee (cf. [10], p. 418) if $\hat{\theta} = e^{2u}\theta$ is another pseudohermitian structure and $\hat{g}$ the corresponding Fefferman metric, then $\hat{g} = e^{2(u \circ \pi)} g$.

3. – Pontrjagin forms

Let $I^\ell(GL(2n))$ be the space of all invariant polynomials of degree ℓ, i.e. symmetric multilinear maps $P : \mathbf{gl}(2n)^\ell \to \mathbb{R}$ which are $ad(GL(2n))$-invariant. Here $\mathbf{gl}(2n)$ is the Lie algebra of $GL(2n) = GL(2n, \mathbb{R})$. Also, if $\mathcal{G}$ is a linear space then $\mathcal{G}^\ell = \mathcal{G} \otimes \cdots \otimes \mathcal{G}$ (ℓ terms). Let $Q_\ell \in I^\ell(GL(2n))$, $1 \leq \ell \leq 2n$, be the natural generators of the ring of invariant polynomials on $\mathbf{gl}(2n)$ (cf. [2], p. 57, for the explicit expressions of the Q_ℓ). Let $(M, T_{1,0}(M))$ be a strictly pseudoconvex CR manifold of CR dimension $n-1$ and θ a pseudohermitian structure on M so that L_θ is positive definite. Let g be the Fefferman metric of (M, θ). Let $F(C(M)) \to C(M)$ be the principal $GL(2n)$-bundle of all linear frames on $C(M)$ and $\omega \in \Gamma^\infty(T^*(F(C(M))) \otimes \mathbf{gl}(2n))$ the connection 1-form (of the Levi-Civita connection) of the Lorentz manifold $(C(M), g)$. Then:

THEOREM 1. *The characteristic forms* $Q_{2\ell+1}\left(\Omega^{2\ell+1}\right)$ *vanish for any* $0 \leq \ell \leq n-1$.

Here $\Omega = D\omega$ is the curvature 2-form of ω. Also, for any $P \in I^\ell(GL(2n))$ we set $P(\Omega^\ell) = P \circ \Omega^\ell$ where $\Omega^\ell = \Omega \wedge \cdots \wedge \Omega$ (ℓ terms). Let us prove Theorem 1. To this end, let $\mathcal{L}(C(M)) \to C(M)$ be the principal $O(2n-1, 1)$-bundle of all Lorentz frames, i.e. $u = (c, \{X_i\}) \in \mathcal{L}(C(M))$ if $g_c(X_i, X_j) = \epsilon_i \delta_{ij}$ where $\epsilon_\alpha = 1$, $1 \leq \alpha \leq 2n-1$ and $\epsilon_{2n} = -1$, $c \in C(M)$. Here $O(2n-1, 1)$ is the Lorentz group. Let $\mathbf{o}(2n-1, 1)$ be its Lie algebra. By hypothesis:

$$\omega_u(T_u(\mathcal{L}(C(M)))) \subseteq \mathbf{o}(2n-1, 1)$$

i.e. $\epsilon\omega_u(X) + \omega_u(X)^t\epsilon = 0$ for any $X \in T_u(\mathcal{L}(C(M)))$, $u \in \mathcal{L}(C(M))$. Here $\epsilon = diag(\epsilon_1, \cdots, \epsilon_{2n})$. Let $\{E^i_j\}$ be the canonical basis of $\mathbf{gl}(2n)$ and set $\omega = \omega^i_j \otimes E^j_i$, $\Omega = \Omega^i_j \otimes E^j_i$. We claim that:

$$\epsilon^i \Omega^i_j + \epsilon^j \Omega^j_i = 0 \tag{2}$$

at all points of $\mathcal{L}(C(M))$, as a form $F(C(M)$. Here $\epsilon^i = \epsilon_i$. As Ω is horizontal, it suffices to check (2) on horizontal vectors (hence tangent to $\mathcal{L}(C(M))$). We have:

$$\begin{aligned} \epsilon^i \Omega^i_j &= \epsilon^i (d\omega^i_j + \omega^i_k \wedge \omega^k_j) \\ &= d(-\epsilon^j \omega^j_i) + \sum_k (-\epsilon^k \omega^k_i) \wedge \omega^k_j = -\epsilon^j \Omega^j_i \end{aligned}$$

on $T_u(\mathcal{L}(C(M)))$ for any $u \in \mathcal{L}(C(M))$, etc. Next, note that for any $A \in \mathbf{o}(2n-1, 1)$ one has i) $\mathrm{tr}(A) = 0$, ii) $\mathrm{tr}(AB) = 0$, for any $B \in \mathcal{M}_{2n}(\mathbb{R})$ satisfying $B = \epsilon B^t \epsilon$, and iii) $\mathrm{tr}(A^{2\ell+1}) = 0$. Then:

$$\mathrm{tr}(A_1 \cdots A_{2\ell+1}) = 0 \tag{3}$$

for any $A_1, \cdots, A_{2\ell+1} \in \mathbf{o}(2n-1, 1)$ (the proof is by induction over ℓ). Since $Q_{2\ell+1}(\Omega^{2\ell+1})$ is invariant, we need only show that it vanishes at the points of $\mathcal{L}(C(M))$. But at these points the range of $\Omega^{2\ell+1}$ lies (by (2)-(3)) in the kernel of $Q_{2\ell+1}$. Our Theorem 1 is proved.

Let $P \in I^\ell(GL(2n))$. The *transgression form* $TP(\omega)$ is given by:

$$TP(\omega) = \ell \int_0^1 P(\omega \wedge \Omega_t^{\ell-1}) dt$$

where $\Omega_t = t\Omega + (1/2)t(t-1)[\omega, \omega]$, $0 \leq t \leq 1$. By Chern-Weil theory (cf. e.g. [8], vol. II, p. 297) one has $P(\Omega^\ell) = dTP(\omega)$. By Theorem 1, the transgression forms $TQ_{2\ell+1}(\omega)$ are closed, hence we get the cohomology classes $[TQ_{2\ell+1}(\omega)] \in H^{4\ell+1}(F(C(M)), \mathbb{R})$. Note that:

$$[TQ_{2\ell+1}(\omega)] \in \ker(j^*) \tag{4}$$

where $j^* : H^{4\ell+1}(F(C(M)), \mathbb{R}) \to H^{4\ell+1}(\mathcal{L}(C(M)), \mathbb{R})$ is induced by $j : \mathcal{L}(C(M)) \subset F(C(M))$. Indeed $TQ_{2\ell+1}(\omega)$ may be written as:

$$TQ_{2\ell+1}(\omega) = \sum_{i=0}^{2\ell} B_i Q_{2\ell+1}(\omega \wedge [\omega, \omega]^i \wedge \Omega^{2\ell-i})$$

for some constants $B_i > 0$. As $j^*\omega$ is $\mathbf{o}(2n-1, 1)$-valued, the same argument as in the proof of Theorem 1 shows that $j^*TQ_{2\ell+1}(\omega) = 0$, q.e.d. One has to work with $j^*\omega$ (rather than ω at a point of $\mathcal{L}(C(M))$) because ω (unlike its curvature form) is not horizontal.

If g_0 is a riemannian metric on $C(M)$ with connection 1-form ω_0 and $O(C(M)) \to C(M)$ is the principal $O(2n)$-bundle of orthonormal (with respect to g_0) frames on $C(M)$, then orthonormalization of frames gives a deformation retract $F(C(M)) \to O(C(M))$ and hence (cf. Proposition 4.3 in [2], p. 58) the corresponding transgression forms $TQ_{2\ell+1}(\omega_0)$ are exact. As to the Lorentz case, in general (4) need not imply exactness of $TQ_{2\ell+1}(\omega)$. For instance $\mathbb{R}^2_1$ is a Lorentz manifold for which the homomorphism $j^* : H^1(F(\mathbb{R}^2_1), \mathbb{R}) \to H^1(\mathcal{L}(\mathbb{R}^2_1), \mathbb{R})$ (induced by $j : \mathcal{L}(\mathbb{R}^2_1) \subset F(\mathbb{R}^2_1)$) has a nontrivial kernel. Here $\mathbb{R}^N_\nu = (\mathbb{R}^N, \langle\ ,\ \rangle_{N-\nu,\nu})$ and $\langle\ ,\ \rangle_{N-\nu,\nu} = \sum_{i=1}^{N-\nu} x_i y_i - \sum_{i=N-\nu+1}^{N} x_i y_i$. Indeed, as both $F(\mathbb{R}^2_1)$ and $\mathcal{L}(\mathbb{R}^2_1)$ are trivial bundles j^* may be identified with the homomorphism $j^* : H^1(GL(2), \mathbb{R}) \to H^1(O(1,1), \mathbb{R})$ (induced by $j : O(1,1) \subset GL(2)$). The Lorentz group $O(1,1)$ has four components, each diffeomorphic to $\mathbb{R}$. Hence $H^1(O(1,1)) = 0$. Moreover $O(2) \subset GL(2)$ is a homotopy equivalence, hence $\ker(j^*) = H^1(GL(2), \mathbb{R}) = H^1(O(2), \mathbb{R}) = \mathbb{R} \oplus \mathbb{R}$ (as $O(2)$ has two components, each diffeomorphic to S^1).

At this point, we may state the following:

THEOREM 2. *Let M be a strictly pseudoconvex CR manifold of CR dimension $n-1$ and $P \in I^{\ell}(GL(2n))$. Then $P(\Omega^{\ell})$ is a CR invariant of M. Moreover, if $P(\Omega^{\ell}) = 0$, then the cohomology class $[TP(\omega)] \in H^{2\ell-1}(F(C(M)), \mathbb{R})$ is a CR invariant of M. In particular $[TQ_{2\ell+1}(\omega)] \in H^{4\ell+1}(F(C(M)), \mathbb{R})$ is a CR invariant.*

4. – Applications

Let M be a strictly pseudoconvex CR manifold. Assume that M is realizable as a real hypersurface in $\mathbb{C}^n$. If $\varphi : M \to \mathbb{C}^n$ is the given immersion, then $\eta = \varphi^* dz^1 \wedge \cdots \wedge dz^n$ is a nowhere zero global $(n, 0)$-form on M, hence $C(M)$ is a trivial bundle. By work of C.L. Fefferman [4], there is a smooth defining function ψ of M satisfying the complex Monge-Ampère equation:

$$J(\psi) \equiv \det \begin{pmatrix} \psi & \partial\psi/\partial\overline{z}^k \\ \partial\psi/\partial z^j & \partial^2\psi/\partial z^j \partial\overline{z}^k \end{pmatrix} = 1$$

to second order along M, so that F^*h is the Fefferman metric of $(M, \hat{\theta})$, $\hat{\theta} = \frac{i}{2}\varphi^*(\overline{\partial} - \partial)\psi$, where h is the Lorentz metric given by:

$$h = -\frac{i}{n+1} j^*\{(\partial - \overline{\partial})\psi\} \odot d\gamma + j^*\left\{ \frac{\partial^2\psi}{\partial z^j \partial\overline{z}^k} dz^j \odot d\overline{z}^k \right\}$$

and $F : C(M) \approx M \times S^1$ the diffeomorphism induced by η. Also γ is a local coordinate on S^1 and $j : M \times S^1 \subset \mathbb{C}^{n+1}$. Let θ be any pseudohermitian structure on M (so that L_θ is positive definite). Then $\hat{\theta} = e^{2u}\theta$ for some smooth function u on M, and an inspection of (1) shows that F^*h and g are conformally equivalent Lorentz metrics. On the other hand $h = j^*G$ where G is the semi-riemannian metric on $\mathbb{C}^n \times \mathbb{C}_*$ given by:

$$G = |\zeta|^{2/(n+1)} \left\{ \frac{\psi}{(n+1)^2} |\zeta|^{-2} d\zeta \odot d\overline{\zeta} + \frac{\partial^2\psi}{\partial z^j \partial\overline{z}^k} dz^j \odot d\overline{z}^k \right.$$
$$\left. + \frac{1}{n+1}\left((\partial\psi) \odot \frac{d\overline{\zeta}}{\overline{\zeta}} + \frac{d\zeta}{\zeta} \odot (\overline{\partial}\psi) \right) \right\}$$

where $(z, \zeta) = (z^1, \cdots, z^n, \zeta)$ are complex coordinates. Summing up, if M is realizable then $(C(M), g)$ admits a global conformal immersion in $(\mathbb{C}^n \times \mathbb{C}_*, G)$, hence (in view of Theorem 5.14 in [2], p. 64) it is reasonable to expect that some of the CR invariants furnished by Theorem 2 are obstructions towards the global embeddability of a given, abstract, CR manifold M. While we leave this as an open problem, we address the following simpler situation. Assume M to be equivalent to S^{2n-1}. Then $C(M)$ is diffeomorphic to the Hopf manifold $H^n = S^{2n-1} \times S^1$. On the other hand, note that $I_{n+1} = \{\zeta \in \mathbb{C} : \zeta^{n+1} = 1\}$ acts freely on $\mathbb{C}^n \times \mathbb{C}_*$ as a properly discontinuous group of complex analytic transformations. Hence the quotient space $V_{n+1} = (\mathbb{C}^n \times \mathbb{C}_*)/I_{n+1}$ is a complex $(n+1)$-dimensional manifold. Consider the biholomorphism $p : V_{n+1} \to \mathbb{C}^n \times \mathbb{C}_*$ given by $p([z,\zeta]) = (z/\zeta, \zeta^{n+1})$ for any $[z,\zeta] \in V_{n+1}$ and set $\phi_0 = p^{-1} \circ j \circ F$. Next:

$$G_0 = \sum_{j=1}^{n} dz^j \odot d\bar{z}^j - d\zeta \odot d\bar{\zeta} \tag{5}$$

is I_{n+1}-invariant, hence gives rise to a globally defined semi-riemannian metric of index 2 on V_{n+1}. Note that (V_{n+1}, G_0) is locally isometric to $\mathbb{R}^{2n+2}_2$.

LEMMA 1. *$\phi_0 : (C(M), g) \to (V_{n+1}, G_0)$ is a conformal immersion.*

Indeed, let $\psi(z) = |z|^2 - 1$. A calculation then shows that $G_0 = p^*G$. Finally, it may be seen that $F : (C(M), g) \to (H^n, h)$ is a conformal diffeomorphism.

Let $P_i \in I^{2i}(GL(2n))$ be given by:

$$\det\left(\lambda I_{2n} - \frac{1}{2\pi}A\right) = \sum_{i=0}^{n} P_i(A \otimes \cdots \otimes A)\lambda^{2n-2i} + Q(\lambda^{2n-odd})$$

i.e. the invariant polynomials obtained by ignoring the powers λ^{2n-odd}. We obtain the following:

THEOREM 3. *Let M be a strictly pseudoconvex CR manifold of CR dimension $n-1$ and θ a pseudohermitian structure on M so that L_θ is positive definite. Let g be the Fefferman metric of (M,θ). Let ω be the connection 1-form of g and Ω its curvature 2-form. If M is CR equivalent to S^{2n-1} then $P_1(\Omega^2) = 0$ and $[TP_1(\omega)] \in H^3(F(C(M)), \mathbb{Z})$, provided $n \geq 3$.*

To prove Theorem 3, we study the geometry of the second fundamental form of the immersion $\phi = p^{-1} \circ j : H^n \to (\mathbb{C}^n \times \mathbb{C}_*, G)$. Set $C_n = \sqrt{n+1}/\sqrt{2(n+1)}$. The tangent vector fields ξ_a given by:

$$\xi_1 = C_n\left(z^j \frac{\partial}{\partial z^j} + \bar{z}^j \frac{\partial}{\partial \bar{z}^j} + \zeta \frac{\partial}{\partial \zeta} + \bar{\zeta}\frac{\partial}{\partial \bar{\zeta}}\right)$$

$$\zeta_2 = C_n\left(z^j \frac{\partial}{\partial z^j} + \bar{z}^j \frac{\partial}{\partial \bar{z}^j} - (n+2)\left(\zeta \frac{\partial}{\partial \zeta} + \bar{\zeta}\frac{\partial}{\partial \bar{\zeta}}\right)\right)$$

are such that $G(\xi_1, \xi_2) = 0$, $G(\xi_1, \xi_1) = 1$ and $G(\xi_2, \xi_2) = -1$, and form a frame of the normal bundle of ϕ. Since p is a biholomorphism (with the inverse $p^{-1}(z,\zeta) = [z\zeta^{1/(n+1)}, \zeta^{1/(n+1)}]$) we have:

$$p_* \frac{\partial}{\partial z^j} = \zeta^{-1/(n+1)} \frac{\partial}{\partial z^j}$$

$$p_* \frac{\partial}{\partial \zeta} = \zeta^{-1/(n+1)} \left(-z^j \frac{\partial}{\partial z^j} + (n+1)\zeta \frac{\partial}{\partial \zeta}\right).$$

By (5) the Christoffel symbols of the Levi-Civita connection ∇^0 of (V_{n+1}, G_0) vanish. The Levi-Civita connection ∇ of $(\mathbb{C}^n \times \mathbb{C}_*, G)$ is related to ∇^0 by:

$$p_*\left(\nabla^0_X Y\right) = \nabla_{p_* X} p_* Y$$

for any $X, Y \in T(V_{n+1})$. A calculation shows that:

$$\nabla_{\frac{\partial}{\partial z^j}} \frac{\partial}{\partial z^k} = 0; \quad \nabla_{\frac{\partial}{\partial \zeta}} \frac{\partial}{\partial \zeta} = -\frac{n}{n+1}\frac{1}{\zeta}\frac{\partial}{\partial \zeta}$$
$$\nabla_{\frac{\partial}{\partial \zeta}} \frac{\partial}{\partial z^j} = \frac{1}{n+1}\frac{1}{\zeta}\frac{\partial}{\partial z^j}.$$

Tangent vector fields on H^n are of the form $X+Y$ with $X = A^j\partial/\partial z^j + \overline{A^j}\partial/\partial\overline{z}^j$ and $Y = B\partial/\partial\zeta + \overline{B}\partial/\partial\overline{\zeta}$ satisfying $A^j\overline{z}_j + \overline{A^j}z_j = 0$, respectively $B\overline{\zeta} + \overline{B}\zeta = 0$. Here $z^j = z_j$. It follows that:

$$\nabla_X \xi_1 = C_n\frac{n+2}{n+1}X, \quad \nabla_X \xi_2 = -\frac{C_n}{n+1}X \tag{6}$$

$$\nabla_Y \xi_1 = \frac{C_n}{n+1}\left\{Y + B\overline{\zeta}z^j\frac{\partial}{\partial z^j} + \overline{B}\zeta\overline{z}^j\frac{\partial}{\partial \overline{z}^j}\right\} \tag{7}$$

$$\nabla_Y \xi_2 = \frac{C_n}{n+1}\left\{-(n+2)Y + B\overline{\zeta}z^j\frac{\partial}{\partial z^j} + \overline{B}\zeta\overline{z}^j\frac{\partial}{\partial \overline{z}^j}\right\}. \tag{8}$$

Let $A_a = A_{\xi_a}$ be the Weingarten operator corresponding to the normal section ξ_a. We shall need the following:

LEMMA 2. *The first Pontrjagin form of* (H^n, h) *is:*

$$\frac{1}{4\pi^2}\Psi_{12} \wedge \Psi_{12}$$

where (with respect to a local coordinate system (x^i) *on* H^n*):*

$$\Psi_{12} = h\left(\frac{\partial}{\partial x^i}, A_1 A_2 \frac{\partial}{\partial x^j}\right) dx^i \wedge dx^j.$$

We shall prove Lemma 2 later on. Recall the Ricci equation (of the given immersion ϕ, cf. e.g. (2.7) in [13], p. 22):

$$G(R(X,Y)\xi, \xi') = G(R^\perp(X,Y)\xi, \xi') + h([A_\xi, A_{\xi'}]X, Y)$$

where R, $R^\perp$ denote respectively the curvature tensor fields of $(\mathbb{C}^n \times \mathbb{C}_*, G)$ and of the normal connection. As a consequence of (6)-(8) ξ_a are parallel in the normal bundle, hence the immersion ϕ has a flat normal connection ($R^\perp = 0$). On the other hand $R = 0$ (because (V_{n+1}, G_0) is flat) and the Ricci equation shows that the Weingarten operators A_a commute. Then $\Psi_{12} = 0$ and our Lemmas 1 and 2 together with Theorem 2 yield $P_1(\Omega^2) = 0$.

Let $q : H^3(F(C(M)), \mathbb{R}) \to H^3(F(C(M)), \mathbb{R}/\mathbb{Z})$ be the natural homomorphism. By Theorem 3.16 in [2], p. 56, since $P_1(\Omega^2) = 0$, there is a cohomology class $\alpha \in H^3(C(M), \mathbb{R}/\mathbb{Z})$ so that $p_F^* \alpha = q([TP_1(\omega)])$, where $p_F : F(C(M)) \to C(M)$ is the projection. Yet, for the Hopf manifold $H^3(H^n, \mathbb{R}/\mathbb{Z}) = 0$ provided $n \geq 3$, hence $[TP_1(\omega)] \in \ker(q)$ and then by the exactness of the Bockstein sequence:

$$\cdots \to H^3(F(C(M)), \mathbb{Z}) \to H^3(F(C(M)), \mathbb{R}) \to$$
$$\to H^3(F(C(M)), \mathbb{R}/\mathbb{Z}) \to H^4(F(C(M)), \mathbb{R}) \to \cdots$$

it follows that $[TP_1(\omega)]$ is an integral class.

5. – Proof of Theorem 2

Let $\varphi \in \Gamma^\infty(T^*(F(C(M))) \otimes \mathbb{R}^{2n})$ be the canonical 1-form and set $\varphi = \varphi^i \otimes e_i$, where $\{e_i\}$ is the canonical basis in $\mathbb{R}^{2n}$. Moreover, let $E_i = B(e_i)$ be the corresponding standard horizontal vector fields (cf. e.g. [8], vol. I, p. 119). Let $u : M \to \mathbb{R}$ be a C^∞ function and let $\hat{g}$ be the Fefferman metric of $(M, e^{2u}\theta)$. Let $\hat{\omega}$ be the corresponding connection 1-form. Then:

$$\hat{\omega}^i_j = \omega^i_j + d(u \circ \rho)\delta^i_j + E_j(u \circ \rho)\varphi^i - \epsilon_i E_i(u \circ \rho)\epsilon_j \varphi^j \tag{9}$$

at all points of $\mathcal{L}(C(M))$, as forms on $F(C(M))$. Here $\rho = \pi \circ p_F$. The proof is to relate the Levi-Civita connections of the conformally equivalent Fefferman metrics g and $\hat{g}$, followed by a translation of the result in principal bundle terminology. We omit the details. Consider the 1-parameter family of Lorentz metrics $g(s) = e^{2s(u \circ \pi)} g$, $0 \leq s \leq 1$, on $C(M)$. Let $\omega(s)$ be the corresponding connection 1-form and set $\omega' = \frac{d}{ds}\{\omega(s)\}_{s=0}$. By (9) (applied to $s(u \circ \rho)$ instead of $u \circ \rho$) we obtain:

$$\omega'^i_j = d(u \circ \rho)\delta^i_j + E_i(u \circ \rho)\varphi^i - \epsilon_i E_i(u \circ \rho)\epsilon_j \varphi^j \tag{10}$$

at all points of $\mathcal{L}(C(M))$, as forms on $F(C(M))$. Let $P \in I^\ell(GL(2n))$. We wish to show that $P(\Omega^\ell)$ is invariant under any transformation $\hat{\theta} = e^{2u}\theta$. Note that a relation of the form:

$$T\,P(\hat{\omega}) = T\,P(\omega) + exact \tag{11}$$

yields $P(\hat{\Omega}^\ell) = P(\Omega^\ell)$, hence we only need to prove (11). Since the Q_ℓ generate $I(GL(2n))$ we may assume that P is a monomial in the Q_ℓ. Using

Proposition 3.7 in [2], p. 53, an inductive argument shows that it is sufficient to prove (11) for $P = Q_\ell$. It is enough to prove that:

$$\frac{d}{ds}\{T Q_\ell(\omega(s))\} = exact\,. \tag{12}$$

Since each point on the curve $s \mapsto g(s)$ is the initial point of another such curve, it suffices to prove (12) at $s = 0$. By Proposition 3.8 in [2], p. 53, we know that:

$$\frac{d}{ds}\{T Q_\ell(\omega(s))\}_{s=0} = \ell\, Q_\ell(\omega' \wedge \Omega^{\ell-1}) + exact$$

hence it is enough to show that $Q_\ell(\omega' \wedge \Omega^{\ell-1}) = exact$. Using (10) and the identity:

$$Q_\ell(\psi \wedge \Omega^{\ell-1}) = \sum_{i_1,\cdots,i_\ell} \psi^{i_1}_{i_2} \wedge \Omega^{i_2}_{i_3} \wedge \cdots \wedge \Omega^{i_\ell}_{i_1}$$

(cf. (4.2) in [2], p. 57) for any $\mathbf{gl}(2n)$-valued form ψ on $F(C(M))$, we may conduct the following calculation:

$$\begin{aligned} Q_\ell(\omega' \wedge \Omega^{\ell-1}) &= \sum \omega'^{i_1}_{i_2} \wedge \Omega^{i_2}_{i_3} \wedge \cdots \wedge \Omega^{i_\ell}_{i_1} \\ &= \sum d(u \circ \rho) \wedge \Omega^{i_2}_{i_3} \wedge \cdots \wedge \Omega^{i_\ell}_{i_2} \\ &+ \sum \left(E_{i_2}(u \circ \rho)\varphi^{i_1} - \epsilon_{i_1} E_{i_1}(u \circ \rho)\epsilon_{i_2}\varphi^{i_2}\right) \wedge \Omega^{i_2}_{i_3} \wedge \cdots \wedge \Omega^{i_\ell}_{i_1} \end{aligned}$$

Recall the structure equations, cf. e.g. [8], vol. I, p. 121. As g is Lorentz, ω is torsion free. Hence $\varphi^{i_1} \wedge \Omega^{i_\ell}_{i_1} = 0$. This and (2) also yield $\epsilon_{i_2}\varphi^{i_2} \wedge \Omega^{i_2}_{i_3} = 0$. Hence:

$$Q_\ell(\omega' \wedge \Omega^{\ell-1}) = d(u \circ \rho) \wedge Q_{\ell-1}(\Omega^{\ell-1}) = exact$$

(because $dQ_{\ell-1}(\Omega^{\ell-1}) = 0$) at all points of $\mathcal{L}(C(M))$, as a form on $F(C(M))$. This suffices because both $Q_\ell(\omega' \wedge \Omega^{\ell-1})$ and $(u \circ \rho)Q_{\ell-1}(\Omega^{\ell-1})$ are invariant forms.

6. – Proof of Lemma 2

Recall (cf. e.g. [8], vol. II, p. 313) that:

$$P_\ell(\Omega^{2\ell}) = c_\ell \sum \delta^{j_1\cdots j_{2\ell}}_{i_1\cdots i_{2\ell}} \Omega^{i_1}_{j_1} \wedge \cdots \wedge \Omega^{i_{2\ell}}_{j_{2\ell}}$$

where $c_\ell = 1/\left((2\pi)^{2\ell}(2\ell)!\right)$ and the summation runs over all ordered subsets $(i_1, \cdots, i_{2\ell})$ of $\{1, \cdots, 2n\}$ and all permutations $(j_1, \cdots, j_{2\ell})$ of $(i_1, \cdots, i_{2\ell})$ and

$\delta^{j_1\cdots j_{2\ell}}_{i_1\cdots i_{2\ell}}$ is the sign of the permutation. We need the Gauss equation (cf. e.g. (2.4) in [13], p. 21):

$$R^{\ell}_{kij} = B^a_{jk} A^{\ell}_{ai} - B^a_{ik} A^{\ell}_{aj}$$

where R^{ℓ}_{kij}, B^a_{jk} are respectively the curvature tensor field of (H^n, h) and the second fundamental form of ϕ (with respect to a local coordinate system (U, x^i) on H^n). Also $A_a \partial_i = A^j_{ai} \partial_j$ where ∂_i is short for $\partial/\partial x^i$. The Gauss equation and the identity:

$$R(X, Y)Z = u\left(2\Omega(X^*, Y^*)_u (u^{-1}Z)\right)$$

(cf. [8], vol. I, p. 133) for any $X, Y, Z \in T_x(H^n)$ and some $u \in F(H^n)_x$, furnish:

$$2\Omega^r_s = Y^r_p X^k_s \left(B^a_{jk} A^p_{ai} - B^a_{ik} A^p_{aj}\right) dx^i \wedge dx^j$$

(where $X^i_j : p_F^{-1}(U) \to \mathbb{R}$ are fibre coordinates on $F(H^n)$ and $(Y^i_j) = (X^i_j)^{-1}$). Using:

$$B^a_{jk} = A^r_{aj} h_{rk}$$

a calculation leads to:

$$2P_1(\Omega^2) = -c_1 \Big(B^{a_1}_{j_1 k_1} A^{k_2}_{a_1 p_1} B^{a_2}_{j_2 k_2} A^{k_1}_{a_2 p_2} - B^{a_1}_{p_1 k_1} A^{k_2}_{a_1 j_1} B^{a_2}_{j_2 k_2} A^{k_1}_{a_2 p_2}\Big) dx^{p_1} \wedge dx^{j_1} \wedge dx^{p_2} \wedge dx^{j_2}$$

hence:

$$P_1(\Omega^2) = c_1 \sum_{a,b} \Psi_{ab} \wedge \Psi_{ab}$$

where Ψ_{ab} is the 2-form on $F(H^n)$ given by:

$$\Psi_{ab} = h(A_a \partial_i, A_b \partial_j) dx^i \wedge dx^j .$$

Finally, note that $\Psi_{11} = \Psi_{22} = 0$ and $\Psi_{21} = -\Psi_{12}$ and Lemma 2 is proved. Note that the proof works for any codimension two submanifold of a flat riemannian manifold.

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