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Asymptotic Behaviour and Correctors for Dirichlet Problems in Perforated Domains with Homogeneous Monotone Operators

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0. – Introduction

In this paper we study the asymptotic behaviour of the solutions of some nonlinear elliptic equations of monotone type with Dirichlet boundary conditions in perforated domains. Let Ω be a bounded open set in $\mathbb{R}^N$, and let $A\colon H_0^{1,p}(\Omega) \to H^{-1,q}(\Omega)$ be a monotone operator of the form

$$Au = -\mathrm{div}\big(a\,(x, Du)\big)\,,$$

where $1 < p < +\infty$, $1/p + 1/q = 1$, and $a\colon \Omega\times\mathbb{R}^N \to \mathbb{R}^N$ satisfies the classical hypotheses: Carathéodory conditions, local Hölder continuity, and strong monotonicity (see (1.4)–(1.7) below). We suppose, in addition, that $a(x, \cdot)$ is odd and positively homogeneous of degree $p - 1$. Given a sequence (Ω_n) of open sets contained in Ω, we consider for every $f \in H^{-1,q}(\Omega)$ the sequence (u_n) of the solutions of the nonlinear Dirichlet problems

$$u_n \in H_0^{1,p}(\Omega_n)\,, \qquad Au_n = f \quad \text{in } \mathcal{D}'(\Omega_n)\,. \tag{0.1}$$

We extend u_n to Ω by setting $u_n = 0$ on $\Omega\setminus\Omega_n$, and we consider (u_n) as a sequence in $H_0^{1,p}(\Omega)$. The problem is to describe the asymptotic behaviour of the sequence (u_n) as n tends to infinity.

The main theorem of the paper is the following compactness result (Theorem 6.6), which holds without any assumption on the sets Ω_n. For every sequence (Ω_n) of open sets contained in Ω there exist a subsequence, still denoted by (Ω_n), and a non-negative measure μ in the class $\mathcal{M}_0^p(\Omega)$ such that for every $f \in H^{-1,q}(\Omega)$ the solutions u_n of (0.1) converge weakly in $H_0^{1,p}(\Omega)$ to the solution u of the problem

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$$(0.2)\qquad \begin{cases} u \in H_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,, \\ \langle Au, v\rangle + \displaystyle\int_\Omega |u|^{p-2} uv\, d\mu = \langle f, v\rangle \qquad \forall\, v \in H_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,. \end{cases}$$

Here $\langle \cdot, \cdot\rangle$ denotes the duality pairing between $H^{-1,q}(\Omega)$ and $H_0^{1,p}(\Omega)$, while $\mathcal{M}_0^p(\Omega)$ is the class of all non-negative Borel measures on Ω with values in $[0, +\infty]$ which vanish on all sets of $(1, p)$-capacity zero. Moreover, we prove (Theorem 6.8) that (u_n) converges to u strongly in $H_0^{1,r}(\Omega)$ for every $r < p$, and that $\big(a\,(x, Du_n)\big)$ converges to $a\,(x, Du)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ and strongly in $L^s(\Omega, \mathbb{R}^N)$ for every $s < q$. Finally the energy $\big(a\,(x, Du_n)\,, Du_n\big)$ converges to $\big(a\,(x, Du)\,, Du\big) + |u|^p\mu$ weakly* in the sense of Radon measures on Ω.

To prove this compactness result we observe (Remark 2.4) that all problems of the form (0.1) can be written as problems of the form (0.2) with a suitable choice of the measure $\mu = \mu_{\Omega_n}$ in the class $\mathcal{M}_0^p(\Omega)$. Actually we prove (Theorem 6.5) that, for every sequence (μ_n) of measures of the class $\mathcal{M}_0^p(\Omega)$, there exist a subsequence, still denoted by (μ_n), and a measure $\mu \in \mathcal{M}_0^p(\Omega)$ such that for every $f \in H^{-1,q}(\Omega)$ the solutions u_n of the problems

$$(0.3)\qquad \begin{cases} u_n \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \\ \langle Au_n, v\rangle + \displaystyle\int_\Omega |u_n|^{p-2} u_n v\, d\mu_n = \langle f, v\rangle \qquad \forall\, v \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega) \end{cases}$$

converge weakly in $H_0^{1,p}(\Omega)$ to the solution u of (0.2).

In our proof an essential role is played by the solution w_n of the Dirichlet problem

$$w_n \in H_0^{1,p}(\Omega_n)\,, \qquad Aw_n = 1 \quad \text{in } \mathcal{D}'(\Omega_n)\,,$$

or, in the general case (0.3), by the solution w_n of the problem

$$(0.4)\qquad \begin{cases} w_n \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \\ \langle Aw_n, v\rangle + \displaystyle\int_\Omega |w_n|^{p-2} w_n v\, d\mu_n = \int_\Omega v\, dx \quad \forall\, v \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,. \end{cases}$$

As the sequences (u_n) and (w_n) of the solutions of (0.3) and (0.4) are bounded in $H_0^{1,p}(\Omega)$ (Theorem 2.1), we may assume that (u_n) and (w_n) converge weakly in $H_0^{1,p}(\Omega)$ to some functions u and w. By using a result of [4] we prove that (Du_n) and (Dw_n) converge to Du and Dw strongly in $L^r(\Omega, \mathbb{R}^N)$ for every $r < p$ (Theorem 2.11).

The crucial step in the proof of our compactness theorem is the following corrector result (Theorem 3.1), which is interesting in itself: if $f \in L^\infty(\Omega)$, then

$$(0.5)\qquad Du_n = Du + uP_n + R_n \qquad \text{a.e. in } \Omega\,,$$

where

$$P_n(x) = \begin{cases} \dfrac{1}{w(x)}\big(Dw_n(x) - Dw(x)\big), & \text{if } w(x) > 0, \\ 0, & \text{if } w(x) = 0, \end{cases}$$

and

$$R_n \to 0 \qquad \text{strongly in } L^p(\Omega, \mathbb{R}^N).$$

The idea of the proof is to consider the function $\frac{uw_n}{w}$, which, due to the homogeneity of $a(x,\xi)$, satisfies an equation similar to (0.3). We then use $u_n - \frac{uw_n}{w}$ as test function in the difference of (0.3) and of this equation. Using the strong monotonicity of $a(x,\xi)$ we prove that $(u_n - \frac{uw_n}{w})$ tends to 0 strongly in $H_0^{1,p}(\Omega)$ (Theorem 3.7). Actually the previous description of the proof is not accurate, because in general w can vanish on a set of positive Lebesgue measure. For this reason we are forced to replace $\frac{uw_n}{w}$ by $\frac{uw_n}{w\vee\varepsilon}$ with $\varepsilon > 0$, and to consider separately the sets where $w > \varepsilon$ and $w \leq \varepsilon$ (Lemmas 3.4, 3.5, 3.6).

Another important step in our proof is the construction of the measure μ, which depends only on w and on the operator A. We prove (Section 5) that $\nu = 1 - Aw$ is a non-negative Radon measure on Ω which belongs to $H^{-1,q}(\Omega)$, and we define the measure μ by

$$\mu(B) = \begin{cases} \displaystyle\int_B \frac{d\nu}{w^{p-1}}, & \text{if } C_p(B \cap \{w = 0\}) = 0, \\ +\infty, & \text{if } C_p(B \cap \{w = 0\}) > 0, \end{cases} \tag{0.6}$$

where $C_p(E)$ is the $(1,p)$-capacity of E with respect to Ω. We prove also that this measure belongs to $\mathcal{M}_0^p(\Omega)$ (Theorem 5.1).

It remains to prove that u coincides with the solution of (0.2) corresponding to the measure μ defined by (0.6). To give an idea of this part of the proof, we assume now that $2 \leq p < +\infty$ (the case $1 < p < 2$ is more technical) and that $f \in L^\infty(\Omega)$ (the case $f \notin L^\infty(\Omega)$ needs a further approximation). Given $\varphi \in C_0^\infty(\Omega)$, we take $v = w_n^{p-1}\varphi$ as test function in (0.3). We would like to take $v = \left|\frac{u}{w}\right|^{p-2}\frac{u}{w}w_n^{p-1}\varphi$ as test function in (0.4). This is not always possible, thus as before we replace w by $w \vee \varepsilon$ for some $\varepsilon > 0$ and we take $v = \left|\frac{u}{w\vee\varepsilon}\right|^{p-2}\frac{u}{w\vee\varepsilon}w_n^{p-1}\varphi$ as test function in (0.4). Subtracting the two equations we obtain

$$\begin{aligned} &\langle Au_n, w_n^{p-1}\varphi\rangle - \Big\langle Aw_n, \left|\frac{u}{w\vee\varepsilon}\right|^{p-2}\frac{u}{w\vee\varepsilon}w_n^{p-1}\varphi\Big\rangle \\ &\quad + \int_\Omega |u_n|^{p-2}u_n w_n^{p-1}\varphi\, d\mu_n - \int_\Omega \left|\frac{uw_n}{w\vee\varepsilon}\right|^{p-2}\frac{uw_n}{w\vee\varepsilon}w_n^{p-1}\varphi\, d\mu_n \\ &= \int_\Omega f w_n^{p-1}\varphi\, dx - \int_\Omega \left|\frac{u}{w\vee\varepsilon}\right|^{p-2}\frac{u}{w\vee\varepsilon}w_n^{p-1}\varphi\, dx. \end{aligned}$$

Using (0.5) and the homogeneity of A, as well as some technical lemmas proved in Section 4, we pass to the limit first as $n \to \infty$ and then as $\varepsilon \to 0$, obtaining

$$\langle Au, w^{p-1}\varphi\rangle - \langle Aw, |u|^{p-2}u\varphi\rangle = \int_\Omega f w^{p-1}\varphi\, dx - \int_\Omega |u|^{p-2}u\varphi\, dx.$$

As by definition $\nu = 1 - Aw$, we get

$$\langle Au, w^{p-1}\varphi\rangle + \int_\Omega |u|^{p-2} u\varphi \, d\nu = \int_\Omega f w^{p-1}\varphi \, dx ,$$

and thus (0.6) gives

$$\langle Au, w^{p-1}\varphi\rangle + \int_\Omega |u|^{p-2} u w^{p-1}\varphi \, d\mu = \int_\Omega f w^{p-1}\varphi \, dx \qquad \forall\varphi \in C_0^\infty(\Omega) .$$

Since the set $\{w^{p-1}\varphi : \varphi \in C_0^\infty(\Omega)\}$ is dense in $H_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)$ (Proposition 5.5), it follows that u is the solution of (0.2).

In the case $1 < p < 2$ the main difficulty lies in the fact that the function $|u|^{p-2}u$ does not belong to $H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for every $u \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. We solve this problem by considering a suitable locally Lipschitz modification, near the origin, of the function $t \mapsto |t|^{p-2}t$, which enables us to complete the proof following the ideas of the case $2 \leq p < +\infty$.

When $p = 2$ and A is a symmetric linear elliptic operator, the compactness result proved in the present paper has already been obtained by Γ-convergence techniques in [2], [1], [23], [7], [39]. These results were recently extended to the vectorial case by [27].

When $1 < p < +\infty$ and A is the subdifferential of a convex functional $\Psi(u) = \int_\Omega \psi(x, Du)\, dx$ defined on $H_0^{1,p}(\Omega)$, with $\psi(x, \cdot)$ even and positively homogeneous of degree p, the compactness result was proved in [20] by Γ-convergence techniques. The general case where $\psi(x, \cdot)$ is not homogeneous can be obtained by putting together the results of [2], [15], [16] on obstacle problems. Further reference on this subject can be found in the book [18] and in the recent paper [21], which contains a wide bibliography on the linear case.

Another method used in the study of this type of problems was introduced in [31] and [37], where the asymptotic behaviour of the solutions of (0.1) in the linear case was investigated in various interesting situations, under some assumptions involving the geometry or the capacity of the closed sets $\Omega \backslash \Omega_n$. The case of general Leray-Lions operators in $H_0^{1,p}(\Omega)$, with A possibly non homogeneous, was treated in [40]–[45], [32], [26] under similar assumptions.

The linear case was also investigated in [14] by using test functions as in the energy method introduced by Tartar [46] in the study of homogenization problems for elliptic operators. In this paper some restrictive hypotheses on the sets Ω_n were made in terms of the test functions, and a corrector result was obtained. The same method was then extended to a more general situation including also the case where $1 < p < +\infty$ and A is the p-Laplacian (see [2], [33], [38], [11]). The linear problem with a perturbation with quadratic growth with respect to the gradient was recently solved in [8] and [9].

A new decisive step was achieved in [21], where a compactness result without any hypothesis on the sets (Ω_n) was proved in the (not necessarily symmetric) linear case by introducing a new sequence of test functions which were defined as the solutions of equations (0.4) above with $p = 2$. Our paper follows the same lines as far as the test functions are concerned, but presents an alternative method of proof for the compactness and corrector results.

In conclusion, the compactness result proved in the present paper is new when the homogeneous monotone operator A is nonlinear and is not the sub-differential of a convex functional. The corrector result, the strong convergence of the solutions u_n in $H_0^{1,r}(\Omega)$ for $r < p$, and the convergence of the fields $a(x, Du_n)$ are new for all nonlinear homogeneous monotone operators, including the case of subdifferentials of convex functionals: they are indeed obtained without any hypothesis on the sets Ω_n.

An expository version of the present work, restricted to the case $p = 2$ and including also a simplified, but almost self-contained version of the proofs, has been published in [25].

The results of the present paper have recently been extended to the case of a monotone operator without homogeneity conditions in [10] and in [12], where the case of systems is also solved.

1. – Notation and preliminary results

Sobolev spaces and capacity. Let us fix a bounded open subset Ω of $\mathbb{R}^N$ and two real numbers p and q, with $1 < p < +\infty$, $1 < q < +\infty$, and $1/q + 1/p = 1$. The space $\mathcal{D}'(\Omega)$ of distributions in Ω is the dual of the space $C_0^\infty(\Omega)$. The space $H_0^{1,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev space $H^{1,p}(\Omega)$, and the space $H^{-1,q}(\Omega)$ is the dual of $H_0^{1,p}(\Omega)$. On $H_0^{1,p}(\Omega)$ we consider the norm

$$\|u\|^p_{H_0^{1,p}(\Omega)} = \int_\Omega |Du|^p dx\,,$$

and $H^{-1,q}(\Omega)$ is endowed with the corresponding dual norm.

For every subset E of Ω the $(1, p)$*-capacity* of E in Ω, denoted by $C_p(E)$, is defined as the infimum of $\int_\Omega |Du|^p\,dx$ over the set of all functions $u \in H_0^{1,p}(\Omega)$ such that $u \geq 1$ a.e. in a neighbourhood of E.

We say that a property $\mathcal{P}(x)$ holds *quasi everywhere* (abbreviated as *q.e.*) in a set E if it holds for all $x \in E$ except for a subset N of E with $C_p(N) = 0$. The expression *almost everywhere* (abbreviated as *a.e.*) refers, as usual, to the Lebesgue measure. A function $u\colon \Omega \to \mathbb{R}$ is said to be *quasi continuous* if for every $\varepsilon > 0$ there exists a set $A \subseteq \Omega$, with $C_p(A) < \varepsilon$, such that the restriction of u to $\Omega \backslash A$ is continuous.

It is well known that every $u \in H^{1,p}(\Omega)$ has a quasi continuous representative, which is uniquely defined up to a set of capacity zero. In the sequel we shall always identify u with its quasi continuous representative, so that the pointwise values of a function $u \in H^{1,p}(\Omega)$ are defined quasi everywhere in Ω. We recall that, if a sequence (u_n) converges to u strongly in $H_0^{1,p}(\Omega)$, then a subsequence of (u_n) converges to u q.e. in Ω. For all these properties of quasi continuous representatives of Sobolev functions we refer to [47], Section 3, and [30], Section 4.

A subset U of Ω is said to be *quasi open* if for every $\varepsilon > 0$ there exists an open subset V of Ω, with $C_p(V) < \varepsilon$, such that $U \cup V$ is open.

We shall frequently use the following lemma about the approximation of the characteristic function of a quasi open set. We recall that the characteristic function 1_E of a set $E \subseteq \Omega$ is defined by $1_E(x) = 1$, if $x \in E$, and by $1_E(x) = 0$, if $x \in \Omega \setminus E$.

LEMMA 1.1. *For every quasi open subset U of Ω there exists an increasing sequence (v_n) of non-negative functions of $H_0^{1,p}(\Omega)$ which converges to 1_U quasi everywhere in Ω.*

PROOF. See [16], Lemma 1.5, or [21], Lemma 2.1. □

Measures. By a *Radon measure* on Ω we mean a continuous linear functional on the space $C_0(\Omega)$ of all continuous functions with compact support in Ω. It is well known that for every Radon measure λ there exists a countably additive set function μ, defined on the family of all relatively compact Borel subsets of Ω, such that $\lambda(u) = \int_\Omega u \, d\mu$ for every $u \in C_0(\Omega)$. We shall always identify the functional λ with the set function μ.

By a *non-negative Borel measure* on Ω we mean a non-negative, countably additive set function defined in the Borel σ-field of Ω with values in $[0, +\infty]$. It is well known that every non-negative Borel measure which is finite on all compact subsets of Ω is a non-negative Radon measure. We shall always identify a non-negative Borel measure with its completion (see, e.g., [28], Section 13). If μ is a non-negative Borel measure on Ω, we shall use $L^r_\mu(\Omega)$, $1 \le r \le +\infty$, to denote the usual Lebesgue space with respect to the measure μ. We adopt the standard notation $L^r(\Omega)$ when μ is the Lebesgue measure.

We denote by $\mathcal{M}_0^p(\Omega)$ the set of all non-negative Borel measures μ on Ω such that

(i) $\mu(B) = 0$ for every Borel set $B \subseteq \Omega$ with $C_p(B) = 0$,

(ii) $\mu(B) = \inf\{\mu(U) : U \text{ quasi open}, B \subseteq U\}$ for every Borel set $B \subseteq \Omega$.

Property (ii) is a weak regularity property of the measure μ. Since any quasi open set differs from a Borel set by a set of C_p-capacity zero, every quasi open set is μ-measurable for every non-negative Borel measure μ which satisfies (i). Therefore $\mu(U)$ is well defined when U is quasi open, and condition (ii) makes sense.

REMARK 1.2. Our class $\mathcal{M}_0^p(\Omega)$ coincides with the class $\mathcal{M}_p^*(\Omega)$ introduced in [20]. It is slightly different from the classes $\mathcal{M}_p$ and $\mathcal{M}_0^p(\Omega)$ considered in [19] and [36], where condition (ii) is not present. It is well known that every non-negative Radon measure satisfies (ii), while there are examples of non-negative Borel measures which satisfy (i), but do not satisfy (ii). Condition (ii) will be essential in the uniqueness result given by Lemma 5.4, which is used in the proof of the uniqueness of the γ^A-limit (Remark 6.4). □

For every open set $U \subseteq \Omega$ we consider the Borel measure μ_U defined by

$$\mu_U(B) = \begin{cases} 0, & \text{if } C_p(B \backslash U) = 0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.1}$$

As U is open, it is easy to see that this measure belongs to the class $\mathcal{M}_0^p(\Omega)$. The measures μ_U will be useful in the study the asymptotic behaviour of sequences of Dirichlet problems in varying domains (see Remark 2.4 and Theorem 6.6).

If $\mu \in \mathcal{M}_0^p(\Omega)$, then the space $H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ is well defined, since all functions in $H_0^{1,p}(\Omega)$ are defined μ-almost everywhere in Ω. It is easy to see that $H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ is a Banach space with the norm $\|u\|^p_{H_0^{1,p}(\Omega)\cap L_\mu^p(\Omega)} = \|u\|^p_{H_0^{1,p}(\Omega)} + \|u\|^p_{L_\mu^p(\Omega)}$ (see [5], Proposition 2.1).

Finally, we say that a Radon measure ν on Ω belongs to $H^{-1,q}(\Omega)$ if there exists $f \in H^{-1,q}(\Omega)$ such that

$$\langle f, \varphi \rangle = \int_\Omega \varphi \, d\nu \qquad \forall \varphi \in C_0^\infty(\Omega), \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-1,q}(\Omega)$ and $H_0^{1,p}(\Omega)$. We shall always identify f and ν. Note that, by the Riesz theorem, for every non-negative functional $f \in H^{-1,q}(\Omega)$ there exists a non-negative Radon measure ν such that (1.2) holds. It is well known that every non-negative Radon measure which belongs to $H^{-1,q}(\Omega)$ belongs also to $\mathcal{M}_0^p(\Omega)$.

The monotone operator. Let $a\colon \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Borel function satisfying the following homogeneity condition:

$$a(x, t\xi) = |t|^{p-2} t \, a(x, \xi) \tag{1.3}$$

for every $x \in \Omega$, for every $t \in \mathbb{R}$, and for every $\xi \in \mathbb{R}^N$, with the convention $|t|^{p-2}t = 0$ for $t = 0$ and $1 < p < 2$. In the case $2 \leq p < +\infty$ we assume that there exist two constants $c_0 > 0$ and $c_1 > 0$ such that

$$\big(a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2\big) \geq c_0 |\xi_1 - \xi_2|^p, \tag{1.4}$$

$$|a(x, \xi_1) - a(x, \xi_2)| \leq c_1 \big(|\xi_1| + |\xi_2|\big)^{p-2} |\xi_1 - \xi_2| \tag{1.5}$$

for every $x \in \Omega$ and for every ξ_1, $\xi_2 \in \mathbb{R}^N$, where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^N$. In the case $1 < p \leq 2$ we assume that there exist two constants $c_0 > 0$ and $c_1 > 0$ such that

$$\big(a(x, \xi_1) - a(x, \xi_2), \xi_1 - \xi_2\big) \geq c_0 \big(|\xi_1| + |\xi_2|\big)^{p-2} |\xi_1 - \xi_2|^2, \tag{1.6}$$

$$|a(x, \xi_1) - a(x, \xi_2)| \leq c_1 |\xi_1 - \xi_2|^{p-1} \tag{1.7}$$

for every $x \in \Omega$ and for every $\xi_1, \xi_2 \in \mathbb{R}^N$ with $\xi_1 \neq \xi_2$.

In particular, (1.3) implies that

$$a(x, -\xi) = -a(x, \xi) \tag{1.8}$$

for every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$, hence

$$a(x, 0) = 0 \tag{1.9}$$

for every $x \in \Omega$, while (1.4)-(1.7) and (1.9) imply that

$$\big(a(x, \xi), \xi\big) \geq c_0 |\xi|^p, \tag{1.10}$$

$$|a(x, \xi)| \leq c_1 |\xi|^{p-1} \tag{1.11}$$

for every $x \in \Omega$ and for every $\xi \in \mathbb{R}^N$.

We now define the operator $A: H^{1,p}(\Omega) \to H^{-1,q}(\Omega)$ by $Au = -\mathrm{div}\big(a(x, Du)\big)$, i.e.,

$$\langle Au, v\rangle = \int_\Omega (a(x, Du), Dv)\, dx \tag{1.12}$$

for every $u \in H^{1,p}(\Omega)$ and for every $v \in H_0^{1,p}(\Omega)$.

This operator is strongly monotone on $H_0^{1,p}(\Omega)$. The model case is the p-Laplacian $-\Delta_p u = -\mathrm{div}\big(|Du|^{p-2} Du\big)$, which corresponds to the choice $a(x, \xi) = |\xi|^{p-2}\xi$. Conditions (1.4)-(1.7) are satisfied in this case, since

$$\begin{gathered} \big(|\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2, \xi_1 - \xi_2\big) \geq 2^{2-p}|\xi_1 - \xi_2|^p, \\ \big|\, |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2 \big| \leq (p-1)\big(|\xi_1| + |\xi_2|\big)^{p-2}|\xi_1 - \xi_2| \end{gathered} \tag{1.13}$$

for $2 \leq p < +\infty$, while

$$\begin{gathered} \big(|\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2, \xi_1 - \xi_2\big) \geq \big(|\xi_1| + |\xi_2|\big)^{p-2}|\xi_1 - \xi_2|^2, \\ \big|\, |\xi_1|^{p-2}\xi_1 - |\xi_2|^{p-2}\xi_2 \big| \leq 2^{2-p}|\xi_1 - \xi_2|^{p-1} \end{gathered} \tag{1.14}$$

for $1 < p \leq 2$ and $\xi_1 \neq \xi_2$.

2. – Relaxed Dirichlet problems

Estimates for the solutions. Let $\mu \in \mathcal{M}_0^p(\Omega)$, $f \in H^{-1,q}(\Omega)$, and let A be the operator defined by (1.12). We shall consider the following *relaxed Dirichlet problem* (see [22] and [23]): find u such that

$$(2.1)\qquad \begin{cases} u \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)\,, \\ \langle Au, v\rangle + \displaystyle\int_\Omega |u|^{p-2} uv\, d\mu = \langle f, v\rangle \qquad \forall\, v \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)\,. \end{cases}$$

The name "relaxed Dirichlet problem" is motivated by Theorem 6.6 and is generally adopted for this kind of problems (see also the density results proved in [23] and [22]). More generally, given $\psi \in H^{1,p}(\Omega) \cap L_\mu^p(\Omega)$, we shall consider the following problem: find u such that

$$(2.2)\qquad \begin{cases} u \in H^{1,p}(\Omega) \cap L_\mu^p(\Omega)\,, \quad u - \psi \in H_0^{1,p}(\Omega)\,, \\ \langle Au, v\rangle + \displaystyle\int_\Omega |u|^{p-2} uv\, d\mu = \langle f, v\rangle \qquad \forall\, v \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)\,. \end{cases}$$

The existence of a solution of (2.2) is given by the following theorem.

THEOREM 2.1. *Let $\mu \in \mathcal{M}_0^p(\Omega)$, $\psi \in H^{1,p}(\Omega) \cap L_\mu^p(\Omega)$, and $f \in H^{-1,q}(\Omega)$. Then problem (2.2) has a unique solution. Moreover the solution u of (2.2) satisfies the estimate*

$$(2.3)\qquad \int_\Omega |Du|^p dx + \int_\Omega |u|^p d\mu \le C\left(\|f\|^q_{H^{-1,q}(\Omega)} + \int_\Omega |D\psi|^p dx + \int_\Omega |\psi|^p d\mu\right),$$

where C is a constant depending only on p, c_0, c_1.

PROOF. Let $B\colon H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega) \to \big(H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)\big)'$ be the operator defined by

$$\langle Bz, v\rangle = \langle A(z+\psi), v\rangle + \int_\Omega |z+\psi|^{p-2}(z+\psi)v\, d\mu$$

for every $z, v \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$. It is easy to see that B is monotone, continuous, and coercive. As f can be identified with an element of $\big(H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)\big)'$, there exists a solution $z \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ of the equation $Bz = f$ in $\big(H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)\big)'$ (see, e.g., [35], Chapter 2, Theorem 2.1). It is clear that $u = z + \psi$ is a solution of (2.2). The uniqueness follows easily from the monotonicity conditions (1.4), (1.6), (1.13), and (1.14) by a standard argument.

Let us prove the estimate (2.3). If we take $v = u - \psi$ as test function in (2.2), we obtain

$$\langle Au, u-\psi\rangle + \int_\Omega |u|^{p-2} u(u-\psi)\, d\mu = \langle f, u-\psi\rangle\,.$$

Therefore, by the coerciveness condition (1.10) and by the boundedness condition (1.11) we have

$$
\begin{aligned}
c_0 \int_\Omega |Du|^p dx + \int_\Omega |u|^p d\mu \le\ & \|f\|_{H^{-1,q}(\Omega)} \|u-\psi\|_{H_0^{1,p}(\Omega)} \\
& + c_1 \left(\int_\Omega |Du|^p dx\right)^{1/q} \left(\int_\Omega |D\psi|^p dx\right)^{1/p} \\
& + \left(\int_\Omega |u|^p d\mu\right)^{1/q} \left(\int_\Omega |\psi|^p d\mu\right)^{1/p},
\end{aligned}
$$

which implies (2.3) by Young's inequality. □

The following lemma will be used to prove the continuous dependence on f of the solution of (2.2).

LEMMA 2.2. *Let $\mu \in \mathcal{M}_0^p(\Omega)$, let $u_1, u_2 \in H^{1,p}(\Omega) \cap L^p_\mu(\Omega)$, let $\varphi \in H^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $\varphi \ge 0$ q.e. in Ω. If $2 \le p < +\infty$, then*

$$
\begin{aligned}
& c_0 \int_\Omega |Du_1 - Du_2|^p \varphi\, dx + 2^{2-p} \int_\Omega |u_1-u_2|^p \varphi\, d\mu \\
(2.5)\qquad & \le \int_\Omega (a(x,Du_1) - a(x,Du_2), Du_1 - Du_2)\, \varphi\, dx \\
& + \int_\Omega \big(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\big)(u_1-u_2)\varphi\, d\mu .
\end{aligned}
$$

If $1 < p \le 2$, then

$$
\begin{aligned}
& c_0 \left(\int_\Omega |Du_1 - Du_2|^p \varphi\, dx\right)^{2/p} + \left(\int_\Omega |u_1-u_2|^p \varphi\, d\mu\right)^{2/p} \\
(2.6)\qquad & \le K(u_1,u_2,\varphi) \int_\Omega (a(x,Du_1) - a(x,Du_2), Du_1 - Du_2)\, \varphi\, dx \\
& + K(u_1,u_2,\varphi) \int_\Omega \big(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\big)(u_1-u_2)\varphi\, d\mu,
\end{aligned}
$$

where $K(u_1,u_2,\varphi)$ stands for

$$
2\left(\int_\Omega |Du_1|^p \varphi\, dx + \int_\Omega |u_1|^p \varphi\, d\mu + \int_\Omega |Du_2|^p \varphi\, dx + \int_\Omega |u_2|^p \varphi\, d\mu\right)^{(2-p)/p}.
$$

PROOF. If $2 \le p < +\infty$, (2.5) follows from the monotonicity conditions (1.4) and (1.13). Let us consider now the case $1 < p \le 2$. Let $z = u_1 - u_2$.

By the monotonicity condition (1.6) and by Hölder's inequality we have

$$
\begin{aligned}
& c_0^{p/2} \int_\Omega |Dz|^p \varphi \, dx \\
& \leq \left(\int_\Omega (|Du_1| + |Du_2|)^p \varphi \, dx \right)^{(2-p)/2} \\
& \quad \cdot \left(c_0 \int_\Omega (|Du_1| + |Du_2|)^{p-2} |Dz|^2 \varphi \, dx \right)^{p/2} \\
& \leq 2^{(p-1)(2-p)/2} \left(\int_\Omega |Du_1|^p \varphi \, dx + \int_\Omega |Du_2|^p \varphi \, dx \right)^{(2-p)/2} \\
& \quad \cdot \left(\int_\Omega (a(x, Du_1) - a(x, Du_2), Dz) \varphi \, dx \right)^{p/2} .
\end{aligned}
$$

Since $2^{(p-1)(2-p)/p} < 2$, we obtain

$$
\begin{aligned}
& c_0 \left(\int_\Omega |Dz|^p \varphi \, dx \right)^{2/p} \\
& \leq 2 \left(\int_\Omega |Du_1|^p \varphi \, dx + \int_\Omega |Du_2|^p \varphi \, dx \right)^{(2-p)/p} \\
& \quad \cdot \int_\Omega (a(x, Du_1) - a(x, Du_2), Dz) \varphi \, dx .
\end{aligned}
\tag{2.7}
$$

By (1.14) and by Hölder's inequality we have

$$
\begin{aligned}
\int_\Omega |z|^p \varphi \, d\mu & \leq \left(\int_\Omega (|u_1| + |u_2|)^p \varphi \, d\mu \right)^{(2-p)/2} \\
& \quad \cdot \left(\int_\Omega (|u_1| + |u_2|)^{p-2} |z|^2 \varphi \, d\mu \right)^{p/2} \\
& \leq 2^{p/2} \left(\int_\Omega |u_1|^p \varphi \, d\mu + \int_\Omega |u_2|^p \varphi \, d\mu \right)^{(2-p)/2} \\
& \quad \cdot \left(\int_\Omega (|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2) z \varphi \, d\mu \right)^{p/2} .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
& \left(\int_\Omega |z|^p \varphi \, d\mu \right)^{2/p} \\
& \leq 2 \left(\int_\Omega |u_1|^p \varphi \, d\mu + \int_\Omega |u_2|^p \varphi \, d\mu \right)^{(2-p)/p} \\
& \quad \cdot \int_\Omega (|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2) z \varphi \, d\mu .
\end{aligned}
\tag{2.8}
$$

The conclusion follows now from (2.7) and (2.8). □

The following theorem shows that the continuous dependence on f of the solutions u of (2.2) is uniform with respect to μ.

THEOREM 2.3. *Let $\mu \in \mathcal{M}_0^p(\Omega)$, let $\psi \in H^{1,p}(\Omega) \cap L^p_\mu(\Omega)$, let f_1, $f_2 \in H^{-1,q}(\Omega)$, and let u_1, u_2 be the solutions of* (2.2) *corresponding to $f = f_1$ and $f = f_2$. If $2 \leq p < +\infty$, then*

$$
\|u_1 - u_2\|^p_{H_0^{1,p}(\Omega)} + \|u_1 - u_2\|^p_{L^p_\mu(\Omega)} \leq C\, \|f_1 - f_2\|^q_{H^{-1,q}(\Omega)}\,, \tag{2.9}
$$

where C is a constant depending only on p and c_0. If $1 < p \leq 2$, then

$$
\|u_1 - u_2\|^2_{H_0^{1,p}(\Omega)} + \|u_1 - u_2\|^2_{L^p_\mu(\Omega)} \leq C\,\Gamma(f_1, f_2, \psi)\, \|f_1 - f_2\|^2_{H^{-1,q}(\Omega)}\,, \tag{2.10}
$$

where

$$
\Gamma(f_1, f_2, \psi) = \left(\|f_1\|^q_{H^{-1,q}(\Omega)} + \|f_2\|^q_{H^{-1,q}(\Omega)} + \int_\Omega |D\psi|^p dx + \int_\Omega |\psi|^p d\mu \right)^{2(2-p)/p},
$$

and C is a constant depending only on p, c_0, c_1.

PROOF. If we use $v = u_1 - u_2$ as test function in the problems solved by by u_1 and u_2, and if we subtract the corresponding equalities, we obtain

$$
\begin{aligned}
&\langle Au_1 - Au_2, u_1 - u_2\rangle + \int_\Omega \left(|u_1|^{p-2} u_1 - |u_2|^{p-2} u_2\right)(u_1 - u_2)\, d\mu \\
&= \langle f_1 - f_2, u_1 - u_2\rangle \leq \|f_1 - f_2\|_{H^{-1,q}(\Omega)} \|u_1 - u_2\|_{H_0^{1,p}(\Omega)}\,.
\end{aligned} \tag{2.11}
$$

If $2 \leq p < +\infty$, from (2.5) and (2.11) we obtain

$$
c_0 \|u_1 - u_2\|^p_{H_0^{1,p}(\Omega)} + 2^{2-p} \|u_1 - u_2\|^p_{L^p_\mu(\Omega)} \leq \|f_1 - f_2\|_{H^{-1,q}(\Omega)} \|u_1 - u_2\|_{H_0^{1,p}(\Omega)}\,,
$$

which implies (2.9) by Young's inequality.

Let us consider now the case $1 < p \leq 2$. From (2.6) and (2.11) we obtain

$$
\begin{aligned}
&c_0\, \|u_1 - u_2\|^2_{H_0^{1,p}(\Omega)} + \|u_1 - u_2\|^2_{L^p_\mu(\Omega)} \\
&\leq K(u_1, u_2, 1)\, \|f_1 - f_2\|_{H^{-1,q}(\Omega)}\, \|u_1 - u_2\|_{H_0^{1,p}(\Omega)}\,,
\end{aligned} \tag{2.12}
$$

where the constant $K(u_1, u_2, 1)$ is given by Lemma 2.2. By (2.3) we have $K(u_1, u_2, 1)^2 \leq C\,\Gamma(f_1, f_2, \psi)$, and thus (2.10) follows from (2.12) by Cauchy's inequality. □

A connection between classical Dirichlet problems on open subsets of Ω and relaxed Dirichlet problems of the form (2.1) is given by the following remark.

REMARK 2.4. If U is an open subset of Ω, and v is a function of $H_0^{1,p}(\Omega)$ such that $v=0$ q.e. in $\Omega\backslash U$, then the restriction of v to U belongs to $H_0^{1,p}(U)$ (see [3], Theorem 4, and [29], Lemma 4). Conversely, if we extend a function $v \in H_0^{1,p}(U)$ by setting $v=0$ in $\Omega\backslash U$, then v is quasi continuous and belongs to $H_0^{1,p}(\Omega)$. Therefore, if μ is the measure μ_U introduced in (1.1), then a function u in $H_0^{1,p}(\Omega)$ is the solution of problem (2.1) if and only if the restriction of u to U is the solution in of the classical boundary value problem

$$u \in H_0^{1,p}(U)\,, \qquad Au = f \quad \text{in } \mathcal{D}'(U)\,,$$

and, in addition, $u=0$ q.e. in $\Omega\backslash U$. □

Comparison principles. The solutions of relaxed Dirichlet problems satisfy the comparison principles given by the following propositions.

PROPOSITION 2.5. *Let $\mu \in \mathcal{M}_0^p(\Omega)$, let $f \in H^{-1,q}(\Omega)$, and let u be the solution of problem* (2.1). *If $f \geq 0$ in Ω, then $u \geq 0$ q.e. in Ω.*

PROOF. Let $v = -(u \wedge 0)$. Then $v \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ and $v \geq 0$ q.e. in Ω. Since $|u|^{p-2}uv \leq 0$ q.e. in Ω and $\langle f, v\rangle \geq 0$, taking v as test function in (2.1) we obtain $\langle Au, v\rangle \geq 0$. Since $Dv = -Du$ a.e. in $\{u<0\}$ and $Dv=0$ a.e. in $\{u \geq 0\}$, by the coerciveness condition (1.10) we have

$$c_0\|v\|^p_{H_0^{1,p}(\Omega)} = c_0 \int_{\{u<0\}} |Du|^p dx \leq \int_{\{u<0\}} \big(a\,(x, Du)\,,\, Du\big)\, dx = -\langle Au, v\rangle\,.$$

As $\langle Au, v\rangle \geq 0$, we obtain that $v=0$ q.e. in Ω, hence $u \geq 0$ q.e. in Ω. □

PROPOSITION 2.6. *Let f_1, f_2 be Radon measures of $H^{-1,q}(\Omega)$, let μ_1, $\mu_2 \in \mathcal{M}_0^p(\Omega)$, and let u_1, u_2 be the solutions of problem* (2.1) *corresponding to f_1, μ_1 and f_2, μ_2. Assume that $0 \leq f_2$, $f_1 \leq f_2$, and $\mu_2 \leq \mu_1$ in Ω. Then $u_1 \leq u_2$ q.e. in Ω.*

PROOF. By Proposition 2.5 we have that $u_2 \geq 0$ q.e. in Ω. Let $v = (u_1-u_2)^+$. Since $0 \leq v \leq u_1^+$ and $\mu_2 \leq \mu_1$, we have $v \in L^p_{\mu_1}(\Omega) \subseteq L^p_{\mu_2}(\Omega)$. As $\int_\Omega |u_2|^{p-2}u_2 v\, d\mu_2 \leq \int_\Omega |u_2|^{p-2}u_2 v\, d\mu_1$, taking v as test function in the problems solved by u_1 and u_2 and subtracting the corresponding equalities, we obtain

$$\langle Au_1 - Au_2, v\rangle + \int_\Omega \big(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\big) v\, d\mu_1 \leq \langle f_1 - f_2, v\rangle \leq 0\,.$$

Since $\big(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2\big)v \geq 0$ q.e. in Ω by (1.13) and (1.14), we get

$$\langle A(u_2+v) - Au_2, v\rangle = \langle Au_1 - Au_2, v\rangle \leq 0\,.$$

From the monotonicity conditions (1.4) and (1.6) it follows that $v=0$ q.e. in Ω and, consequently, $u_1 \leq u_2$ q.e. in Ω. □

Proposition 2.7. *Let f_1, f_2 be Radon measures of $H^{-1,q}(\Omega)$, let $\mu_1, \mu_2 \in \mathcal{M}_0^p(\Omega)$, and let u_1, u_2 be the solutions of problem* (2.1) *corresponding to f_1, μ_1 and f_2, μ_2. If $|f_1| \leq f_2$ and $\mu_2 \leq \mu_1$ in Ω, then $|u_1| \leq u_2$ q.e. in Ω.*

Proof. By Proposition 2.6 we have $u_1 \leq u_2$ q.e. in Ω. By (1.8) the function $-u_1$ is the solution of (2.1) corresponding to $-f_1$ and μ_1, and so, by Proposition 2.6, we have also $-u_1 \leq u_2$ q.e. in Ω. □

Estimates involving auxiliary Radon measures. We consider now some further estimates for the gradients of the solutions u of (2.2) and for the corresponding fields $a(x, Du)$. We begin by proving that, if $f \in L^q(\Omega)$, then the solutions of (2.2) are actually solutions, in the sense of distributions, of a new equation involving a Radon measure λ, which depends on u, μ, f, and whose variation on compact sets can be estimated in terms of $\|f\|_{L^q(\Omega)}$ and $\|Du\|_{L^p(\Omega,\mathbb{R}^N)}$.

Proposition 2.8. *Let $\mu \in \mathcal{M}_0^p(\Omega)$, let $f \in L^q(\Omega)$, and let u be the solution of problem* (2.2) *for some $\psi \in H^{1,p}(\Omega) \cap L^p_\mu(\Omega)$. Let λ, λ_1, λ_2 be the elements of $H^{-1,q}(\Omega)$ defined by $Au + \lambda = f$, $A(u^+) + \lambda_1 = f^+$, $A(-u^-) - \lambda_2 = -f^-$. Then λ, λ_1, λ_2 are Radon measures, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda = \lambda_1 - \lambda_2$, and $|\lambda| \leq \lambda_1 + \lambda_2$. Moreover for every compact set $K \subseteq \Omega$ we have*

$$|\lambda|(K) \leq C_p(K)^{1/p}\left(2c_1\|Du\|^{p-1}_{L^p(\Omega,\mathbb{R}^N)} + c_{p,\Omega}\|f\|_{L^q(\Omega)}\right), \tag{2.13}$$

where c_1 is the constant in (1.11) *and $c_{p,\Omega}$ is a constant depending only on p and Ω.*

Proof. Let $v \in H_0^{1,p}(\Omega)$ with $v \geq 0$ q.e. in Ω and let $v_n = (\frac{1}{n}v) \wedge u^+$. Then $v_n \geq 0$ q.e. in Ω and $v_n \in H_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)$. As $|u|^{p-2}uv_n \geq 0$ q.e. in Ω and $fv_n \leq f^+v_n$ a.e. in Ω, by taking v_n as test function in problem (2.2) we obtain $\langle Au, v_n\rangle \leq \int_\Omega f^+v_n\,dx \leq \frac{1}{n}\int_\Omega f^+v\,dx$. Since $Dv_n = \frac{1}{n}Dv$ a.e. in $\{v < nu^+\}$ and $Dv_n = Du^+$ a.e. in $\{v \geq nu^+\}$, we have

$$\frac{1}{n}\int_{\{v<nu^+\}}\big(a(x,Du), Dv\big)\,dx + \int_{\{v\geq nu^+\}}\big(a(x,Du), Du^+\big)\,dx \leq \frac{1}{n}\int_\Omega f^+v\,dx\,.$$

As $(a(x,Du), Du^+) = (a(x,Du^+), Du^+) \geq 0$, we get

$$\int_{\{v<nu^+\}}\big(a(x,Du), Dv\big)\,dx \leq \int_\Omega f^+v\,dx\,.$$

Taking the limit as $n \to \infty$, we obtain

$$\int_{\{u^+>0\}}\big(a(x,Du), Dv\big)\,dx \leq \int_\Omega f^+v\,dx\,.$$

Since $Du^+ = Du$ a.e. in $\{u^+ > 0\}$ and $Du^+ = 0$ a.e. in $\{u^+ = 0\}$, from (1.9) it follows that

$$\int_\Omega\big(a(x,Du^+), Dv\big)\,dx \leq \int_\Omega f^+v\,dx$$

for every $v \in H_0^{1,p}(\Omega)$ with $v \geq 0$ q.e. in Ω. This implies that $\lambda_1 \geq 0$, hence λ_1 is a Radon measure.

In a similar way we deduce that λ_2 is a non-negative Radon measure. By (1.9) we have $Au = A(u^+) + A(-u^-)$, hence $\lambda = \lambda_1 - \lambda_2$, which implies $|\lambda| \leq \lambda_1 + \lambda_2$. By the upper bound (1.11) we have

$$\|\lambda_1\|_{H^{-1,q}(\Omega)} \leq c_1 \|Du^+\|^{p-1}_{L^p(\Omega,\mathbb{R}^N)} + C\,\|f^+\|_{L^q(\Omega)}\,,$$

where C depends only on p and Ω. The same estimate holds for λ_2. To prove (2.13), for every $\varepsilon > 0$ we fix a function $z \in H_0^{1,p}(\Omega)$ such that $z \geq 0$ q.e. in Ω, $z \geq 1$ q.e. in a neighbourhood of K, and $\|z\|^p_{H_0^{1,p}(\Omega)} < C_p(K) + \varepsilon$. Then

$$\begin{aligned}
|\lambda|(K) &\leq \int_\Omega z\,d\lambda_1 + \int_\Omega z\,d\lambda_2 \\
&\leq \|z\|_{H_0^{1,p}(\Omega)}\Big(\|\lambda_1\|_{H^{-1,q}(\Omega)} + \|\lambda_2\|_{H^{-1,q}(\Omega)}\Big) \\
&\leq 2\,\big(C_p(K)+\varepsilon\big)^{1/p}\Big(c_1\|Du\|^{p-1}_{L^p(\Omega,\mathbb{R}^N)} + C\,\|f\|_{L^q(\Omega)}\Big)\,.
\end{aligned}$$

Taking the limit as $\varepsilon \to 0$ we obtain (2.13). □

REMARK 2.9. Under the assumptions of Proposition 2.8, if $f \geq 0$ then $u = u^+$ (Proposition 2.5) and $\lambda = \lambda_1$. Therefore in this case $\lambda \geq 0$ and hence $Au \leq f$ in Ω in the sense of $H^{-1,q}(\Omega)$. □

The following theorem, together with Proposition 2.8, will be used in the proof of the main result of this section (Theorem 2.11): if u_n are the solutions of (2.2) corresponding to some measures $\mu_n \in \mathcal{M}_0^p(\Omega)$, and (u_n) converges weakly in $H^{1,p}(\Omega)$, then (u_n) converges strongly in $H^{1,r}(\Omega)$ for every $r < p$.

THEOREM 2.10. *Let (g_n) be a sequence in $H^{-1,q}(\Omega)$, let (λ_n) be a sequence of Radon measures, and, for every $n \in \mathbb{N}$, let $u_n \in H^{1,p}(\Omega)$ be a solution of the equation*

$$Au_n = g_n + \lambda_n \quad \text{in } \mathcal{D}'(\Omega)\,.$$

Assume that (u_n) converges weakly in $H^{1,p}(\Omega)$ to some function u, (g_n) converges strongly in $H^{-1,q}(\Omega)$, and (λ_n) is bounded in the space of Radon measures, i.e., for every compact set $K \subseteq \Omega$ there exists a constant C_K such that

$$|\lambda_n|(K) \leq C_K \qquad \forall n \in \mathbb{N}\,.$$

Then (u_n) converges to u strongly in $H^{1,r}(\Omega)$ for every $r < p$, and $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ and strongly in $L^s(\Omega, \mathbb{R}^N)$ for every $s < q$.

PROOF. By Theorem 2.1 of [4] the sequence (u_n) converges to u strongly in $H^{1,r}(\Omega)$ for every $r < p$. Let us fix a subsequence, still denoted by (u_n), such that (Du_n) converges to Du pointwise a.e. in Ω. As $a(x,\cdot)$ is continuous, we deduce that $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ pointwise a.e. in Ω. By the upper bound (1.11) this sequence is bounded in $L^q(\Omega, \mathbb{R}^N)$, and so, by the dominated convergence theorem, $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ and strongly in $L^s(\Omega, \mathbb{R}^N)$ for every $s < q$. □

As a consequence of Proposition 2.8 and Theorem 2.10 we have the following result.

THEOREM 2.11. *Let (g_n) be a sequence in $H^{-1,q}(\Omega)$ which converges strongly to some $g \in H^{-1,q}(\Omega)$, let (μ_n) be a sequence in $\mathcal{M}_0^p(\Omega)$, and let (ψ_n) be a sequence in $H^{1,p}(\Omega)$ such that $\int_\Omega |\psi_n|^p d\mu_n \leq M$ for a suitable constant M independent of n. Assume that the solution u_n of (2.2) corresponding to $\mu = \mu_n$, $f = g_n$, $\psi = \psi_n$ converges weakly in $H^{1,p}(\Omega)$ to some function u. Then (u_n) converges to u strongly in $H^{1,r}(\Omega)$ for every $r < p$, and $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ and strongly in $L^s(\Omega, \mathbb{R}^N)$ for every $s < q$.*

PROOF. Given $\varepsilon \in \,]0,1[$, we fix a function $h \in L^q(\Omega)$ such that $\|h - g\|_{H^{-1,q}(\Omega)} < \varepsilon$ and we consider the solutions z_n of (2.2) corresponding to $\mu = \mu_n$, $f = h$, $\psi = \psi_n$. By Theorem 2.3 we have

$$\|z_n - u_n\|_{H_0^{1,p}(\Omega)} \leq C\,\|h - g_n\|^\alpha_{H^{-1,q}(\Omega)}, \tag{2.14}$$

where $\alpha = 1/(p-1)$ for $p \geq 2$ and $\alpha = 1$ for $1 < p \leq 2$, while C is a constant depending only on p, c_0, c_1, M, $\sup_n \|g_n\|_{H^{-1,q}(\Omega)}$. This implies, in particular, that (z_n) is bounded in $H^{1,p}(\Omega)$. Therefore, passing to a subsequence, we may assume that (z_n) converges weakly in $H^{1,p}(\Omega)$ to some function z, and (2.14) gives

$$\|z - u\|_{H_0^{1,p}(\Omega)} \leq C\,\|h - g\|^\alpha_{H^{-1,q}(\Omega)} < C\,\varepsilon^\alpha . \tag{2.15}$$

By Proposition 2.8 there exists a sequence (λ_n) of Radon measures of $H^{-1,q}(\Omega)$ such that $Az_n + \lambda_n = h$ in Ω. By (2.13) for every compact set $K \subseteq \Omega$ the sequence $\big(|\lambda_n|(K)\big)$ is bounded. Therefore Theorem 2.10 implies that (z_n) converges to z strongly in $H^{1,r}(\Omega)$ for every $r < p$. Using Poincaré's and Hölder's inequalities we obtain

$$\|u_n - u\|_{H^{1,r}(\Omega)} \leq C_{p,\Omega}\|u_n - z_n\|_{H_0^{1,p}(\Omega)} + \|z_n - z\|_{H^{1,r}(\Omega)} + C_{p,\Omega}\|z - u\|_{H_0^{1,p}(\Omega)},$$

which, together with (2.14) and (2.15), gives

$$\limsup_{n\to\infty} \|u_n - u\|_{H^{1,r}(\Omega)} \leq 2\,C_{p,\Omega}\,C\,\varepsilon^\alpha .$$

As ε is arbitrary, we obtain that (u_n) converges to u strongly in $H^{1,r}(\Omega)$. The convergence of $\big(a(x, Du_n)\big)$ to $a(x, Du)$ can be proved as in Theorem 2.10. □

3. – Corrector result

Definition of the corrector. Let (μ_n) be a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $f \in L^\infty(\Omega)$. For every $n \in \mathbb{N}$ let us consider the solution u_n of the problem

$$
(3.1) \qquad \begin{cases} u_n \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \\ \langle Au_n, v\rangle + \displaystyle\int_\Omega |u_n|^{p-2} u_n v \, d\mu_n \\ \qquad = \displaystyle\int_\Omega f v \, dx \quad \forall\, v \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,. \end{cases}
$$

By estimate (2.3) the sequence (u_n) is bounded in $H_0^{1,p}(\Omega)$, thus we may assume that (u_n) converges weakly in $H_0^{1,p}(\Omega)$ to some function u. By Theorem 2.11 the sequence (Du_n) converges to Du weakly in $L^p(\Omega, \mathbb{R}^N)$ and strongly in $L^r(\Omega, \mathbb{R}^N)$ for every $r < p$.

We shall improve this convergence of the gradients and we shall obtain their strong convergence in $L^p(\Omega, \mathbb{R}^N)$ by means of a corrector: we shall define a sequence of Borel functions $P_n\colon \Omega \to \mathbb{R}^N$, depending on the sequence (μ_n), but independent of f, u, u_n, such that, if R_n is defined by

$$
(3.2) \qquad Du_n = Du + uP_n + R_n \qquad \text{a.e. in } \Omega\,,
$$

then the sequence (R_n) converges to 0 strongly in $L^p(\Omega, \mathbb{R}^N)$. This fact means that the oscillations around Du of the sequence of the gradients (Du_n) are determined, up to a remainder which tends to 0 strongly in $L^p(\Omega, \mathbb{R}^N)$, only by the values of the limit function u and by the sequence of correctors (P_n), which depends only on the sequence (μ_n).

In order to construct P_n, let us consider for every $n \in \mathbb{N}$ the solution w_n of the problem

$$
(3.3) \qquad \begin{cases} w_n \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \\ \langle Aw_n, v\rangle + \displaystyle\int_\Omega |w_n|^{p-2} w_n v \, d\mu_n \\ \qquad = \displaystyle\int_\Omega v \, dx \qquad \forall\, v \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,. \end{cases}
$$

By estimate (2.3) the sequence (w_n) is bounded in $H_0^{1,p}(\Omega)$, thus we may assume that (w_n) converges weakly in $H_0^{1,p}(\Omega)$ to some function w. Then we define the Borel function $P_n\colon \Omega \to \mathbb{R}^N$ by

$$
(3.4) \qquad P_n(x) = \begin{cases} \dfrac{1}{w(x)}\big(Dw_n(x) - Dw(x)\big)\,, & \text{if } w(x) > 0, \\ 0\,, & \text{if } w(x) = 0. \end{cases}
$$

We are now in a position to state the main theorem of this section.

THEOREM 3.1. *Let (μ_n) be a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $f \in L^\infty(\Omega)$. For every $n \in \mathbb{N}$, let u_n and w_n be the solutions of problems* (3.1) *and* (3.3). *Assume that (u_n) and (w_n) converge weakly in $H_0^{1,p}(\Omega)$ to some functions u and w, and define P_n and R_n by* (3.4) *and* (3.2). *Then (R_n) converges to* 0 *strongly in $L^p(\Omega, \mathbb{R}^N)$.*

REMARK 3.2. Let w_0 be the unique function of $H_0^{1,p}(\Omega)$ such that $Aw_0 = 1$ in Ω. By the comparison principle (Proposition 2.7) and by the homogeneity condition (1.3) we have $|u_n| \le c\, w_n \le c\, w_0$ q.e. in Ω, with $c = \|f\|_{L^\infty(\Omega)}^{1/(p-1)}$, hence $|u| \le c\, w \le c\, w_0$ q.e. in Ω. As $w_0 \in L^\infty(\Omega)$ (see [34], Chapter 4, Theorem 7.1), the functions u and w belong to $L^\infty(\Omega)$, and the sequences (u_n) and (w_n) are bounded in $L^\infty(\Omega)$. □

REMARK 3.3. Before proving Theorem 3.1, let us observe that, if $f \in L^\infty(\Omega)$, then the sequence (R_n) defined by (3.2) and (3.4) converges to 0 weakly in $L^p(\Omega, \mathbb{R}^N)$ and strongly in $L^r(\Omega, \mathbb{R}^N)$ for every $r < p$. Indeed $\frac{u}{w} \in L^\infty(\{w > 0\})$ by Remark 3.2 and

$$(3.5)\qquad \begin{aligned} R_n &= Du_n - Du \quad \text{in } \{w = 0\}\,, \\ R_n &= (Du_n - Du) - \tfrac{u}{w}(Dw_n - Dw) \quad \text{in } \{w > 0\}\,, \end{aligned}$$

while $(Du_n - Du)$ and $(Dw_n - Dw)$ converge to 0 weakly in $L^p(\Omega, \mathbb{R}^N)$ and strongly in $L^r(\Omega, \mathbb{R}^N)$ for every $r < p$ by the definition of u and w and by Theorem 2.11. The result of Theorem 3.1 is thus to assert that the convergence of (R_n) is actually strong in $L^p(\Omega, \mathbb{R}^N)$.

The corrector result of Theorem 3.1 is formally equivalent to the strong convergence of $(u_n - \frac{uw_n}{w})$ to 0 in $H^{1,p}(\Omega)$. This assertion, which is only formal since w can vanish on a set of positive Lebesgue measure, becomes correct in $H^{1,p}(U)$ if U is some open subset of Ω such that $w > \varepsilon > 0$ in U (see Lemma 3.4 and (3.7)). □

Three lemmas. To prove Theorem 3.1 we use the following lemmas.

LEMMA 3.4. *Assume that the hypotheses of Theorem 3.1 are satisfied. For every $\varepsilon > 0$ define $U_\varepsilon = \{w > \varepsilon\} \cap \{|u| > \varepsilon w\}$. Then for every $\varepsilon > 0$ the functions $\frac{uw_n}{w \vee \varepsilon}$ belong to $H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)$ and one has*

$$(3.6)\qquad \lim_{n\to\infty} \left(\int_{U_\varepsilon} \left| Du_n - D\left(\frac{uw_n}{w \vee \varepsilon}\right)\right|^p dx + \int_{U_\varepsilon} \left| u_n - \frac{uw_n}{w \vee \varepsilon}\right|^p d\mu_n \right) = 0\,.$$

Note that $w \vee \varepsilon = w$ in U_ε and thus (3.6) implies formally that $(Du_n - D(\frac{uw_n}{w}))$ converges to 0 strongly in $L^p(U_\varepsilon, \mathbb{R}^N)$. As $D(\frac{uw_n}{w})$ is not always well defined, the rigorous formulation of the previous statement is that (3.6) yields

$$Du_n - \frac{w_n}{w} Du - \frac{u}{w}\left(Dw_n - \frac{w_n}{w} Dw\right) \longrightarrow 0$$

strongly in $L^p(U_\varepsilon, \mathbb{R}^N)$. Since $(\frac{w_n}{w})$ is bounded in $L^\infty(U_\varepsilon)$ (Remark 3.2) and (up to a subsequence) converges to 1 almost everywhere in U_ε, while $\frac{u}{w} \in L^\infty(U_\varepsilon)$ (Remark 3.2), we conclude that for $\varepsilon > 0$ fixed (3.6) implies

$$Du_n - \left(Du + \frac{u}{w}(Dw_n - Dw)\right) \longrightarrow 0 \tag{3.7}$$

strongly in $L^p(U_\varepsilon, \mathbb{R}^N)$.

PROOF OF LEMMA 3.4. Define for every $\varepsilon > 0$ and for every $n \in \mathbb{N}$ the functions

$$u_n^\varepsilon = \frac{uw_n}{w \vee \varepsilon} \qquad \text{and} \qquad r_n^\varepsilon = u_n - u_n^\varepsilon .$$

First step. In this step we shall prove that these functions belong to $H_0^{1,p}(\Omega) \cap L^\infty(\Omega) \cap L^p_{\mu_n}(\Omega)$ and study their convergence as $n \to \infty$ for $\varepsilon > 0$ fixed.

The functions u and $\frac{1}{w\vee\varepsilon}$ belong to $H^{1,p}(\Omega) \cap L^\infty(\Omega)$, while the sequences (u_n) and (w_n) are bounded in $L^\infty(\Omega)$ (Remark 3.2) and converge to u and w weakly in $H_0^{1,p}(\Omega)$. Since any function in $H^{1,p}(\Omega) \cap L^\infty(\Omega)$ belongs to $L^\infty_\mu(\Omega)$ for any $\mu \in \mathcal{M}_0^p(\Omega)$, $\frac{u}{w\vee\varepsilon}$ belongs to $L^\infty_\mu(\Omega)$. Thus u_n^ε and r_n^ε belong to $H_0^{1,p}(\Omega) \cap L^\infty(\Omega) \cap L^p_{\mu_n}(\Omega)$ and (u_n^ε) and (r_n^ε) are bounded in $L^\infty(\Omega)$ and converge to $\frac{uw}{w\vee\varepsilon}$ and $u - \frac{uw}{w\vee\varepsilon}$ weakly in $H_0^{1,p}(\Omega)$. By Theorem 2.11 the sequences (u_n) and (w_n) converge to u and w strongly in $H_0^{1,r}(\Omega)$ for every $r < p$, and so (u_n^ε) converges to $\frac{uw}{w\vee\varepsilon}$ strongly in $H_0^{1,r}(\Omega)$. Therefore, passing to a subsequence, we may assume that (u_n), (w_n), (Du_n), (Dw_n), (Du_n^ε) converge to u, w, Du, Dw, $D(\frac{uw}{w\vee\varepsilon})$ almost everywhere in Ω. Then the upper bound (1.11) and the continuity of $a(x,\cdot)$ imply that $\big(a(x, Du_n)\big)$, $\big(a(x, Dw_n)\big)$, $\big(a\left(x, Du_n^\varepsilon\right)\big)$ converge to $a(x, Du)$, $a(x, Dw)$, $a\left(x, D(\frac{uw}{w\vee\varepsilon})\right)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ and almost everywhere in Ω. As $u - \frac{uw}{w\vee\varepsilon} = 0$ a.e. in U_ε, we obtain that (r_n^ε) converges to 0 strongly in $L^p(U_\varepsilon)$, (Dr_n^ε) converges to 0 weakly in $L^p(U_\varepsilon, \mathbb{R}^N)$, and $\big(a\left(x, Du_n^\varepsilon\right)\big)$ converges to $a(x, Du)$ weakly in $L^q(U_\varepsilon, \mathbb{R}^N)$.

Let us fix a function $\varphi \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ such that $0 \le \varphi \le 1$ q.e. in Ω, $\varphi = 1$ q.e. in $U_{2\varepsilon}$, $\varphi = 0$ q.e. in $\Omega \backslash U_\varepsilon$, and hence $D\varphi = 0$ a.e. in $\Omega \backslash U_\varepsilon$. For instance, we can take $\varphi = \Phi_\varepsilon(w)\Phi_\varepsilon(\frac{|u|}{w\vee\varepsilon})$, where $\Phi_\varepsilon : \mathbb{R} \to \mathbb{R}$ is the Lipschitz function defined by $\Phi_\varepsilon(t) = 0$ for $t \le \varepsilon$, $\Phi_\varepsilon(t) = \frac{t}{\varepsilon} - 1$ for $\varepsilon \le t \le 2\varepsilon$, $\Phi_\varepsilon(t) = 1$ for $t \ge 2\varepsilon$. By the previous remarks the sequence $(r_n^\varepsilon \varphi)$ converges to 0 weakly in $H_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$.

Second step. Let us define

$$\begin{aligned} \mathcal{E}_n^\varepsilon = {} & \int_\Omega \left(a(x, Du_n) - a\left(x, Du_n^\varepsilon\right), Dr_n^\varepsilon\right) \varphi \, dx \\ & + \int_\Omega \left(|u_n|^{p-2} u_n - \left|u_n^\varepsilon\right|^{p-2} u_n^\varepsilon\right) r_n^\varepsilon \varphi \, d\mu_n . \end{aligned}$$

In this step we shall prove that for $\varepsilon > 0$ fixed we have

$$\lim_{n\to\infty} \mathcal{E}_n^\varepsilon = 0 . \tag{3.8}$$

For that we write $\mathcal{E}_n^\varepsilon$ as

$$\begin{aligned}
\mathcal{E}_n^\varepsilon &= \int_\Omega \left(a\,(x, Du_n) - a\left(x, Du_n^\varepsilon\right), D(r_n^\varepsilon \varphi)\right) dx + \int_\Omega |u_n|^{p-2} u_n r_n^\varepsilon \varphi \, d\mu_n \\
&\quad - \int_\Omega |u_n^\varepsilon|^{p-2} u_n^\varepsilon r_n^\varepsilon \varphi \, d\mu_n - \int_{U_\varepsilon} \left(a\,(x, Du_n) - a\left(x, Du_n^\varepsilon\right), D\varphi\right) r_n^\varepsilon \, dx \\
&= \int_\Omega \left(a\,(x, Du_n), D(r_n^\varepsilon \varphi)\right) dx + \int_\Omega |u_n|^{p-2} u_n r_n^\varepsilon \varphi \, d\mu_n \\
&\quad - \int_\Omega \left(a\left(x, \frac{u}{w \vee \varepsilon} Dw_n\right), D(r_n^\varepsilon \varphi)\right) dx - \int_\Omega |u_n^\varepsilon|^{p-2} u_n^\varepsilon r_n^\varepsilon \varphi \, d\mu_n \\
&\quad + \int_{U_\varepsilon} \left(a\left(x, \frac{u}{w \vee \varepsilon} Dw_n\right) - a\left(x, Du_n^\varepsilon\right), D(r_n^\varepsilon \varphi)\right) dx \\
&\quad - \int_{U_\varepsilon} \left(a\,(x, Du_n) - a\left(x, Du_n^\varepsilon\right), D\varphi\right) r_n^\varepsilon \, dx .
\end{aligned} \tag{3.9}$$

Since $\varphi = 0$ q.e. in $\Omega \backslash U_\varepsilon$, the function $z = \left|\frac{u}{w\vee\varepsilon}\right|^{p-2} \frac{u}{w\vee\varepsilon}\varphi$ belongs to $H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. This is trivial if $p \geq 2$. When $1 < p < 2$ it is enough to observe that $z = \Psi_\varepsilon(\frac{u}{w\vee\varepsilon})\,\varphi$, where Ψ_ε is the locally Lipschitz function defined by (4.3) below. By the homogeneity property (1.3) we obtain

$$\begin{aligned}
&- \int_\Omega \left(a\left(x, \frac{u}{w \vee \varepsilon} Dw_n\right), D(r_n^\varepsilon \varphi)\right) dx \\
&= - \int_\Omega \left(a\,(x, Dw_n), \left|\frac{u}{w \vee \varepsilon}\right|^{p-2} \frac{u}{w \vee \varepsilon} D(r_n^\varepsilon \varphi)\right) dx \\
&= - \int_\Omega \left(a\,(x, Dw_n), D\left(\left|\frac{u}{w \vee \varepsilon}\right|^{p-2} \frac{u}{w \vee \varepsilon} r_n^\varepsilon \varphi\right)\right) dx \\
&\quad + (p-1) \int_{U_\varepsilon} \left(a\,(x, Dw_n), D\left(\frac{u}{w \vee \varepsilon}\right)\right) \left|\frac{u}{w \vee \varepsilon}\right|^{p-2} r_n^\varepsilon \varphi \, dx .
\end{aligned}$$

As $w_n \frac{u}{w\vee\varepsilon} = u_n^\varepsilon$, taking $v = \left|\frac{u}{w\vee\varepsilon}\right|^{p-2} \frac{u}{w\vee\varepsilon} r_n^\varepsilon \varphi$ in (3.3) we get

$$\begin{aligned}
&- \int_\Omega \left(a\,(x, Dw_n), D\left(\left|\frac{u}{w \vee \varepsilon}\right|^{p-2} \frac{u}{w \vee \varepsilon} r_n^\varepsilon \varphi\right)\right) dx \\
&- \int_\Omega |u_n^\varepsilon|^{p-2} u_n^\varepsilon r_n^\varepsilon \varphi \, d\mu_n \\
&= - \int_{U_\varepsilon} \left|\frac{u}{w \vee \varepsilon}\right|^{p-2} \frac{u}{w \vee \varepsilon} r_n^\varepsilon \varphi \, dx ,
\end{aligned}$$

and so, taking $v = r_n^\varepsilon \varphi$ in (3.1), from (3.9) we obtain

$$
\begin{aligned}
\mathcal{E}_n^\varepsilon &= \int_{U_\varepsilon} f r_n^\varepsilon \varphi\, dx - \int_{U_\varepsilon} \left|\frac{u}{w \vee \varepsilon}\right|^{p-2} \frac{u}{w \vee \varepsilon} r_n^\varepsilon \varphi\, dx \\
&\quad + (p-1) \int_{U_\varepsilon} \left(a(x, Dw_n), D\left(\frac{u}{w \vee \varepsilon}\right)\right) \left|\frac{u}{w \vee \varepsilon}\right|^{p-2} r_n^\varepsilon \varphi\, dx \\
&\quad + \int_{U_\varepsilon} \left(a\left(x, \frac{u}{w \vee \varepsilon} Dw_n\right) - a\left(x, Du_n^\varepsilon\right), D(r_n^\varepsilon \varphi)\right) dx \\
&\quad - \int_{U_\varepsilon} \left(a(x, Du_n) - a\left(x, Du_n^\varepsilon\right), D\varphi\right) r_n^\varepsilon\, dx \\
&= \mathcal{J}_n^1 - \mathcal{J}_n^2 + \mathcal{J}_n^3 + \mathcal{J}_n^4 - \mathcal{J}_n^5 .
\end{aligned}
$$

Since $|\frac{u}{w \vee \varepsilon}|^{p-2} \in L^\infty(U_\varepsilon)$, and (r_n^ε) is bounded in $L^\infty(\Omega)$ and converges to 0 strongly in $L^p(U_\varepsilon)$, while the sequences $(a(x, Dw_n))$, $(a(x, Du_n))$, $(a(x, Du_n^\varepsilon))$ converge weakly in $L^q(U_\varepsilon, \mathbb{R}^N)$, it follows that the sequences $(\mathcal{J}_n^1)$, $(\mathcal{J}_n^2)$, $(\mathcal{J}_n^3)$, $(\mathcal{J}_n^5)$ tend to 0. To conclude the proof of (3.8) it is enough to show that

$$
\lim_{n\to\infty} \int_{U_\varepsilon} \left(a\left(x, \frac{u}{w \vee \varepsilon} Dw_n\right) - a\left(x, Du_n^\varepsilon\right), D(r_n^\varepsilon \varphi)\right) dx = 0 . \tag{3.10}
$$

Since (Dw_n) and (Du_n^ε) converge to Dw and Du almost everywhere in U_ε, it follows that

$$
\begin{aligned}
&\lim_{n\to\infty} \left(a\left(x, \frac{u}{w \vee \varepsilon} Dw_n\right) - a\left(x, Du_n^\varepsilon\right)\right) \\
&= a\left(x, \frac{u}{w \vee \varepsilon} Dw\right) - a(x, Du) \quad \text{a.e. in } U_\varepsilon .
\end{aligned} \tag{3.11}
$$

Let us prove that $|a(x, \frac{u}{w\vee\varepsilon} Dw_n) - a(x, Du_n^\varepsilon)|^q$ is equi-integrable. Let us consider the case $2 \le p < +\infty$. Since $\frac{u}{w\vee\varepsilon} \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and (w_n) is bounded in $L^\infty(\Omega)$, by the continuity condition (1.5) there exists a constant C such that

$$
\begin{aligned}
&\left| a\left(x, \frac{u}{w \vee \varepsilon} Dw_n\right) - a\left(x, Du_n^\varepsilon\right)\right|^q \\
&\le c_1^q \left(2\left|\frac{u}{w \vee \varepsilon} Dw_n\right| + \left|w_n D\left(\frac{u}{w \vee \varepsilon}\right)\right|\right)^{q(p-2)} \left|w_n D\left(\frac{u}{w \vee \varepsilon}\right)\right|^q \\
&\le C\left(|Dw_n|^{q(p-2)} \left|D\left(\frac{u}{w \vee \varepsilon}\right)\right|^q + \left|D\left(\frac{u}{w \vee \varepsilon}\right)\right|^p\right) .
\end{aligned} \tag{3.12}
$$

By Hölder's inequality for every measurable set $E \subseteq \Omega$ we have

$$\int_E |Dw_n|^{q(p-2)} \left|D(\frac{u}{w \vee \varepsilon})\right|^q dx$$
$$\leq \left(\int_\Omega |Dw_n|^p dx\right)^{(p-2)/(p-1)} \left(\int_E \left|D\left(\frac{u}{w \vee \varepsilon}\right)\right|^p dx\right)^{q/p}.$$

By (3.12) this inequality shows that $\left|a\left(x, \frac{u}{w\vee\varepsilon} Dw_n\right) - a\left(x, Du_n^\varepsilon\right)\right|^q$ is equi-integrable.

In the case $1 < p \leq 2$, by the continuity condition (1.7) we have

$$\left|a\left(x, \frac{u}{w \vee \varepsilon} Dw_n\right) - a\left(x, Du_n^\varepsilon\right)\right|^q \leq c_1^q \left|w_n D\left(\frac{u}{w \vee \varepsilon}\right)\right|^p,$$

and so the sequence $\left|a\left(x, \frac{u}{w\vee\varepsilon} Dw_n\right) - a\left(x, Du_n^\varepsilon\right)\right|^q$ is dominated by an integrable function, since $\frac{u}{w\vee\varepsilon} \in H_0^{1,p}(\Omega)$ and (w_n) is bounded in $L^\infty(\Omega)$.

Therefore, in both cases $\left|a\left(x, \frac{u}{w\vee\varepsilon} Dw_n\right) - a\left(x, Du_n^\varepsilon\right)\right|^q$ is equi-integrable, and thus, by the dominated convergence theorem, (3.11) implies that $\left(a\left(x, \frac{u}{w\vee\varepsilon} Dw_n\right) - a\left(x, Du_n^\varepsilon\right)\right)$ converges to $a\left(x, \frac{u}{w\vee\varepsilon} Dw\right) - a\left(x, Du\right)$ strongly in $L^q(\Omega, \mathbb{R}^N)$. As $\left(D(r_n^\varepsilon \varphi)\right)$ converges to 0 weakly in $L^p(\Omega, \mathbb{R}^N)$, we obtain (3.10), which implies (3.8).

Third step. If $2 \leq p < +\infty$, then (2.5) gives

$$c_0 \int_\Omega |Dr_n^\varepsilon|^p \varphi\, dx + 2^{2-p} \int_\Omega |r_n^\varepsilon|^p \varphi\, d\mu_n \leq \mathcal{E}_n^\varepsilon. \tag{3.13}$$

If $1 < p \leq 2$, we note that the sequences $\left(\|u_n\|_{L^p_{\mu_n}(\Omega)}\right)$ and $\left(\|w_n\|_{L^p_{\mu_n}(\Omega)}\right)$ are bounded by estimate (2.3). Since u and $\frac{1}{w\vee\varepsilon}$ belong to $H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, we conclude that $\left(\|u_n^\varepsilon\|_{L^p_{\mu_n}(\Omega)}\right)$ is bounded too. Since (u_n) and (u_n^ε) are bounded in $H_0^{1,p}(\Omega)$, by (2.6) there exists a constant K such that

$$\left(\int_\Omega |Dr_n^\varepsilon|^p \varphi\, dx\right)^{2/p} + \left(\int_\Omega |r_n^\varepsilon|^p \varphi\, d\mu_n\right)^{2/p} \leq K\, \mathcal{E}_n^\varepsilon. \tag{3.14}$$

Taking (3.13) and (3.14) into account, in both cases we obtain from (3.8) that

$$\lim_{n\to\infty} \left(\int_\Omega |Dr_n^\varepsilon|^p \varphi\, dx + \int_\Omega |r_n^\varepsilon|^p \varphi d\mu_n\right) = 0,$$

hence

$$\lim_{n\to\infty} \left(\int_{U_{2\varepsilon}} |Dr_n^\varepsilon|^p dx + \int_{U_{2\varepsilon}} |r_n^\varepsilon|^p d\mu_n\right) = 0. \tag{3.15}$$

As $w \vee 2\varepsilon = w \vee \varepsilon$ q.e. in $U_{2\varepsilon}$, we have $r_n^\varepsilon = u_n - \frac{uw_n}{w\vee 2\varepsilon}$ q.e. in $U_{2\varepsilon}$ and $Dr_n^\varepsilon = Du_n - D(\frac{uw_n}{w\vee 2\varepsilon})$ a.e. in $U_{2\varepsilon}$. Therefore (3.15) implies (3.6) with 2ε replaced by ε. □

LEMMA 3.5. *Let $f \in L^\infty(\Omega)$ and, for every $n \in \mathbb{N}$, let u_n be the solution of* (3.1). *For every $\varepsilon > 0$ define $V_\varepsilon = \{w \le \varepsilon\}$. Then*

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \left(\int_{V_\varepsilon} |Du_n|^p dx + \int_{V_\varepsilon} |u_n|^p d\mu_n \right) = 0 . \tag{3.16}$$

PROOF. For every $\varepsilon > 0$ let $\Phi^\varepsilon : \mathbb{R} \to \mathbb{R}$ be the Lipschitz function defined at the end of the first step of the proof of Lemma 3.4, and let $z^\varepsilon \in H^{1,p}(\Omega) \cap L^\infty(\Omega)$ be the function defined by $z^\varepsilon = 1 - \Phi^\varepsilon(w)$. As $z^\varepsilon \ge 0$ q.e. in Ω and $z^\varepsilon = 1$ q.e. in V_ε, by (1.10) and (3.1) we have

$$\begin{aligned}
& c_0 \int_{V_\varepsilon} |Du_n|^p dx + \int_{V_\varepsilon} |u_n|^p d\mu_n \\
& \le \int_\Omega \big(a(x, Du_n), Du_n\big) z^\varepsilon dx + \int_\Omega |u_n|^p z^\varepsilon d\mu_n \\
& = \int_\Omega \big(a(x, Du_n), D(u_n z^\varepsilon)\big)\, dx + \int_\Omega |u_n|^p z^\varepsilon d\mu_n - \int_\Omega \big(a(x, Du_n), Dz^\varepsilon\big) u_n\, dx \\
& = \int_\Omega f u_n z^\varepsilon dx - \int_\Omega \big(a(x, Du_n), Dz^\varepsilon\big) u_n\, dx .
\end{aligned}$$

Since (u_n) converges to u strongly in $L^p(\Omega)$ and is bounded in $L^\infty(\Omega)$ (Remark 3.2), while $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ (Theorem 2.11), we can take the limit of the the last two terms as $n \to \infty$, obtaining

$$\begin{aligned}
& (c_0 \wedge 1) \limsup_{n\to\infty} \left(\int_{V_\varepsilon} |Du_n|^p dx + \int_{V_\varepsilon} |u_n|^p d\mu_n \right) \\
& \le \int_\Omega f u z^\varepsilon dx - \int_\Omega \big(a(x, Du), Dz^\varepsilon\big) u\, dx .
\end{aligned} \tag{3.17}$$

As (z^ε) is bounded in $L^\infty(\Omega)$ and converges to the characteristic function of $\{w = 0\}$ as $\varepsilon \to 0$, while $u = 0$ a.e. in $\{w = 0\}$ (Remark 3.2), we have that (uz^ε) converges to 0 strongly in $L^p(\Omega)$. On the other hand, since $|u| \le c\, w$ q.e. in Ω by Remark 3.2, we have

$$\int_\Omega |u|^p |Dz^\varepsilon|^p dx \le \frac{c^p}{\varepsilon^p} \int_{\{\varepsilon < w < 2\varepsilon\}} w^p |Dw|^p dx \le (2c)^p \int_{\{\varepsilon < w < 2\varepsilon\}} |Dw|^p dx ,$$

so that (uDz^ε) converges to 0 strongly in $L^p(\Omega, \mathbb{R}^N)$. Taking the limit in (3.17) as $\varepsilon \to 0$ we obtain (3.16). □

LEMMA 3.6. *Let $f \in L^\infty(\Omega)$ and, for every $n \in \mathbb{N}$, let u_n be the solution of* (3.1). *For every $\varepsilon > 0$ define $W_\varepsilon = \{w > \varepsilon\} \cap \{|u| \le \varepsilon\, w\}$. Then*

$$\lim_{\varepsilon \to 0} \limsup_{n\to\infty} \left(\int_{W_\varepsilon} |Du_n|^p dx + \int_{W_\varepsilon} |u_n|^p d\mu_n \right) = 0 \tag{3.18}$$

for every $\varepsilon > 0$.

PROOF. For every $\varepsilon > 0$ let $\Phi^\varepsilon : \mathbb{R} \to \mathbb{R}$ be the Lipschitz function defined at the end of the first step of the proof of Lemma 3.4. As $\frac{u}{w\vee\varepsilon} \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (Remark 3.2), the function $z^\varepsilon = 1 - \Phi^\varepsilon(\frac{|u|}{w\vee\varepsilon})$ belongs to $H^{1,p}(\Omega) \cap L^\infty(\Omega)$. As $z^\varepsilon \ge 0$ q.e. in Ω and $z^\varepsilon = 1$ q.e. in W_ε, by (1.10) and (3.1) we have

$$\begin{aligned} c_0 \int_{W_\varepsilon} |Du_n|^p dx + \int_{W_\varepsilon} |u_n|^p d\mu_n &\le \int_\Omega \big(a(x, Du_n), Du_n\big) z^\varepsilon\, dx \\ + \int_\Omega |u_n|^p z^\varepsilon d\mu_n &= \int_\Omega \big(a(x, Du_n), D(u_n z^\varepsilon)\big)\, dx + \int_\Omega |u_n|^p z^\varepsilon d\mu_n \\ - \int_\Omega \big(a(x, Du_n), Dz^\varepsilon\big) u_n\, dx &= \int_\Omega f u_n z^\varepsilon\, dx - \int_\Omega \big(a(x, Du_n), Dz^\varepsilon\big) u_n\, dx\,. \end{aligned}$$

Since $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ (Theorem 2.11) and (u_n) is bounded in $L^\infty(\Omega)$ and converges to u strongly in $L^p(\Omega)$ (Remark 3.2), we can take the limit of the the last two terms as $n \to \infty$, obtaining

$$\begin{aligned} &(c_0 \wedge 1) \limsup_{n\to\infty} \left(\int_{W_\varepsilon} |Du_n|^p dx + \int_{W_\varepsilon} |u_n|^p d\mu_n \right) \\ &\le \int_\Omega f u z^\varepsilon dx - \int_\Omega \big(a(x, Du), Dz^\varepsilon\big) u\, dx\,. \end{aligned} \tag{3.19}$$

As (z^ε) is bounded in $L^\infty(\Omega)$ and converges to the characteristic function of $\{u = 0\}$, we have that (uz^ε) converges to 0 strongly in $L^p(\Omega)$. Moreover,

$$\begin{aligned} \int_\Omega |u|^p |Dz^\varepsilon|^p dx &\le \frac{1}{\varepsilon^p} \int_{\{0<|u|\le 2\varepsilon(w\vee\varepsilon)\}} |u|^p \left| D\left(\frac{|u|}{w \vee \varepsilon}\right)\right|^p dx \\ &\le \frac{2^{p-1}}{\varepsilon^p} \int_{\{0<|u|\le 2\varepsilon(w\vee\varepsilon)\}} \left(\left(\frac{|u|}{w\vee\varepsilon}\right)^p |Du|^p + \left(\frac{|u|}{w\vee\varepsilon}\right)^{2p} |Dw|^p \right) dx \\ &\le 2^{2p-1} \int_{\{0<|u|\le 2\varepsilon(w\vee\varepsilon)\}} \big(|Du|^p + (2\varepsilon)^p |Dw|^p\big) dx\,, \end{aligned}$$

and so (uDz^ε) converges to 0 strongly in $L^p(\Omega, \mathbb{R}^N)$ as $\varepsilon \to 0$. Therefore (3.18) follows from (3.19) by taking the limit $\varepsilon \to 0$. □

PROOF OF THEOREM 3.1. Recall that $U_\varepsilon = \{w > \varepsilon\} \cap \{|u| > \varepsilon\, w\}$, $V_\varepsilon = \{w \le \varepsilon\}$, and $W_\varepsilon = \{w > \varepsilon\} \cap \{|u| \le \varepsilon\, w\}$, which for $\varepsilon > 0$ allows us to write

$$\int_\Omega |R_n|^p dx = \int_{U_\varepsilon} |R_n|^p dx + \int_{V_\varepsilon} |R_n|^p dx + \int_{W_\varepsilon} |R_n|^p dx .$$

Since $R_n = Du_n - Du - \frac{u}{w}(Dw_n - Dw)$ in $\{w > 0\}$, we deduce from (3.7) that for $\varepsilon > 0$ fixed

$$\lim_{n\to\infty} \int_{U_\varepsilon} |R_n|^p dx = 0 .$$

On the other hand we shall prove that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \int_{V_\varepsilon} |R_n|^p dx , \tag{3.20}$$

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \int_{W_\varepsilon} |R_n|^p dx . \tag{3.21}$$

Since $|u| \le c\, w$ q.e. in Ω (Remark 3.2), we deduce from (3.5) that $|R_n| \le |Du_n - Du| + c|Dw_n - Dw|$ q.e. in Ω. Therefore

$$\begin{aligned} 4^{1-p} \limsup_{n\to\infty} \int_{V_\varepsilon} |R_n|^p dx &\le \limsup_{n\to\infty} \int_{V_\varepsilon} |Du_n|^p dx + \int_{V_\varepsilon} |Du|^p dx \\ &+ c^p \limsup_{n\to\infty} \int_{V_\varepsilon} |Dw_n|^p dx + c^p \int_{V_\varepsilon} |Dw|^p dx \end{aligned}$$

for every $\varepsilon > 0$. As $|u| \le c\, w$ (Remark 3.2), we have $Du = Dw = 0$ a.e. in $\{w = 0\}$. This fact, together with Lemma 3.5 (applied to the sequences (u_n) and (w_n)), allows us to obtain (3.20) from the previous inequality.

As $|u| \le \varepsilon\, w$ q.e. in W_ε, by (3.5) we have $|R_n| \le |Du_n - Du| + \varepsilon\, |Dw_n - Dw|$ q.e. in W_ε. Therefore

$$\begin{aligned} 4^{1-p} \limsup_{n\to\infty} \int_{W_\varepsilon} |R_n|^p dx &\le \limsup_{n\to\infty} \int_{W_\varepsilon} |Du_n|^p dx + \int_{W_\varepsilon} |Du|^p dx \\ &+ \varepsilon^p \limsup_{n\to\infty} \int_\Omega |Dw_n|^p dx + \varepsilon^p \int_\Omega |Dw|^p dx . \end{aligned}$$

As the characteristic function of W_ε converges to the characteristic function of $\{w > 0\} \cap \{u = 0\}$, and $Du = 0$ a.e. in $\{u = 0\}$, the previous inequality together with Lemma 3.6 yields (3.21), which concludes the proof of the theorem. □

Theorem 3.1 explains how to correct Du_n in order to obtain strong convergence (see (3.2)). Note that the function $Du + uP_n$ is not a gradient in general. The following theorem gives a corrector result in $H_0^{1,p}(\Omega)$ for the function u_n.

THEOREM 3.7. *Let (μ_n) a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $f \in L^\infty(\Omega)$. Assume that the solutions u_n and w_n of problems* (3.1) *and* (3.3) *converge weakly in $H_0^{1,p}(\Omega)$ to some functions u and w. Then for every $\varepsilon > 0$ we have*

$$u_n = \frac{u w_n}{w \vee \varepsilon} + r_n^\varepsilon ,$$

with $\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \|r_n^\varepsilon\|_{H_0^{1,p}(\Omega)} = 0.$

PROOF. Recall that $U_\varepsilon = \{w > \varepsilon\} \cap \{|u| > \varepsilon\, w\}$, $V_\varepsilon = \{w \le \varepsilon\}$, and $W_\varepsilon = \{w > \varepsilon\} \cap \{|u| \le \varepsilon\, w\}$. Then

$$\int_\Omega |Dr_n^\varepsilon|^p dx = \int_{U_\varepsilon} |Dr_n^\varepsilon|^p dx + \int_{V_\varepsilon} |Dr_n^\varepsilon|^p dx + \int_{W_\varepsilon} |Dr_n^\varepsilon|^p dx . \tag{3.22}$$

Since, by Lemma 3.4, (Dr_n^ε) converges to 0 strongly in $L^p(U_\varepsilon, \mathbb{R}^N)$ as $n \to \infty$, we have only to estimate the last two terms of (3.22). As

$$Dr_n^\varepsilon = Du_n - \frac{u}{w \vee \varepsilon} Dw_n - \frac{w_n}{w \vee \varepsilon} Du + \frac{u w_n}{(w \vee \varepsilon)^2} D(w \vee \varepsilon) ,$$

and $|u| \le c\, w$ q.e. in Ω (Remark 3.2), we have

$$4^{1-p} |Dr_n^\varepsilon|^p \le |Du_n|^p + c^p |Dw_n|^p + \left(\frac{w_n}{w \vee \varepsilon}\right)^p |Du|^p + c^p \left(\frac{w_n}{w \vee \varepsilon}\right)^p |Dw|^p .$$

Since (w_n) is bounded in $L^\infty(\Omega)$ (Remark 3.2) and converges to w weakly in $H_0^{1,p}(\Omega)$, we obtain

$$\begin{aligned} 4^{1-p} \limsup_{n\to\infty} \int_{V_\varepsilon} |Dr_n^\varepsilon|^p dx &\le \limsup_{n\to\infty} \int_{V_\varepsilon} |Du_n|^p dx \\ &+ c^p \limsup_{n\to\infty} \int_{V_\varepsilon} |Dw_n|^p dx + \int_{V_\varepsilon} |Du|^p dx + c^p \int_{V_\varepsilon} |Dw|^p dx . \end{aligned}$$

As $|u| \le c\, w$ q.e. in Ω, we have $Du = 0$ a.e. in $\{w = 0\}$, and so the last two terms tend to 0 as $\varepsilon \to 0$. Therefore, Lemma 3.5 (applied to the sequences (u_n) and (w_n)) implies that

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \int_{V_\varepsilon} |Dr_n^\varepsilon|^p dx = 0 .$$

Since $w = w \vee \varepsilon$ and $|u| \le \varepsilon\, w$ q.e. in W_ε, we have

$$\begin{aligned} 4^{1-p} |Dr_n^\varepsilon|^p \le\ & |Du_n|^p + \varepsilon^p |Dw_n|^p + \left(\frac{w_n}{w}\right)^p |Du|^p \\ & + \varepsilon^p \left(\frac{w_n}{w}\right)^p |Dw|^p \quad \text{q.e. in } W_\varepsilon , \end{aligned}$$

and thus

$$4^{1-p} \limsup_{n\to\infty} \int_{W_\varepsilon} |Dr_n^\varepsilon|^p dx \le \limsup_{n\to\infty} \int_{W_\varepsilon} |Du_n|^p dx$$
$$+ \varepsilon^p \limsup_{n\to\infty} \int_\Omega |Dw_n|^p dx + \int_{W_\varepsilon} |Du|^p dx + \varepsilon^p \int_\Omega |Dw|^p dx\,.$$

As the characteristic function of W_ε converges to the characteristic function of $\{w > 0\} \cap \{u = 0\}$, and $Du = 0$ a.e. in $\{u = 0\}$, the term $\int_{W_\varepsilon} |Du|^p dx$ tends to 0 as $\varepsilon \to 0$. Therefore, by Lemma 3.6 we have

$$\lim_{\varepsilon\to 0} \limsup_{n\to\infty} \int_{W_\varepsilon} |Dr_n^\varepsilon|^p dx = 0\,,$$

which concludes the proof of the theorem. □

The case $f \notin L^\infty(\Omega)$ requires a further approximation (see [11]) and will be considered at the end of Section 6.

4. – Estimates based on the corrector result

Let (μ_n) a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $f \in L^\infty(\Omega)$. Assume that the solutions u_n and w_n of problems (3.1) and (3.3) converge weakly in $H_0^{1,p}(\Omega)$ to some functions u and w. In this section we shall study the behaviour of the sequences

$$(4.1) \qquad \langle Au_n, w_n^\beta \varphi\rangle - \left\langle Aw_n, \left|\frac{u}{w\vee\varepsilon}\right|^{p-2} \frac{u}{w\vee\varepsilon} w_n^\beta \varphi \right\rangle,$$

$$(4.2) \qquad \int_\Omega |u_n|^{p-2} u_n w_n^\beta \varphi\, d\mu_n - \int_\Omega \left|\frac{u}{w\vee\varepsilon}\right|^{p-2} \frac{u}{w\vee\varepsilon} w_n^{p-1+\beta} \varphi\, d\mu_n\,,$$

where $\beta \ge (p-1)\vee 1$ and $\varphi \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. As explained in the introduction, these estimates will be useful in the proof of the main results of Section 6.

Since for $1 < p < 2$ the function $|\frac{u}{w\vee\varepsilon}|^{p-2} \frac{u}{w\vee\varepsilon}$ does not belong to $H_0^{1,p}(\Omega)$ (note that the derivative of the function $t \mapsto |t|^{p-2} t$ is singular at the origin), formula (4.1) is not correct and we are forced to introduce the locally Lipschitz function $\Psi_\varepsilon : \mathbb{R} \to \mathbb{R}$ defined by

$$(4.3) \qquad \Psi_\varepsilon(t) = \begin{cases} |t|^{p-2} t\,, & \text{if } |t| > \varepsilon, \\ \varepsilon^{p-2} t\,, & \text{if } |t| \le \varepsilon, \end{cases}$$

and to replace $|\frac{u}{w\vee\varepsilon}|^{p-2} \frac{u}{w\vee\varepsilon}$ by $\Psi_\varepsilon(\frac{u}{w\vee\varepsilon})$ in (4.1) and (4.2). This leads to some technical estimates that will be used in Section 6.

We begin with an estimate in the set $U_\varepsilon = \{w > \varepsilon\} \cap \{|u| > \varepsilon\, w\}$.

LEMMA 4.1. *Let (μ_n) a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $f \in L^\infty(\Omega)$. Assume that the solutions u_n and w_n of problems* (3.1) *and* (3.3) *converge weakly in $H_0^{1,p}(\Omega)$ to some functions u and w. Let $\varepsilon > 0$, let $\beta \geq 1$, and let $v_\varepsilon \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be the function defined by $v_\varepsilon = \Psi_\varepsilon(\frac{u}{w \vee \varepsilon})$, where Ψ_ε is given by* (4.3). *Then the sequence*

$$\Big(a\,(x, Du_n)\,, D(w_n^\beta)\Big) - \Big(a\,(x, Dw_n)\,, D(v_\varepsilon w_n^\beta)\Big)$$

converges weakly in $L^1(U_\varepsilon)$, as $n \to \infty$, to the function

$$\Big(a\,(x, Du)\,, D(w^\beta)\Big) - \Big(a\,(x, Dw)\,, D(v_\varepsilon w^\beta)\Big)\,.$$

PROOF. By Theorem 3.1 we have

$$Du_n = Du + \frac{u}{w}Dw_n - \frac{u}{w}Dw + R_n \qquad \text{a.e. in } U_\varepsilon\,, \tag{4.4}$$

where (R_n) converges to 0 strongly in $L^p(\Omega, \mathbb{R}^N)$. Since $v_\varepsilon = \left|\frac{u}{w}\right|^{p-2}\frac{u}{w}$ a.e. in U_ε, by the homogeneity property (1.3) for a.e. $x \in U_\varepsilon$ we have

$$\begin{aligned}&\Big(a\,(x, Du_n)\,, D(w_n^\beta)\Big) - \Big(a\,(x, Dw_n)\,, D(v_\varepsilon w_n^\beta)\Big)\\ &= \beta\,(a\,(x, Du_n)\,, Dw_n)\,w_n^{\beta-1}\\ &\quad - \beta\,(a\,(x, Dw_n)\,, Dw_n)\,w_n^{\beta-1}\left|\frac{u}{w}\right|^{p-2}\frac{u}{w} - (a\,(x, Dw_n)\,, Dv_\varepsilon)\,w_n^\beta\\ &= \beta\left(a\,(x, Du_n) - a\left(x, \frac{u}{w}Dw_n\right), Dw_n\right)w_n^{\beta-1} - (a\,(x, Dw_n)\,, Dv_\varepsilon)\,w_n^\beta.\end{aligned} \tag{4.5}$$

Similarly, we obtain

$$\begin{aligned}&\Big(a\,(x, Du)\,, D(w^\beta)\Big) - \Big(a\,(x, Dw)\,, D(v_\varepsilon w^\beta)\Big)\\ &= \beta\left(a\,(x, Du) - a\left(x, \frac{u}{w}Dw\right), Dw\right)w^{\beta-1} - (a\,(x, Dw)\,, Dv_\varepsilon)\,w^\beta\,.\end{aligned} \tag{4.6}$$

By Theorem 2.11 the sequences (u_n) and (w_n) converge to u and w strongly in $H_0^{1,r}(\Omega)$ for every $r < p$, and so, passing to a subsequence, we may assume that (u_n), (w_n), (Du_n), (Dw_n) converge to u, w, Du, Dw almost everywhere in Ω. As $a\,(x, \cdot)$ is continuous for a.e. $x \in \Omega$, this implies that $\big(a\,(x, Du_n)\big)$ and $\big(a\left(x, \frac{u}{w}Dw_n\right)\big)$ converge to $a\,(x, Du)$ and $a\left(x, \frac{u}{w}Dw\right)$ almost everywhere in U_ε. We want to prove that $\big(a\,(x, Du_n) - a\left(x, \frac{u}{w}Dw_n\right)\big)$ converges to $a\,(x, Du) - a\left(x, \frac{u}{w}Dw\right)$ strongly in $L^q(U_\varepsilon, \mathbb{R}^N)$. To this aim it is enough to show that the sequence $\left|a\,(x, Du_n) - a\left(x, \frac{u}{w}Dw_n\right)\right|^q$ is equi-integrable. Let us consider first the case $2 \leq p < +\infty$. Since $\frac{u}{w} \in L^{\infty}(U_{\varepsilon})$ (Remark 3.2), by (1.5) and (4.4) there exists a constant C such that

$$\begin{aligned}
&\left|a\left(x, Du_n\right) - a\left(x, \frac{u}{w} Dw_n\right)\right|^q \\
(4.7) \qquad &\leq c_1^q \left(2\left|\frac{u}{w} Dw_n\right| + \left|Du - \frac{u}{w} Dw + R_n\right|\right)^{q(p-2)} \left|Du - \frac{u}{w} Dw + R_n\right|^q \\
&\leq C\left(|Dw_n|^{q(p-2)}\left|Du - \frac{u}{w} Dw + R_n\right|^q + \left|Du - \frac{u}{w} Dw + R_n\right|^p\right).
\end{aligned}$$

for a.e. $x \in U_{\varepsilon}$. By Hölder's inequality for every measurable set $E \subseteq U_{\varepsilon}$ we have

$$\begin{aligned}
&\int_E |Dw_n|^{q(p-2)} \left|Du - \frac{u}{w} Dw + R_n\right|^q dx \\
&\leq \left(\int_{\Omega} |Dw_n|^p dx\right)^{(p-2)/(p-1)} \left(\int_E \left|Du - \frac{u}{w} Dw + R_n\right|^p dx\right)^{q/p}.
\end{aligned}$$

As (Dw_n) is bounded in $L^p(\Omega, \mathbb{R}^N)$ and (R_n) converges to 0 strongly in $L^p(\Omega, \mathbb{R}^N)$, the previous inequality, together with (4.7), shows that $\left|a\left(x, Du_n\right) - a\left(x, \frac{u}{w} Dw_n\right)\right|^q$ is equi-integrable in U_{ε}. In the case $1 < p \leq 2$, by the continuity condition (1.7) we have

$$\left|a\left(x, Du_n\right) - a\left(x, \frac{u}{w} Dw_n\right)\right|^q \leq c_1^q \left|Du_n - \frac{u}{w} Dw_n\right|^p = c_1^q \left|Du - \frac{u}{w} Dw + R_n\right|^p.$$

As (R_n) converges to 0 strongly in $L^p(\Omega, \mathbb{R}^N)$, the sequence $\left|a\left(x, Du_n\right) - a\left(x, \frac{u}{w} Dw_n\right)\right|^q$ is equi-integrable in U_{ε}. Therefore, in both cases the dominated convergence theorem implies that $\left(a\left(x, Du_n\right) - a\left(x, \frac{u}{w} Dw_n\right)\right)$ converges to $a\left(x, Du\right) - a\left(x, \frac{u}{w} Dw\right)$ strongly in $L^q(U_{\varepsilon}, \mathbb{R}^N)$. As (Dw_n) converges to Dw weakly in $L^p(U_{\varepsilon}, \mathbb{R}^N)$, and (w_n) is bounded in $L^{\infty}(\Omega)$ (Remark 3.2) and converges to w almost everywhere in Ω, it follows that

$$\beta\left(a\left(x, Du_n\right) - a\left(x, \frac{u}{w} Dw_n\right), Dw_n\right) w_n^{\beta-1}$$

converges to

$$\beta\left(a\left(x, Du\right) - a\left(x, \frac{u}{w} Dw\right), Dw\right) w^{\beta-1}$$

weakly in $L^1(U_{\varepsilon})$. As $\left(a\left(x, Dw_n\right)\right)$ converges to $a\left(x, Dw\right)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ (Theorem 2.11), the sequence $\left(\left(a\left(x, Dw_n\right), Dv_{\varepsilon}\right) w_n^{\beta}\right)$ converges to $(a(x, Dw), Dv_{\varepsilon}) w^{\beta}$ weakly in $L^1(\Omega)$. The conclusion follows now from (4.5) and (4.6). $\square$

LEMMA 4.2. *Let (μ_n) a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $f \in L^\infty(\Omega)$. Assume that the solutions u_n and w_n of problems* (3.1) *and* (3.3) *converge weakly in $H_0^{1,p}(\Omega)$ to some functions u and w. Let $\beta \geq 1$, let $\varphi \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, and, for every $\varepsilon > 0$, let $v_\varepsilon \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be the function defined by $v_\varepsilon = \Psi_\varepsilon(\frac{u}{w \vee \varepsilon})$, where Ψ_ε is given by* (4.3). *Then*

$$\langle Au_n, w_n^\beta \varphi\rangle - \langle Aw_n, v_\varepsilon w_n^\beta \varphi\rangle = \langle Au, w^\beta \varphi\rangle - \langle Aw, v_\varepsilon w^\beta \varphi\rangle + \mathcal{R}_n^\varepsilon,$$

with $\lim\limits_{\varepsilon\to 0} \limsup\limits_{n\to\infty} |\mathcal{R}_n^\varepsilon| = 0.$

PROOF. For every $\varepsilon > 0$ we define $U_\varepsilon = \{w > \varepsilon\} \cap \{|u| > \varepsilon\, w\}$, $V_\varepsilon = \{w \leq \varepsilon\}$, and $W_\varepsilon = \{w > \varepsilon\} \cap \{|u| \leq \varepsilon\, w\}$. Then

$$\langle Au_n, w_n^\beta \varphi\rangle - \langle Aw_n, v_\varepsilon w_n^\beta \varphi\rangle = \mathcal{A}_n^\varepsilon + \mathcal{B}_n^\varepsilon + \mathcal{C}_n^\varepsilon,$$

where

$$\begin{aligned}
\mathcal{A}_n^\varepsilon &= \int_{U_\varepsilon} \Big(a(x, Du_n), D(w_n^\beta)\Big)\varphi\, dx - \int_{U_\varepsilon} \Big(a(x, Dw_n), D(v_\varepsilon w_n^\beta)\Big)\varphi\, dx,\\
\mathcal{B}_n^\varepsilon &= \int_{V_\varepsilon \cup W_\varepsilon} \Big(a(x, Du_n), D(w_n^\beta)\Big)\varphi\, dx - \int_{V_\varepsilon \cup W_\varepsilon} \Big(a(x, Dw_n), D(v_\varepsilon w_n^\beta)\Big)\varphi\, dx,\\
\mathcal{C}_n^\varepsilon &= \int_\Omega (a(x, Du_n), D\varphi)\, w_n^\beta\, dx - \int_\Omega (a(x, Dw_n), D\varphi)\, v_\varepsilon w_n^\beta\, dx.
\end{aligned}$$

Similarly, we define $\mathcal{A}^\varepsilon$, $\mathcal{B}^\varepsilon$, $\mathcal{C}^\varepsilon$ by replacing u_n with u and w_n with w, so that

$$\langle Au, w^\beta \varphi\rangle - \langle Aw, v_\varepsilon w^\beta \varphi\rangle = \mathcal{A}^\varepsilon + \mathcal{B}^\varepsilon + \mathcal{C}^\varepsilon.$$

By Lemma 4.1 we have

$$\lim_{n\to\infty} \mathcal{A}_n^\varepsilon = \mathcal{A}^\varepsilon \qquad \forall\, \varepsilon > 0. \tag{4.8}$$

Since $\big(a(x, Du_n)\big)$ and $\big(a(x, Dw_n)\big)$ converge to $a(x, Du)$ and $a(x, Dw)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ (Theorem 2.11), and (w_n) is bounded in $L^\infty(\Omega)$ (Remark 3.2) and converges to w strongly in $L^p(\Omega)$, while $v_\varepsilon \in L^\infty(\Omega)$ (Remark 3.2), we conclude that

$$\lim_{n\to\infty} \mathcal{C}_n^\varepsilon = \mathcal{C}^\varepsilon \qquad \forall\, \varepsilon > 0. \tag{4.9}$$

Let us consider now the term $\mathcal{B}_n^\varepsilon - \mathcal{B}^\varepsilon$. For every measurable set $B \subseteq \Omega$ we define

$$\begin{aligned}
\mathcal{J}_n^1(B) &= \beta \int_B (a(x, Du_n), Dw_n)\, w_n^{\beta-1} \varphi\, dx,\\
\mathcal{J}_n^{\varepsilon,2}(B) &= \beta \int_B (a(x, Dw_n), Dw_n)\, w_n^{\beta-1} v_\varepsilon \varphi\, dx,\\
\mathcal{J}_n^{\varepsilon,3}(B) &= \int_B (a(x, Dw_n), Dv_\varepsilon)\, w_n^\beta \varphi\, dx.
\end{aligned}$$

Similarly, we define $\mathcal{J}^1(B)$, $\mathcal{J}^{\varepsilon,2}(B)$, $\mathcal{J}^{\varepsilon,3}(B)$ by replacing u_n with u and w_n with w. Clearly we have

$$
\begin{aligned}
|\mathcal{B}_n^\varepsilon - \mathcal{B}^\varepsilon| \le\ & |\mathcal{J}_n^1(V_\varepsilon \cup W_\varepsilon)| + |\mathcal{J}^1(V_\varepsilon \cup W_\varepsilon)| + |\mathcal{J}_n^{\varepsilon,2}(V_\varepsilon)| + |\mathcal{J}^{\varepsilon,2}(V_\varepsilon)| \\
& + |\mathcal{J}_n^{\varepsilon,2}(W_\varepsilon)| + |\mathcal{J}^{\varepsilon,2}(W_\varepsilon)| + |\mathcal{J}_n^{\varepsilon,3}(V_\varepsilon \cup W_\varepsilon) - \mathcal{J}^{\varepsilon,3}(V_\varepsilon \cup W_\varepsilon)|\,.
\end{aligned} \tag{4.10}
$$

Since $\beta \ge 1$, the sequence $(w_n^{\beta-1})$ is bounded in $L^\infty(\Omega)$ (Remark 3.2). Moreover, as $|u| \le c\, w$ (Remark 3.2), by (4.3) we have $|v_\varepsilon| \le (c \vee \varepsilon)^{p-1}$ q.e. in Ω. Therefore, by the upper bound (1.11) there exists a constant K such that for $\varepsilon \le 1$ we have

$$
\begin{aligned}
|\mathcal{J}_n^1(V_\varepsilon \cup W_\varepsilon)| + |\mathcal{J}_n^{\varepsilon,2}(V_\varepsilon)| \ &\le\ K \int_{V_\varepsilon \cup W_\varepsilon} |Du_n|^{p-1} |Dw_n|\, dx + K \int_{V_\varepsilon} |Dw_n|^p dx \\
&\le\ K \left(\int_{V_\varepsilon \cup W_\varepsilon} |Du_n|^p dx \right)^{1/q} \left(\int_\Omega |Dw_n|^p dx \right)^{1/p} + K \int_{V_\varepsilon} |Dw_n|^p dx\,,
\end{aligned}
$$

and thus Lemmas 3.5 and 3.6 imply that

$$
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left(|\mathcal{J}_n^1(V_\varepsilon \cup W_\varepsilon)| + |\mathcal{J}_n^{\varepsilon,2}(V_\varepsilon)| \right) = 0\,. \tag{4.11}
$$

A similar estimate shows that

$$
\lim_{\varepsilon \to 0} \left(|\mathcal{J}^1(V_\varepsilon \cup W_\varepsilon)| + |\mathcal{J}^{\varepsilon,2}(V_\varepsilon)| \right) = 0\,. \tag{4.12}
$$

Since $|u| \le \varepsilon\, w$ q.e. in W_ε, by (4.3) we have $|v_\varepsilon| \le \varepsilon^{p-1}$ q.e. in W_ε. Therefore, the boundedness of $(w_n^{\beta-1})$ in $L^\infty(\Omega)$ (Remark 3.2) and the upper bound (1.11) imply that

$$
|\mathcal{J}_n^{\varepsilon,2}(W_\varepsilon)| \le K \varepsilon^{p-1} \int_\Omega |Dw_n|^p dx
$$

for a suitable constant K. As (w_n) is bounded in $H_0^{1,p}(\Omega)$, we conclude that

$$
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} |\mathcal{J}_n^{\varepsilon,2}(W_\varepsilon)| = 0\,. \tag{4.13}
$$

A similar estimate shows that

$$
\lim_{\varepsilon \to 0} |\mathcal{J}^{\varepsilon,2}(W_\varepsilon)| = 0\,. \tag{4.14}
$$

Since $\big(a(x, Dw_n)\big)$ converges to $a(x, Dw)$ weakly in $L^q(\Omega, \mathbb{R}^N)$, and (w_n) is bounded in $L^\infty(\Omega)$ (Remark 3.2) and converges to w strongly in $L^p(\Omega)$, we conclude that

$$
\lim_{n \to \infty} \mathcal{J}_n^{\varepsilon,3}(V_\varepsilon \cup W_\varepsilon) = \mathcal{J}^{\varepsilon,3}(V_\varepsilon \cup W_\varepsilon) \qquad \forall \varepsilon > 0\,. \tag{4.15}
$$

From (4.10)–(4.15) we obtain

$$
\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \left| \mathcal{B}_n^\varepsilon - \mathcal{B}^\varepsilon \right| = 0\,, \tag{4.16}
$$

As $\mathcal{R}_n^\varepsilon = \mathcal{A}_n^\varepsilon - \mathcal{A}^\varepsilon + \mathcal{B}_n^\varepsilon - \mathcal{B}^\varepsilon + \mathcal{C}_n^\varepsilon - \mathcal{C}^\varepsilon$, the conclusion follows from (4.8), (4.9), (4.16). □

LEMMA 4.3. *Let (μ_n) a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $f \in L^\infty(\Omega)$. Assume that the solutions u_n and w_n of problems* (3.1) *and* (3.3) *converge weakly in $H_0^{1,p}(\Omega)$ to some functions u and w. Let $\varepsilon > 0$, let $\beta \geq (p-1) \vee 1$, and let $u_n^\varepsilon = \frac{u w_n}{w \vee \varepsilon}$ as in the proof of Lemma* 3.4. *Then*

$$\int_{U_\varepsilon} |u_n|^{p-2} u_n w_n^\beta \varphi \, d\mu_n - \int_{U_\varepsilon} |u_n^\varepsilon|^{p-2} u_n^\varepsilon w_n^\beta \varphi \, d\mu_n$$

tends to 0 *as $n \to \infty$ for every $\varphi \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.*

PROOF. Let $\varphi \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ and let $r_n^\varepsilon = u_n - u_n^\varepsilon$ as in the proof of Lemma 3.4. Since the sequences (u_n) and (u_n^ε) are bounded in $L^\infty(\Omega)$ (Remark 3.2), by (1.13) and (1.14) there exists a constant C such that $\left||u_n|^{p-2} u_n \varphi - |u_n^\varepsilon|^{p-2} u_n^\varepsilon \varphi\right| \leq C |r_n^\varepsilon|^{(p-1)\wedge 1}$. Moreover, since $\beta \geq (p-1) \vee 1$ and (w_n) is bounded in $L^\infty(\Omega)$ (Remark 3.2), there exists a constant K such that $w_n^\beta \leq K \, w_n^{(p-1)\vee 1}$. Therefore

$$\begin{aligned}
&\left| \int_{U_\varepsilon} |u_n|^{p-2} u_n w_n^\beta \varphi \, d\mu_n - \int_{U_\varepsilon} |u_n^\varepsilon|^{p-2} u_n^\varepsilon w_n^\beta \varphi \, d\mu_n \right| \\
&\leq CK \int_{U_\varepsilon} |r_n^\varepsilon|^{(p-1)\wedge 1} w_n^{(p-1)\vee 1} d\mu_n \\
&\leq CK \left(\int_{U_\varepsilon} |r_n^\varepsilon|^p \, d\mu_n \right)^{1/(p\vee q)} \left(\int_\Omega w_n^p \, d\mu_n \right)^{1/(p\wedge q)} .
\end{aligned}$$

The conclusion follows now from estimate (2.3) and from Lemma 3.4. □

LEMMA 4.4. *Let (μ_n) be a sequence of measures of $\mathcal{M}_0^p(\Omega)$, let $f \in L^\infty(\Omega)$, and let $\varphi \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Assume that the solutions u_n and w_n of problems* (3.1) *and* (3.3) *converge weakly in $H_0^{1,p}(\Omega)$ to some functions u and w. For every $\varepsilon > 0$ let $v_\varepsilon \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be the function defined by $v_\varepsilon = \Psi_\varepsilon(\frac{u}{w \vee \varepsilon})$, where Ψ_ε is given by* (4.3). *Let $\beta \geq (p-1) \vee 1$ and let*

$$\mathcal{E}_n^\varepsilon = \int_\Omega |u_n|^{p-2} u_n w_n^\beta \varphi \, d\mu_n - \int_\Omega v_\varepsilon w_n^{p+\beta-1} \varphi \, d\mu_n .$$

Then $\lim\limits_{\varepsilon \to 0} \limsup\limits_{n \to \infty} |\mathcal{E}_n^\varepsilon| = 0.$

PROOF. For every $\varepsilon > 0$ we define $U_\varepsilon = \{w > \varepsilon\} \cap \{|u| > \varepsilon \, w\}$, $V_\varepsilon = \{w \leq \varepsilon\}$, and $W_\varepsilon = \{w > \varepsilon\} \cap \{|u| \leq \varepsilon \, w\}$. As $w_n^{p-1} v_\varepsilon = |u_n^\varepsilon|^{p-2} u_n^\varepsilon$ q.e. in U_ε, by Lemma 4.3 for every $\varepsilon > 0$ the sequence

$$\int_{U_\varepsilon} |u_n|^{p-2} u_n w_n^\beta \varphi \, d\mu_n - \int_{U_\varepsilon} v_\varepsilon w_n^{p+\beta-1} \varphi \, d\mu_n$$

tends to 0 as $n \to \infty$. As φ is bounded, to conclude the proof it is enough to show that

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{V_\varepsilon\cup W_\varepsilon}|u_n|^{p-1}w_n^\beta d\mu_n = 0\,, \tag{4.17}$$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{V_\varepsilon}|v_\varepsilon|w_n^{p+\beta-1}d\mu_n = 0\,, \tag{4.18}$$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_{W_\varepsilon}|v_\varepsilon|w_n^{p+\beta-1}d\mu_n\,. \tag{4.19}$$

Since $\beta \ge 1$, we have $\|w_n^{\beta-1}\|_{L^\infty(\Omega)} \le K$ for a suitable constant K (Remark 3.2). Therefore

$$\begin{aligned}\int_{V_\varepsilon\cup W_\varepsilon}|u_n|^{p-1}w_n^\beta d\mu_n &\le K\int_{V_\varepsilon\cup W_\varepsilon}|u_n|^{p-1}w_n d\mu_n\\ &\le K\left(\int_{V_\varepsilon\cup W_\varepsilon}|u_n|^p d\mu_n\right)^{1/q}\left(\int_\Omega w_n^p d\mu_n\right)^{1/p},\end{aligned}$$

and thus (4.17) follows from (2.3) and from Lemmas 3.5 and 3.6. Moreover, as $|u| \le c\,w$ (Remark 3.2), by (4.3) we have $|v_\varepsilon| \le (c\vee\varepsilon)^{p-1}$ q.e. in Ω. Therefore

$$\int_{V_\varepsilon}|v_\varepsilon|w_n^{p+\beta-1}d\mu_n \le (c\vee\varepsilon)^{p-1}K\int_{V_\varepsilon}w_n^p d\mu_n\,,$$

and so (4.18) follows from Lemma 3.5. Since $|u| \le \varepsilon\,w$ q.e. in W_ε, by (4.3) we have $|v_\varepsilon| \le \varepsilon^{p-1}$ q.e. in W_ε. Therefore

$$\int_{W_\varepsilon}|v_\varepsilon|w_n^{p+\beta-1}d\mu_n \le \varepsilon^{p-1}K\int_\Omega w_n^p d\mu_n\,,$$

and thus (4.19) follows from (2.3). □

5. – Constructing the measure from the solution corresponding to f = 1

In this section we shall study the properties of the set $\mathcal{K}(\Omega)$ of the functions w such that

$$w \in H_0^{1,p}(\Omega)\,,\quad w \ge 0 \text{ q.e. in } \Omega\,,\quad\text{and } Aw \le 1 \text{ in } \mathcal{D}'(\Omega)\,.$$

The results obtained in the present section will be crucial in the proof of Theorems 6.3 and 6.5. They do not depend on the results of Sections 3 and 4, but only on those of Sections 1 and 2.

By the coerciveness condition (1.10) for every $w \in \mathcal{K}(\Omega)$ we have

$$c_0 \int_\Omega |Dw|^p dx \leq \langle Aw, w\rangle \leq \int_\Omega w\, dx\,.$$

This shows that $\mathcal{K}(\Omega)$ is bounded and hence weakly relatively compact in $H_0^{1,p}(\Omega)$. Let w_0 be the solution of the Dirichlet problem

$$w_0 \in H_0^{1,p}(\Omega)\,, \qquad Aw_0 = 1 \quad \text{in } \mathcal{D}'(\Omega)\,.$$

By the comparison principle (Proposition 2.6) we have $0 \leq w \leq w_0$ q.e. in Ω for every $w \in \mathcal{K}(\Omega)$. As $w_0 \in L^\infty(\Omega)$ (see [34], Chapter 4, Theorem 7.1), the set $\mathcal{K}(\Omega)$ is also bounded in $L^\infty(\Omega)$.

Given $w \in \mathcal{K}(\Omega)$, we define

$$\nu = 1 - Aw\,.$$

By the definition of $\mathcal{K}(\Omega)$ we have $\nu \geq 0$ in $\mathcal{D}'(\Omega)$, hence ν is a non-negative Radon measure. As $Aw \in H^{-1,q}(\Omega)$, we have $\nu \in H^{-1,q}(\Omega)$.

Our aim in this section is to prove the following theorem, which characterizes $\mathcal{K}(\Omega)$ as the set of the solutions of all relaxed Dirichlet problem corresponding to $f = 1$.

THEOREM 5.1. *The set $\mathcal{K}(\Omega)$ is compact in the weak topology of $H_0^{1,p}(\Omega)$. A function $w \in H_0^{1,p}(\Omega)$ belongs to $\mathcal{K}(\Omega)$ if and only if there exists $\mu \in \mathcal{M}_0^p(\Omega)$ such that w is the solution of the problem*

$$\text{(5.1)} \qquad \begin{cases} w \in H_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,, \\ \langle Aw, v\rangle + \displaystyle\int_\Omega |w|^{p-2} wv\, d\mu = \int_\Omega v\, dx \quad \forall\, v \in H_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)\,. \end{cases}$$

The measure $\mu \in \mathcal{M}_0^p(\Omega)$ is uniquely determined by $w \in \mathcal{K}(\Omega)$. More precisely, for every $w \in \mathcal{K}(\Omega)$ and for every Borel set $B \subseteq \Omega$ we have

$$\text{(5.2)} \qquad \mu(B) = \begin{cases} \displaystyle\int_B \frac{d\nu}{w^{p-1}}\,, & \textit{if} \quad C_p(B \cap \{w = 0\}) = 0, \\ +\infty\,, & \textit{if} \quad C_p(B \cap \{w = 0\}) > 0, \end{cases}$$

where ν is the non-negative measure of $H^{-1,q}(\Omega)$ defined by $\nu = 1 - Aw$.

Note that in view of (5.2) we have

$$\text{(5.3)} \qquad \nu(B \cap \{w > 0\}) = \int_B w^{p-1}\, d\mu$$

for every Borel set $B \subseteq \Omega$.

To prove the theorem we need some technical results. We begin with a penalization lemma.

LEMMA 5.2. *Let $\mu \in \mathcal{M}_0^p(\Omega)$ and let $u \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$. For every $n \in \mathbb{N}$ let $u_n \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ be the solution of the problem*

$$(5.4)\qquad \begin{cases} u_n \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)\,, \\ \langle Au_n, v\rangle + \int_\Omega |u_n|^{p-2} u_n v\, d\mu + n\int_\Omega |u_n|^{p-2} u_n v\, dx = n\int_\Omega |u|^{p-2} uv\, dx \\ \forall\, v \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega). \end{cases}$$

Then (u_n) converges to u strongly in $H_0^{1,p}(\Omega)$ and in $L_\mu^p(\Omega)$.

PROOF. Taking $v = u_n - u$ as test function in (5.4) we obtain

$$\begin{aligned} &\langle Au_n, u_n - u\rangle + \int_\Omega |u_n|^{p-2} u_n(u_n - u)\, d\mu \\ &\quad + n\int_\Omega \big(|u_n|^{p-2} u_n - |u|^{p-2} u\big)(u_n - u)\, dx = 0\,, \end{aligned}$$

hence

$$(5.5)\qquad \begin{aligned} &\langle Au_n - Au, u_n - u\rangle + \int_\Omega \big(|u_n|^{p-2} u_n - |u|^{p-2} u\big)(u_n - u)\, d\mu \\ &\quad + n\int_\Omega \big(|u_n|^{p-2} u_n - |u|^{p-2} u\big)(u_n - u)\, dx \\ &= -\langle Au, u_n - u\rangle - \int_\Omega |u|^{p-2} u(u_n - u)\, d\mu\,. \end{aligned}$$

If $2 \le p < +\infty$, from (2.5) we get

$$(5.6)\qquad \begin{aligned} &c_0\|u_n - u\|^p_{H_0^{1,p}(\Omega)} + 2^{2-p}\|u_n - u\|^p_{L_\mu^p(\Omega)} + n\,2^{2-p}\|u_n - u\|^p_{L^p(\Omega)} \\ &\le -\langle Au, u_n - u\rangle - \int_\Omega |u|^{p-2} u(u_n - u)\, d\mu\,, \end{aligned}$$

hence

$$\begin{aligned} &c_0\|u_n - u\|^p_{H_0^{1,p}(\Omega)} + 2^{2-p}\|u_n - u\|^p_{L_\mu^p(\Omega)} + n\,2^{2-p}\|u_n - u\|^p_{L^p(\Omega)} \\ &\le \|Au\|_{H^{-1,q}(\Omega)}\|u_n - u\|_{H_0^{1,p}(\Omega)} + \|u\|^{p/q}_{L_\mu^p(\Omega)}\|u_n - u\|_{L_\mu^p(\Omega)}\,. \end{aligned}$$

By using Young's inequality we obtain

$$\begin{aligned} &\frac{c_0}{q}\|u_n - u\|^p_{H_0^{1,p}(\Omega)} + \frac{2^{2-p}}{q}\|u_n - u\|^p_{L_\mu^p(\Omega)} + n\,2^{2-p}\|u_n - u\|^p_{L^p(\Omega)} \\ &\le \frac{c_0^{1-q}}{q}\|Au\|^q_{H^{-1,q}(\Omega)} + \frac{2}{q}\|u\|^p_{L_\mu^p(\Omega)}\,. \end{aligned}$$

This shows that (u_n) converges to u weakly in $H_0^{1,p}(\Omega)$ and in $L^p_\mu(\Omega)$, and by (5.6) this implies also that (u_n) converges to u strongly in $H_0^{1,p}(\Omega)$ and in $L^p_\mu(\Omega)$.

Assume now that $1 < p < 2$. If we apply (2.8) to the Lebesgue measure we obtain

$$\|u_n - u\|^2_{L^p(\Omega)} \leq 2\left(\|u_n\|^{2-p}_{L^p(\Omega)} + \|u\|^{2-p}_{L^p(\Omega)}\right)\int_\Omega (|u_n|^{p-2}u_n - |u|^{p-2}u)(u_n - u)\,dx.$$

This inequality, together with (2.6) and (5.5), gives

$$\begin{aligned}
&c_0\|u_n - u\|^2_{H_0^{1,p}(\Omega)} + \|u_n - u\|^2_{L^p_\mu(\Omega)} + n\|u_n - u\|^2_{L^p(\Omega)} \\
(5.7)\quad &\leq H(u_n,u)\left(\langle Au, u - u_n\rangle + \int_\Omega |u|^{p-2}u(u-u_n)\,d\mu\right) \\
&\leq H(u_n,u)\left(\|Au\|_{H^{-1,q}(\Omega)}\|u_n - u\|_{H_0^{1,p}(\Omega)} + \|u\|^{p-1}_{L^p_\mu(\Omega)}\|u_n - u\|_{L^p_\mu(\Omega)}\right),
\end{aligned}$$

where

$$\begin{aligned}
H(u_n,u) &= 2\left(\|u_n\|^{2-p}_{H_0^{1,p}(\Omega)} + \|u_n\|^{2-p}_{L^p_\mu(\Omega)} + \|u_n\|^{2-p}_{L^p(\Omega)}\right. \\
&\quad \left. + \|u\|^{2-p}_{H_0^{1,p}(\Omega)} + \|u\|^{2-p}_{L^p_\mu(\Omega)} + \|u\|^{2-p}_{L^p(\Omega)}\right) \\
&\leq 2\left(\|u_n - u\|^{2-p}_{H_0^{1,p}(\Omega)} + \|u_n - u\|^{2-p}_{L^p_\mu(\Omega)} + \|u_n - u\|^{2-p}_{L^p(\Omega)}\right) \\
&\quad + 4\left(\|u\|^{2-p}_{H_0^{1,p}(\Omega)} + \|u\|^{2-p}_{L^p_\mu(\Omega)} + \|u\|^{2-p}_{L^p(\Omega)}\right).
\end{aligned}$$

By using Young's inequality we obtain

$$\begin{aligned}
&\|u_n - u\|^2_{H_0^{1,p}(\Omega)} + \|u_n - u\|^2_{L^p_\mu(\Omega)} + n\|u_n - u\|^2_{L^p(\Omega)} \\
&\leq K_1(u)\left(1 + \|u_n - u\|^{3-p}_{H_0^{1,p}(\Omega)} + \|u_n - u\|^{3-p}_{L^p_\mu(\Omega)} + \|u_n - u\|^{3-p}_{L^p(\Omega)}\right) \\
&\leq \frac{1}{2}\left(\|u_n - u\|^2_{H_0^{1,p}(\Omega)} + \|u_n - u\|^2_{L^p_\mu(\Omega)} + \|u_n - u\|^2_{L^p(\Omega)}\right) + K_2(u),
\end{aligned}$$

where $K_1(u)$ and $K_2(u)$ are constants depending on u. This implies that

$$\|u_n - u\|^2_{H_0^{1,p}(\Omega)} + \|u_n - u\|^2_{L^p_\mu(\Omega)} + (2n-1)\|u_n - u\|^2_{L^p(\Omega)} \leq 2K_2(u),$$

and shows that (u_n) converges to u weakly in $H_0^{1,p}(\Omega)$ and in $L^p_\mu(\Omega)$. Therefore $(H(u_n,u))$ is bounded and by (5.7) (u_n) converges to u strongly in $H_0^{1,p}(\Omega)$ and in $L^p_\mu(\Omega)$. $\square$

Condition (ii) in the definition of $\mathcal{M}_0^p(\Omega)$ (see Remark 1.2) plays a crucial role in the following lemma, that will be used in the proof of the uniqueness result given by Lemma 5.4.

LEMMA 5.3. *Let $\mu \in \mathcal{M}_0^p(\Omega)$ and let w be the solution of problem* (5.1). *Then $\mu(B) = +\infty$ for every Borel set $B \subseteq \Omega$ with $C_p(B \cap \{w = 0\}) > 0$.*

PROOF. Let $u \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$, with $0 \le u \le 1$ q.e. in Ω, and, for every $n \in \mathbb{N}$, let $u_n \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega)$ be the solution of problem (5.4). By the comparison principles (Propositions 2.5 and 2.6) and by the homogeneity condition (1.3) we have $0 \le u_n \le n^{1/(p-1)} w$ q.e. in Ω, hence $u_n = 0$ q.e. in $\{w = 0\}$. Since by Lemma 5.2 (u_n) converges to u in $H_0^{1,p}(\Omega)$, we have $u = 0$ q.e. in $\{w = 0\}$.

Let U be a quasi open subset of Ω such that $\mu(U) < +\infty$. By Lemma 1.1 there exists an increasing sequence (z_n) in $H_0^{1,p}(\Omega)$ converging to 1_U quasi everywhere in Ω and such that $0 \le z_n \le 1_U$ q.e. in Ω for every $n \in \mathbb{N}$. As $\mu(U) < +\infty$, each function z_n belongs to $L_\mu^p(\Omega)$, hence $z_n = 0$ q.e. on $\{w = 0\}$ by the previous step. This implies that $C_p(U \cap \{w = 0\}) = 0$.

Let us consider a Borel set B with $C_p(B \cap \{w = 0\}) > 0$. For every quasi open set U containing B we have $C_p(U \cap \{w = 0\}) > 0$, hence $\mu(U) = +\infty$ by the previous step of the proof, and so $\mu(B) = +\infty$ by property (ii) in the definition of $\mathcal{M}_0^p(\Omega)$. □

We are now in a position to prove the following uniqueness result.

LEMMA 5.4. *Let λ and μ be measures of $\mathcal{M}_0^p(\Omega)$. Assume that there exists a function $w \in H_0^{1,p}(\Omega) \cap L_\lambda^p(\Omega) \cap L_\mu^p(\Omega)$ such that*

$$(5.8) \quad \langle Aw, v\rangle + \int_\Omega |w|^{p-2} wv\, d\lambda = \int_\Omega v\, dx \quad \forall v \in H_0^{1,p}(\Omega) \cap L_\lambda^p(\Omega),$$

$$(5.9) \quad \langle Aw, v\rangle + \int_\Omega |w|^{p-2} wv\, d\mu = \int_\Omega v\, dx \quad \forall v \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega).$$

Then $\lambda = \mu$.

PROOF. From the comparison principle (Proposition 2.5) we know that $w \ge 0$ q.e. in Ω. Let us consider the measures λ_0 and μ_0 defined for every Borel set $B \subseteq \Omega$ by

$$\lambda_0(B) = \int_B w^{p-1}\, d\lambda, \qquad \mu_0(B) = \int_B w^{p-1}\, d\mu.$$

We want to prove that $\lambda_0 = \mu_0$. For every $\varepsilon > 0$ let λ_ε and μ_ε be the measures defined by

$$\lambda_\varepsilon(B) = \int_{B\cap\{w>\varepsilon\}} w^{p-1}\, d\lambda, \qquad \mu_\varepsilon(B) = \int_{B\cap\{w>\varepsilon\}} w^{p-1}\, d\mu.$$

To prove that $\lambda_0 = \mu_0$ it is enough to show that $\lambda_\varepsilon = \mu_\varepsilon$ for every $\varepsilon > 0$. Let us fix $\varepsilon > 0$. As $w \in L_\lambda^p(\Omega) \cap L_\mu^p(\Omega)$, λ_ε and μ_ε are bounded measures.

Therefore it is enough to show that $\lambda_\varepsilon(U) = \mu_\varepsilon(U)$ for every open set $U \subseteq \Omega$. Let us fix U and let $U_\varepsilon = U \cap \{w > \varepsilon\}$. As U_ε is quasi open, by Lemma 1.1 there exists an increasing sequence (z_n) in $H_0^{1,p}(\Omega)$ converging to 1_{U_ε} quasi everywhere in Ω and such that $0 \le z_n \le 1_{U_\varepsilon}$ q.e. in Ω for every $n \in \mathbb{N}$. As $w \in L^p_\lambda(\Omega) \cap L^p_\mu(\Omega)$ and $w > \varepsilon$ q.e. in U_ε, we have $\lambda(U_\varepsilon) < +\infty$ and $\mu(U_\varepsilon) < +\infty$, hence $z_n \in L^p_\lambda(\Omega) \cap L^p_\mu(\Omega)$ for every $n \in \mathbb{N}$. From (5.8) and (5.9) we get

$$\int_\Omega w^{p-1} z_n \, d\lambda = \int_\Omega w^{p-1} z_n \, d\mu .$$

Taking the limit as $n \to \infty$ we obtain

$$\lambda_\varepsilon(U) = \int_{U_\varepsilon} w^{p-1} \, d\lambda = \int_{U_\varepsilon} w^{p-1} \, d\mu = \mu_\varepsilon(U) .$$

This shows that $\lambda_\varepsilon = \mu_\varepsilon$ for every $\varepsilon > 0$, hence $\lambda_0 = \mu_0$.

For every Borel set B contained in $\{w > 0\}$ we have

$$\lambda(B) = \int_B \frac{1}{w^{p-1}} \, d\lambda_0 = \int_B \frac{1}{w^{p-1}} \, d\mu_0 = \mu(B) .$$

If B is Borel set contained in $\{w = 0\}$ and $C_p(B) > 0$, then $\lambda(B) = \mu(B) = +\infty$ by Lemma 5.3. If $C_p(B) = 0$, then $\lambda(B) = \mu(B) = 0$ by the definition of $\mathcal{M}_0^p(\Omega)$. Therefore $\lambda(B) = \lambda(B \cap \{w > 0\}) + \lambda(B \cap \{w = 0\}) = \mu(B \cap \{w > 0\}) + \mu(B \cap \{w = 0\}) = \mu(B)$ for every Borel set $B \subseteq \Omega$. □

Proof of Theorem 5.1. Let us prove that $\mathcal{K}(\Omega)$ is weakly compact in $H_0^{1,p}(\Omega)$. Let (w_n) be a sequence in $\mathcal{K}(\Omega)$. As $\mathcal{K}(\Omega)$ is bounded in $H_0^{1,p}(\Omega)$, we may assume that (w_n) converges weakly in $H_0^{1,p}(\Omega)$ to a function w, and we have only to prove that $w \in \mathcal{K}(\Omega)$. As $w_n \ge 0$ q.e. in Ω, we have also $w \ge 0$ q.e. in Ω. To prove that $Aw \le 1$ in Ω, we consider the measures $\nu_n = 1 - Aw_n$. By the definition of $\mathcal{K}(\Omega)$, (ν_n) is a sequence of non-negative Radon measures which belong to $H^{-1,q}(\Omega)$. Since (w_n) is bounded in $H_0^{1,p}(\Omega)$, the upper bound (1.11) implies that (ν_n) is bounded in $H^{-1,q}(\Omega)$, and thus, since ν_n is non-negative, the sequence $(\nu_n(K))$ is bounded for every compact set $K \subseteq \Omega$. By Theorem 2.10 the sequence $\big(a(x, Dw_n)\big)$ converges to $a(x, Dw)$ weakly in $L^q(\Omega, \mathbb{R}^N)$, and hence (Aw_n) converges to Aw weakly in $H^{-1,q}(\Omega)$. As $Aw_n \le 1$ in $\mathcal{D}'(\Omega)$, we conclude that $Aw \le 1$ in $\mathcal{D}'(\Omega)$, hence $w \in \mathcal{K}(\Omega)$.

In the rest of the proof we follow the lines of Theorem 1 of [13] and of Proposition 3.4 of [21]. Let $\mu \in \mathcal{M}_0^p(\Omega)$ and let w be a solution of (5.1). Then $w \ge 0$ q.e. in Ω by the comparison principle (Proposition 2.5). From Proposition 2.8 and Remark 2.9 we deduce that $Aw \le 1$ in $\mathcal{D}'(\Omega)$, and we conclude that $w \in \mathcal{K}(\Omega)$.

Conversely, assume that $w \in \mathcal{K}(\Omega)$. Let $\nu = 1 - Aw$ and let μ be the measure defined by (5.2). We first prove that $\mu \in \mathcal{M}_0^p(\Omega)$. Since the measure ν

is non-negative and belongs to $H^{-1,q}(\Omega)$, we have $\nu(B)=0$ and thus $\mu(B)=0$ for every Borel set $B\subseteq\Omega$ with $C_p(B)=0$. It remains to prove that

$$\mu(B)=\inf\{\mu(U): U \text{ quasi open}\,, B\subseteq U\} \tag{5.10}$$

for every Borel set $B\subseteq\Omega$ with $\mu(B)<+\infty$. For every $n\in\mathbb{N}$ let μ_n be the measure on Ω defined by $\mu_n(B)=\mu(B\cap\{w>\frac{1}{n}\})$. Note that $\mu_n(\Omega)=\mu(\{w>\frac{1}{n}\})\le n^{p-1}\nu(\{w>\frac{1}{n}\})\le n^p\int_\Omega w\,d\nu=n^p\langle 1-Aw,w\rangle<+\infty$. Let us fix a Borel set $B\subseteq\Omega$ with $\mu(B)<+\infty$. By the definition of μ we have $C_p(B\cap\{w=0\})=0$. For every $n\ge 2$ let $B_n=B\cap\{\frac{1}{n}<w\le\frac{1}{n-1}\}$, and let $B_1=B\cap\{1<w\}$, so that $\mu(B)=\sum_n\mu(B_n)$. Since $\mu_n(\Omega)<+\infty$, for every $\varepsilon>0$ and for every $n\in\mathbb{N}$ there exists an open set V_n, with $B_n\subseteq V_n\subseteq\Omega$, such that $\mu_n(V_n)<\mu_n(B_n)+\varepsilon 2^{-n}=\mu(B_n)+\varepsilon 2^{-n}$. Let $U_n=V_n\cap\{w>\frac{1}{n}\}$. As w is quasi continuous, the set U_n is quasi open. Moreover $B_n\subseteq U_n$ and $\mu(U_n)=\mu_n(V_n)<\mu(B_n)+\varepsilon 2^{-n}$. Let $U_0=B\cap\{w=0\}$ and let U be the union of all sets U_n for $n\ge 0$. Then U is quasi open, contains B, and $\mu(U)<\mu(B)+\varepsilon$. Since $\varepsilon>0$ is arbitrary, this proves (5.10).

Let us prove that w is a solution of (5.1). By (5.2) we have

$$\int_\Omega w^p d\mu=\int_{\{w>0\}} w^p d\mu=\int_{\{w>0\}} w\,d\nu=\langle 1-Aw,w\rangle<+\infty\,,$$

hence $w\in L^p_\mu(\Omega)$. Let $v\in H^{1,p}_0(\Omega)\cap L^p_\mu(\Omega)$. By (5.2) we have $v=0$ q.e. in $\{w=0\}$. By the definitions of μ and ν we have

$$\begin{aligned}\langle Aw,v\rangle+\int_\Omega|w|^{p-2}wv\,d\mu&=\langle Aw,v\rangle+\int_{\{w>0\}}w^{p-1}v\,d\mu\\ =\langle Aw,v\rangle+\int_{\{w>0\}}v\,d\nu\ &=\langle Aw,v\rangle+\int_\Omega v\,d\nu=\int_\Omega v\,dx\,,\end{aligned}$$

which proves (5.1). The uniqueness of μ follows from Lemma 5.4. □

The following density result will be crucial in the proof of Theorem 6.3.

PROPOSITION 5.5. *Let $\mu\in\mathcal{M}^p_0(\Omega)$, let w be the solution of problem* (5.1), *and let $\beta\ge 1$. Then the set $\{w^\beta\varphi:\varphi\in C^\infty_0(\Omega)\}$ is dense in $H^{1,p}_0(\Omega)\cap L^p_\mu(\Omega)$.*

PROOF. First we note that, since $w\in H^{1,p}_0(\Omega)\cap L^\infty(\Omega)\cap L^p_\mu(\Omega)$ (Remark 3.2) and $\beta\ge 1$, the function $w^\beta\varphi$ belongs to $H^{1,p}_0(\Omega)\cap L^p_\mu(\Omega)$ for every $\varphi\in C^\infty_0(\Omega)$. To prove the density, for every $u\in H^{1,p}_0(\Omega)\cap L^p_\mu(\Omega)$ we have to construct a sequence (φ_n) in $C^\infty_0(\Omega)$ such that $(w^\beta\varphi_n)$ converges to u both in $H^{1,p}_0(\Omega)$ and in $L^p_\mu(\Omega)$. Since every function of $H^{1,p}_0(\Omega)\cap L^p_\mu(\Omega)$ can be approximated by truncation both in $H^{1,p}_0(\Omega)$ and in $L^p_\mu(\Omega)$, we may assume that $u\in H^{1,p}_0(\Omega)\cap L^\infty(\Omega)\cap L^p_\mu(\Omega)$ and that $u\ge 0$ q.e. in Ω. For every $n\in\mathbb{N}$ let u_n be the solution of (5.4). By the comparison principles (Propositions 2.5

and 2.6) and by the homogeneity condition (1.3) we have $0 \leq u_n \leq c\,w$ q.e. in Ω, where $c^{p-1} = n\|u\|^{p-1}_{L^\infty(\Omega)}$. By Lemma 5.2 the sequence (u_n) converges to u both in $H^{1,p}_0(\Omega)$ and in $L^p_\mu(\Omega)$. Therefore, in our approximation problem it is not restrictive to assume also that there exists $c > 0$ such that $0 \leq u \leq cw$ q.e. in Ω. Since $\{(u - c\varepsilon)^+ > 0\} \subseteq \{w > \varepsilon\}$, and $(u - c\varepsilon)^+$ converges to u, as $\varepsilon \to 0$, both in $H^{1,p}_0(\Omega)$ and in $L^p_\mu(\Omega)$, we may also assume that there exists $\varepsilon > 0$ such that $\{u > 0\} \subseteq \{w > \varepsilon\}$. Then $u/w^\beta = u/(w \vee \varepsilon)^\beta$. As $u \in H^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, we have $u/w^\beta \in H^{1,p}_0(\Omega) \cap L^\infty(\Omega)$. Therefore there exists a sequence (φ_n) in $C^\infty_0(\Omega)$, bounded in $L^\infty(\Omega)$, which converges to $z = u/w^\beta$ strongly in $H^{1,p}_0(\Omega)$ and quasi everywhere in Ω, hence μ-a.e. in Ω. Since $w \in H^{1,p}_0(\Omega) \cap L^\infty(\Omega)$ and $\beta \geq 1$, the sequence $(w^\beta\varphi_n)$ converges to $w^\beta z = u$ strongly in $H^{1,p}_0(\Omega)$. As $w \in L^p_\mu(\Omega) \cap L^\infty_\mu(\Omega)$ (Remark 3.2) and $\beta \geq 1$, the function w^β belongs to $L^p_\mu(\Omega) \cap L^\infty_\mu(\Omega)$. Since (φ_n) is bounded in $L^\infty(\Omega)$ and converges to $z = u/w^\beta$ μ-a.e. in Ω, by the dominated convergence theorem the sequence $(w^\beta\varphi_n)$ converges to $w^\beta z = u$ strongly in $L^p_\mu(\Omega)$. □

6. – γ^A-convergence: compactness and localization properties

γ^A-convergence. In this section we introduce the notion of γ^A-convergence in $\mathcal{M}^p_0(\Omega)$, which is defined as the convergence of the solutions of the corresponding relaxed Dirichlet problems. When $p = 2$ and A is the Laplace operator $-\Delta$, this notion is defined in [23] in terms of the Γ-convergence of the functionals $\int_\Omega |Du|^2 dx + \int_\Omega u^2 d\mu$ associated with the relaxed Dirichlet problems. For the extension of this definition to the case of symmetric linear operators we refer to [6] and [17]. For the case of non-symmetric linear operators see [21]. The case where A is the subdifferential of a convex functional $\int_\Omega \psi(x, Du)\,dx$ defined on $H^{1,p}_0(\Omega)$, with $1 < p < +\infty$, is studied in [20] by Γ-convergence techniques. The definition we introduce below involves only the solutions of the relaxed Dirichlet problems (2.1), and coincides with the previous definitions in all cases already considered in the literature.

Definition 6.1. Let (μ_n) be a sequence of measures of $\mathcal{M}^p_0(\Omega)$ and let $\mu \in \mathcal{M}^p_0(\Omega)$. We say that (μ_n) γ^A-converges to μ (in Ω) if for every $f \in H^{-1,q}(\Omega)$ the solution u_n of the problem

$$(6.1)\qquad \begin{cases} u_n \in H^{1,p}_0(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \\ \langle Au_n, v\rangle + \displaystyle\int_\Omega |u_n|^{p-2} u_n v\, d\mu_n = \langle f, v\rangle \quad \forall\, v \in H^{1,p}_0(\Omega) \cap L^p_{\mu_n}(\Omega) \end{cases}$$

converges weakly in $H^{1,p}_0(\Omega)$, as $n \to \infty$, to the solution u of the problem

$$(6.2)\qquad \begin{cases} u \in H^{1,p}_0(\Omega) \cap L^p_\mu(\Omega)\,, \\ \langle Au, v\rangle + \int_\Omega |u|^{p-2} uv\, d\mu = \langle f, v\rangle \quad \forall\, v \in H^{1,p}_0(\Omega) \cap L^p_\mu(\Omega)\,. \end{cases}$$

Let us emphasize the fact that the notion of γ^A-limit depends on the operator A. Although the definition depends also on Ω, we shall see in Theorems 6.11 and 6.12 that the boundary condition on $\partial\Omega$ does not play an important role in this problem.

REMARK 6.2. Since the solutions of (6.1) depend continuously on f, uniformly with respect to μ (Theorem 2.3), a sequence (μ_n) γ^A-converges to μ if and only if the solutions of (6.1) converge weakly in $H_0^{1,p}(\Omega)$ to the solution of (6.2) for every f in a dense subset of $H^{-1,q}(\Omega)$. □

Let (μ_n) be a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $\mu \in \mathcal{M}_0^p(\Omega)$. Let w_n and w be the solutions of the problems

$$(6.3)\quad \begin{cases} w_n \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \\ \langle Aw_n, v\rangle + \displaystyle\int_\Omega |w_n|^{p-2}\, w_n v\, d\mu_n = \int_\Omega v\, dx \quad \forall\, v \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \end{cases}$$

$$(6.4)\quad \begin{cases} w \in H_0^{1,p}(\Omega) \cap L^p_{\mu}(\Omega)\,, \\ \langle Aw, v\rangle + \displaystyle\int_\Omega |w|^{p-2}\, w v\, d\mu = \int_\Omega v\, dx \quad \forall\, v \in H_0^{1,p}(\Omega) \cap L^p_{\mu}(\Omega)\,. \end{cases}$$

The following theorem characterizes the γ^A-convergence of a sequence of measures (μ_n) in terms of the weak convergence in $H_0^{1,p}(\Omega)$ of the sequence (w_n).

THEOREM 6.3. *Let (μ_n) be a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and let $\mu \in \mathcal{M}_0^p(\Omega)$. Let w_n and w be the solutions of problems* (6.3) *and* (6.4). *The following conditions are equivalent:*

(a) *(w_n) converges to w weakly in $H_0^{1,p}(\Omega)$;*

(b) *(μ_n) γ^A-converges to μ.*

PROOF. To prove that (b) implies (a) it is enough to take $f = 1$ in the definition of γ^A-convergence. Conversely, assume that (a) holds true. Given $f \in L^\infty(\Omega)$, we consider the solutions u_n of problems (6.1). By estimate (2.3) the sequence (u_n) is bounded in $H_0^{1,p}(\Omega)$, so we may assume that (u_n) converges weakly in $H_0^{1,p}(\Omega)$ to some function u. We want to prove that u is the solution of (6.2). By the comparison principle (Proposition 2.7) and by the homogeneity condition (1.3) we have $|u_n| \le c\, w_n$ q.e. in Ω, with $c = \|f\|_{L^\infty(\Omega)}^{1/(p-1)}$. On letting $n \to \infty$ we get $|u| \le c\, w$ q.e. in Ω.

For every $\varepsilon > 0$ let $\Psi_\varepsilon\colon \mathbb{R} \to \mathbb{R}$ be the locally Lipschitz function defined by (4.3) and let $v_\varepsilon \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ be the function defined by $v_\varepsilon = \Psi_\varepsilon(\frac{u}{w \vee \varepsilon})$. Let us fix $\varphi \in C_0^\infty(\Omega)$ and $\beta \ge (p-1) \vee 1$. As $w_n \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ (Remark 3.2), by taking $v = w_n^\beta \varphi$ in (6.1) and $v = v_\varepsilon w_n^\beta \varphi$ in (6.3) we obtain

$$\begin{aligned} \langle Au_n, w_n^\beta \varphi\rangle + \int_\Omega |u_n|^{p-2}\, u_n w_n^\beta \varphi\, d\mu_n &= \int_\Omega f w_n^\beta \varphi\, dx\,, \\ \langle Aw_n, v_\varepsilon w_n^\beta \varphi\rangle + \int_\Omega |w_n|^{p-2}\, w_n v_\varepsilon w_n^\beta \varphi\, d\mu_n &= \int_\Omega v_\varepsilon w_n^\beta \varphi\, dx\,. \end{aligned}$$

By subtracting the second equation from the first one we get

$$\begin{aligned}(6.5)\qquad &\langle Au_n, w_n^\beta\varphi\rangle - \langle Aw_n, v_\varepsilon w_n^\beta\varphi\rangle \\ &+\int_\Omega |u_n|^{p-2}u_n w_n^\beta\varphi\,d\mu_n - \int_\Omega v_\varepsilon w_n^{\beta+p-1}\varphi\,d\mu_n \\ &= \int_\Omega f w_n^\beta\varphi\,dx - \int_\Omega v_\varepsilon w_n^\beta\varphi\,dx\,.\end{aligned}$$

By Lemmas 4.2 and 4.4 we have

$$\begin{aligned}(6.6)\qquad &\langle Au_n, w_n^\beta\varphi\rangle - \langle Aw_n, v_\varepsilon w_n^\beta\varphi\rangle \\ &+\int_\Omega |u_n|^{p-2}u_n w_n^\beta\varphi\,d\mu_n - \int_\Omega v_\varepsilon w_n^{\beta+p-1}\varphi\,d\mu_n \\ &= \langle Au, w^\beta\varphi\rangle - \langle Aw, v_\varepsilon w^\beta\varphi\rangle + \mathcal{R}_n^\varepsilon\,,\end{aligned}$$

with

$$\lim_{\varepsilon\to0}\limsup_{n\to\infty}|\mathcal{R}_n^\varepsilon| = 0\,.$$

Since (w_n) is bounded in $L^\infty(\Omega)$ (Remark 3.2) and converges to w strongly in $L^p(\Omega)$, for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty}\left(\int_\Omega f w_n^\beta\varphi\,dx - \int_\Omega v_\varepsilon w_n^\beta\varphi\,dx\right) = \int_\Omega f w^\beta\varphi\,dx - \int_\Omega v_\varepsilon w^\beta\varphi\,dx\,.$$

Together with (6.5) and (6.6) this implies that

$$\langle Au, w^\beta\varphi\rangle - \langle Aw, v_\varepsilon w^\beta\varphi\rangle = \int_\Omega f w^\beta\varphi\,dx - \int_\Omega v_\varepsilon w^\beta\varphi\,dx + \mathcal{R}^\varepsilon\,,$$

with

$$\lim_{\varepsilon\to0}|\mathcal{R}^\varepsilon| = 0\,.$$

Let $\nu = 1 - Aw$. By Theorem 5.1 w belongs to $\mathcal{K}(\Omega)$, thus ν is a non-negative Radon measure of $H^{-1,q}(\Omega)$. Therefore we have

$$(6.7)\qquad \langle Au, w^\beta\varphi\rangle + \int_\Omega v_\varepsilon w^\beta\varphi\,d\nu = \int_\Omega f w^\beta\varphi\,dx + \mathcal{R}^\varepsilon\,.$$

As $|u| \le c\,w$ q.e. in Ω (Remark 3.2), by (4.3) we have $|v_\varepsilon| \le (c\vee\varepsilon)^{p-1}$ q.e. in Ω. From definition of Ψ_ε we obtain that $(v_\varepsilon w^\beta)$ converges to $|u|^{p-2}uw^{\beta-p+1}$ quasi everywhere in Ω (recall that $\beta \ge p-1$). Since $\nu \in H^{-1,q}(\Omega)$ and w^β is bounded (Remark 3.2), we have $w^\beta\varphi \in L^1_\nu(\Omega)$, and thus, by the dominated convergence theorem,

$$\lim_{\varepsilon\to0}\int_\Omega v_\varepsilon w^\beta\varphi\,d\nu = \int_\Omega |u|^{p-2}uw^{\beta-p+1}\varphi\,d\nu\,.$$

Therefore (6.7) implies

$$(6.8) \qquad \langle Au, w^\beta \varphi\rangle + \int_\Omega |u|^{p-2} u w^{\beta-p+1} \varphi \, d\nu = \int_\Omega f w^\beta \varphi \, dx \,.$$

As $|u| \leq c\, w$ q.e. in Ω, and $w \in L^p_\mu(\Omega)$, we have $u \in L^p_\mu(\Omega)$ and, by (5.3),

$$\int_\Omega |u|^{p-2} u w^{\beta-p+1} \varphi \, d\nu = \int_{\{w>0\}} |u|^{p-2} u w^{\beta-p+1} \varphi \, d\nu = \int_\Omega |u|^{p-2} u w^\beta \varphi \, d\mu \,,$$

and thus (6.8) gives

$$\langle Au, w^\beta \varphi\rangle + \int_\Omega |u|^{p-2} u w^\beta \varphi \, d\mu = \int_\Omega f w^\beta \varphi \, dx \qquad \forall \varphi \in C_0^\infty(\Omega) \,.$$

Since the set $\{w^\beta \varphi : \varphi \in C_0^\infty(\Omega)\}$ is dense in $H_0^{1,p}(\Omega) \cap L^p_\mu(\Omega)$ (Proposition 5.5), it follows that u coincides with the solution of (6.2). Therefore (μ_n) γ^A-converges to μ by Remark 6.2. □

REMARK 6.4. The uniqueness of the γ^A-limit is an easy consequence of Theorem 6.3. Indeed, if (μ_n) γ^A-converges to μ and λ, then w satisfies (5.8) and (5.9), so that $\mu = \lambda$ by Lemma 5.4. □

Compactness and density properties. The following theorem proves the compactness of $\mathcal{M}_0^p(\Omega)$ with respect to γ^A-convergence.

THEOREM 6.5. *Every sequence of measures of $\mathcal{M}_0^p(\Omega)$ contains a γ^A-convergent subsequence.*

PROOF. Let (μ_n) be a sequence of measures of $\mathcal{M}_0^p(\Omega)$ and, for every $n \in \mathbb{N}$, let w_n be the solution of problem (6.3). By Theorem 5.1 each function w_n belongs to the set $\mathcal{K}(\Omega)$ defined at the beginning of Section 5. Since $\mathcal{K}(\Omega)$ is compact in the weak topology of $H_0^{1,p}(\Omega)$, a subsequence of (w_n) converges weakly in $H_0^{1,p}(\Omega)$ to some function $w \in \mathcal{K}(\Omega)$. By Theorem 5.1 there exists a measure $\mu \in \mathcal{M}_0^p(\Omega)$ such that w is the solution of problem (6.4). The conclusion follows now from Theorem 6.3. □

The case of Dirichlet problems in perforated domains is a particular case of the previous one. It is considered in the following theorem.

THEOREM 6.6. *Let (Ω_n) be an arbitrary sequence of open subsets of Ω. Then there exist a subsequence, still denoted by (Ω_n), and a measure $\mu \in \mathcal{M}_0^p(\Omega)$ such that for every $f \in H^{-1,q}(\Omega)$ the solution u_n of the problem*

$$u_n \in H_0^1(\Omega_n) \,, \qquad Au_n = f \quad \text{in } \mathcal{D}'(\Omega_n) \,,$$

extended by 0 *in $\Omega \setminus \Omega_n$, converges weakly in $H_0^{1,p}(\Omega)$ to the solution u of problem* (6.2).

PROOF. The conclusion follows easily from the compactness theorem (Theorem 6.5) and from the fact that each function u_n can be regarded as the solution of problem (6.1) with $\mu_n = \mu_{\Omega_n}$ (Remark 2.4). □

Using Theorem 6.3 we can prove the following density result in $\mathcal{M}_0^p(\Omega)$. We shall see in Corollary 6.10 that the strong convergence in $H_0^{1,p}(\Omega)$ of the sequence (w_n) implies the strong convergence in $H_0^{1,p}(\Omega)$ of the sequence (u_n) of the solutions of (6.1) for every $f \in H^{-1,q}(\Omega)$.

PROPOSITION 6.7. *Every measure $\mu \in \mathcal{M}_0^p(\Omega)$ is the γ^A-limit of a sequence (μ_n) of Radon measures of $\mathcal{M}_0^p(\Omega)$ such that the solution w_n of* (6.3) *converges strongly in $H_0^{1,p}(\Omega)$ to the solution w of* (6.4).

PROOF. By (5.2) a measure $\mu \in \mathcal{M}_0^p(\Omega)$ is a Radon measure if the solution w of (6.4) satisfies

$$\inf_K w > 0 \qquad \text{for every compact set } K \subseteq \Omega\,. \tag{6.9}$$

Now let $w_0 \in H_0^{1,p}(\Omega)$ be the solution of the equation $Aw_0 = 1$ in Ω. By the strong maximum principle (see [34]) w_0 satisfies (6.9).

Let us fix $\mu \in \mathcal{M}_0^p(\Omega)$ and let $w \in \mathcal{K}(\Omega)$ be the solution of (6.4). For every $n \in \mathbb{N}$, let us define $w_n = w \vee \frac{1}{n} w_0$. It is easy to see that w_n is a non-negative subsolution of the equation $Au = 1$, hence $w_n \in \mathcal{K}(\Omega)$ (see, e.g., [30], Theorem 3.23). Moreover the functions w_n satisfy (6.9) and converge to w strongly in $H_0^{1,p}(\Omega)$. Therefore the measures $\mu_n \in \mathcal{M}_0^p(\Omega)$ associated with w_n according to (5.2) are Radon measures and γ^A-converge to μ by Theorem 6.3. □

Strong convergence and correctors. The following theorem deals with the convergence of solutions, momenta, and energies, when also f varies.

THEOREM 6.8. *Let (μ_n) be a sequence of measures of $\mathcal{M}_0^p(\Omega)$ which γ^A-converges to a measure $\mu \in \mathcal{M}_0^p(\Omega)$ and let (f_n) be a sequence in $H^{-1,q}(\Omega)$ which converges strongly to some $f \in H^{-1,q}(\Omega)$. For every $n \in \mathbb{N}$ let u_n be the solution of the problem*

$$\begin{cases} u_n \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \\ \langle Au_n, v\rangle + \displaystyle\int_\Omega |u_n|^{p-2} u_n v\, d\mu_n = \langle f_n, v\rangle \quad \forall\, v \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \end{cases} \tag{6.10}$$

and let u be the solution of problem (6.2). *Then the sequence (u_n) converges to u weakly in $H_0^{1,p}(\Omega)$ and strongly in $H_0^{1,r}(\Omega)$ for every $r < p$. Moreover $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ and strongly in $L^s(\Omega, \mathbb{R}^N)$ for every $s < q$. Finally the energy $\big(a(x, Du_n), Du_n\big) + |u_n|^p \mu_n$ converges* *to* $\big(a(x, Du), Du\big) + |u|^p \mu$ *weakly* in the sense of Radon measures on* Ω, *i.e.*,

$$
\begin{aligned}
&\lim_{n\to\infty} \left(\int_\Omega (a(x, Du_n), Du_n)\, \varphi\, dx + \int_\Omega |u_n|^p \varphi\, d\mu_n \right) \\
&\quad = \int_\Omega (a(x, Du), Du)\, \varphi\, dx + \int_\Omega |u|^p \varphi\, d\mu
\end{aligned} \tag{6.11}
$$

for every $\varphi \in C_0(\Omega)$.

Proof. For every $n \in \mathbb{N}$, let v_n be the solution of problem (6.1). By Theorem 2.3 the sequence $(u_n - v_n)$ converges to 0 strongly in $H_0^{1,p}(\Omega)$. By the definition of γ^A-convergence, (v_n) converges to u weakly in $H_0^{1,p}(\Omega)$. Therefore (u_n) converges to u weakly in $H_0^{1,p}(\Omega)$. Moreover, by Theorem 2.11, (u_n) converges to u strongly in $H_0^{1,r}(\Omega)$ for every $r < p$, and $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(\Omega, \mathbb{R}^N)$ and strongly in $L^s(\Omega, \mathbb{R}^N)$ for every $s < q$.

By (6.10) for every $\varphi \in C_0^\infty(\Omega)$ we have

$$
\begin{aligned}
&\int_\Omega (a(x, Du_n), Du_n)\, \varphi\, dx + \int_\Omega |u_n|^p \varphi\, d\mu_n \\
&= \int_\Omega (a(x, Du_n), D(u_n\varphi))\, dx + \int_\Omega |u_n|^p \varphi\, d\mu_n - \int_\Omega (a(x, Du_n), D\varphi)\, u\, dx \\
&= \langle f_n, u_n\varphi \rangle - \int_\Omega (a(x, Du_n), D\varphi)\, u_n\, dx\,.
\end{aligned}
$$

Since (f_n) converges to f strongly in $H^{-1,q}(\Omega)$, while (u_n) converges to u weakly in $H_0^{1,p}(\Omega)$ and $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(\Omega, \mathbb{R}^N)$, we conclude that

$$
\begin{aligned}
&\lim_{n\to\infty} \left(\int_\Omega (a(x, Du_n), Du_n)\, \varphi\, dx + \int_\Omega |u_n|^p \varphi\, d\mu_n \right) \\
&= \langle f, u\varphi \rangle - \int_\Omega (a(x, Du), D\varphi)\, u\, dx \\
&= \int_\Omega (a(x, Du), D(u\varphi))\, dx + \int_\Omega |u|^p \varphi\, d\mu - \int_\Omega (a(x, Du), D\varphi)\, u\, dx \\
&= \int_\Omega (a(x, Du), Du)\, \varphi\, dx + \int_\Omega |u|^p \varphi\, d\mu
\end{aligned}
$$

for every $\varphi \in C_0^\infty(\Omega)$ (in the second equality we used the fact that u is the solution of (6.2)). Since the Radon measures $\big(a(x, Du_n), Du_n\big) + |u_n|^p \mu_n$ are non-negative, an easy approximation argument shows that (6.11) holds for every $\varphi \in C_0(\Omega)$. □

We consider now a corrector result for the strong convergence in $H_0^{1,p}(\Omega)$.

THEOREM 6.9. *Under the assumptions of Theorem* 6.8 *let* (P_n) *be the sequence of correctors defined by* (3.4), *where* w_n *and* w *are the solutions of* (6.3) *and* (6.4). *Then for every* $\varepsilon > 0$ *there exists a function* $u^\varepsilon \in H_0^{1,p}(\Omega) \cap L^\infty(\Omega) \cap L^p_\mu(\Omega)$, *with* $\|u - u^\varepsilon\|_{H_0^{1,p}(\Omega)} \le \varepsilon$ *and* $|u^\varepsilon| \le c^\varepsilon w$ *for some constant* $c^\varepsilon \ge 0$, *such that the sequence* (R_n^ε) *defined by*

$$Du_n = Du + u^\varepsilon P_n + R_n^\varepsilon \tag{6.12}$$

satisfies

$$\limsup_{n\to\infty} \|R_n^\varepsilon\|_{L^p(\Omega,\mathbb{R}^N)} \le \varepsilon\,. \tag{6.13}$$

If $f \in L^\infty(\Omega)$, *we can take* $\varepsilon \ge 0$ *and* $u^\varepsilon = u$ *for every* $\varepsilon \ge 0$.

PROOF. When f belongs to $L^\infty(\Omega)$ the corrector result is given by Theorem 3.1. When f belongs to $H^{-1,q}(\Omega)$, for every $\varepsilon > 0$, $\alpha > 0$, $K > 0$ we can choose $f^\varepsilon \in L^\infty(\Omega)$ such that $\|f - f^\varepsilon\|_{H^{-1,q}(\Omega)} \le (K\varepsilon)^\alpha$. If $2 \le p < +\infty$ we choose $\alpha = p-1$ and $K = \frac{1}{2}C^{-1/p}$, where C is the constant which appears in (2.9). If $1 < p \le 2$, we choose $\alpha = 1$ and $K = M \wedge 1$, where $M = \frac{1}{2}C^{-1/2}2^{(p-2)/p}(\|f\|_{H^{-1,q}(\Omega)} + \varepsilon)^{(p-2)/(p-1)}$ and C is the constant which appears in (2.10). We define v_n as the solution of problem (6.1), v_n^ε as the solution of the analogous problem relative to f^ε, and u^ε as the solution of problem (6.2) relative to f^ε. For $\varepsilon > 0$ fixed the sequence (v_n^ε) converges to u^ε weakly in $H_0^{1,p}(\Omega)$. From the uniform continuity (Theorem 2.3) we deduce that

$$\begin{aligned} \|u - u^\varepsilon\|_{H_0^{1,p}(\Omega)} &\le \varepsilon/2\,, \\ \|v_n - v_n^\varepsilon\|_{H_0^{1,p}(\Omega)} &\le \varepsilon/2\,, \end{aligned} \tag{6.14}$$

while Remark 3.2 implies that u^ε belongs to $L^\infty(\Omega)$ and $|u^\varepsilon| \le c^\varepsilon w$ q.e. in Ω, with $c^\varepsilon = \|f^\varepsilon\|_{L^\infty(\Omega)}^{1/(p-1)}$. The corrector result of Theorem 3.1 asserts that the sequence (Q_n^ε) defined by

$$Dv_n^\varepsilon = Du^\varepsilon + u^\varepsilon P_n + Q_n^\varepsilon \tag{6.15}$$

converges to 0 strongly in $L^p(\Omega, \mathbb{R}^N)$ when $\varepsilon > 0$ is fixed and $n \to \infty$. From (6.12) and (6.15) we obtain

$$R_n^\varepsilon = Q_n^\varepsilon + (Du^\varepsilon - Du) + (Dv_n - Dv_n^\varepsilon) + (Du_n - Dv_n)\,.$$

We then deduce (6.13) from (6.14), (6.15) and from the fact that $(u_n - v_n)$ converges to 0 strongly in $H_0^{1,p}(\Omega)$ by Theorem 2.3. □

COROLLARY 6.10. *Under the assumptions of Theorem* 6.8, *if the solution* w_n *of* (6.3) *converges strongly in* $H_0^{1,p}(\Omega)$ *to the solution* w *of* (6.4), *then* u_n *converges to* u *strongly in* $H_0^{1,p}(\Omega)$.

PROOF. For every ε let u^ε be the function introduced by Theorem 6.9. As $\frac{u^\varepsilon}{w}$ is bounded on $\{w > 0\}$, if (w_n) converges to w strongly in $H_0^{1,p}(\Omega)$, then for ε fixed $(u^\varepsilon P_n)$ converges to 0 strongly in $L^p(\Omega, \mathbb{R}^N)$. The conclusion follows now from (6.12) and (6.13). □

Localization properties. We conclude this section by showing the local character of the γ^A-convergence. The following theorem deals with the convergence of local solutions in an open subset U of Ω, without any assumption on the boundary conditions satisfied on ∂U. For every open set $U \subseteq \Omega$ the duality pairing between $H^{-1,q}(U)$ and $H_0^{1,p}(U)$ is denoted by $\langle \cdot, \cdot \rangle_U$. If we replace Ω by U in (1.12), we obtain an operator from $H_0^{1,p}(U)$ to $H^{-1,q}(U)$, which will still be denoted by A.

THEOREM 6.11. *Let (μ_n) be a sequence of measures of $\mathcal{M}_0^p(\Omega)$ which γ^A-converges in Ω to a measure $\mu \in \mathcal{M}_0^p(\Omega)$. Let U be an open subset of Ω, let (f_n) be a sequence in $H^{-1,q}(U)$ which converges strongly to some $f \in H^{-1,q}(U)$, and let (u_n) be a sequence in $H^{1,p}(U)$ which converges weakly to some $u \in H^{1,p}(U)$. Suppose that*

$$
(6.16) \qquad \begin{cases} u_n \in L^p_{\mu_n}(U') \qquad \forall U' \subset\subset U\,, \\ \langle Au_n, v\rangle_U + \displaystyle\int_U |u_n|^{p-2} u_n v \, d\mu_n = \langle f_n, v\rangle_U \\ \qquad\qquad \forall\, v \in H_0^{1,p}(U) \cap L^p_{\mu_n}(U) \quad \textit{with } \operatorname{supp}(v) \subset\subset U\,. \end{cases}
$$

Then

$$
(6.17) \qquad \begin{cases} u \in L^p_{\mu}(U') \qquad \forall U' \subset\subset U\,, \\ \langle Au, v\rangle_U + \displaystyle\int_U |u|^{p-2} u v \, d\mu = \langle f, v\rangle_U \\ \qquad\qquad \forall\, v \in H_0^{1,p}(U) \cap L^p_{\mu_n}(U) \quad \textit{with } \operatorname{supp}(v) \subset\subset U\,. \end{cases}
$$

Moreover, (u_n) converges to u strongly in $H^{1,r}(U)$ for every $r < p$, and $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(U, \mathbb{R}^N)$ and strongly in $L^s(U, \mathbb{R}^N)$ for every $s < q$. Finally the energy $\big(a(x, Du_n), Du_n\big) + |u_n|^p \mu_n$ converges to $\big(a(x, Du), Du\big) + |u|^p \mu$ weakly in the sense of Radon measures on U.*

PROOF. Let us fix an open set $U' \subset\subset U$ and a function $\zeta \in C_0^\infty(U)$ such that $\zeta \geq 0$ in U and $\zeta = 1$ in U'. Using $v = \zeta u_n$ as test function in (6.16) we obtain

$$
\int_{U'} |u_n|^p d\mu_n \leq \langle f_n, \zeta u_n\rangle_U - \langle Au_n, \zeta u_n\rangle_U \leq M
$$

for a suitable constant M. By Theorem 2.11 the sequence (u_n) converges to u strongly in $H^{1,r}(U')$ for every $r < p$, while $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(U', \mathbb{R}^N)$ and strongly in $L^s(U', \mathbb{R}^N)$ for every $s < q$. Since (u_n) is bounded in $H^{1,p}(U)$, and, by the upper bound (1.11), $\big(a(x, Du_n)\big)$ is bounded in

$L^q(U, \mathbb{R}^N)$, from the arbitrariness of $U' \subset\subset U$ we conclude that (u_n) converges to u strongly in $H^{1,r}(U)$ for every $r < p$, while $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(U, \mathbb{R}^N)$ and strongly in $L^s(U, \mathbb{R}^N)$ for every $s < q$.

Let us fix U' and ζ as before and let us consider the function φ defined by $\varphi(x) = \exp(1 - 1/\zeta(x))$, if $\zeta(x) > 0$, and $\varphi(x) = 0$, if $\zeta(x) = 0$. Then $\varphi \in C_0^\infty(U)$, $\varphi \geq 0$ in U, $\varphi = 1$ in U', and $\varphi^{p-1} \in C_0^\infty(U)$. This implies, in particular, that $\varphi^{p-1}v \in H_0^{1,p}(U)$ for every $v \in H^{1,p}(U)$. Let us define $z_n = \varphi u_n$, $z = \varphi u$, and

$$\begin{aligned} \psi_n &= a(x, \varphi Du_n + u_n D\varphi) - a(x, \varphi Du_n)\,, \\ \psi &= a(x, \varphi Du + u D\varphi) \quad - a(x, \varphi Du)\,. \end{aligned}$$

By the homogeneity condition (1.3) for every $v \in H_0^{1,p}(\Omega)$ we have

$$\begin{aligned} &(a(x, Dz_n), Dv) = \big(\psi_n, Dv\big) + (a(x, Du_n), Dv)\,\varphi^{p-1} \\ &= \big(\psi_n, Dv\big) + \Big(a(x, Du_n), D(\varphi^{p-1}v)\Big) - \Big(a(x, Du_n), D(\varphi^{p-1})\Big)\, v\,. \end{aligned}$$

Therefore (6.16) implies that z_n is the solution of the problem

$$\text{(6.18)} \qquad \begin{cases} z_n \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \\ \langle Az_n, v\rangle_\Omega + \displaystyle\int_\Omega |z_h|^{p-2} z_h v\, d\mu_n = \langle g_n, v\rangle_\Omega \\ \qquad\qquad \forall\, v \in H_0^{1,p}(\Omega) \cap L^p_{\mu_n}(\Omega)\,, \end{cases}$$

where g_n is the element of $H^{-1,q}(\Omega)$ defined by

$$\langle g_n, v\rangle_\Omega = \int_U (\psi_n, Dv)\, dx + \langle f_n, \varphi^{p-1}v\rangle_U - \int_U \Big(a(x, Du_n), D(\varphi^{p-1})\Big)\, v\, dx$$

for every $v \in H_0^{1,p}(\Omega)$. Let g be the element of $H^{-1,q}(\Omega)$ defined by

$$\langle g, v\rangle_\Omega = \int_U (\psi, Dv)\, dx + \langle f, \varphi^{p-1}v\rangle_U - \int_U \Big(a(x, Du), D(\varphi^{p-1})\Big)\, v\, dx$$

for every $v \in H_0^{1,p}(\Omega)$. Let us prove that (g_n) converges to g strongly in $H^{-1,q}(\Omega)$. Since $\big(a(x, Du_n)\big)$ converges to $a(x, Du)$ weakly in $L^q(U, \mathbb{R}^N)$ and $\varphi^{p-1} \in C_0^\infty(U)$, the last two terms in the definition of g_n converge strongly in $H^{-1,q}(\Omega)$ to the corresponding terms in the definition of g. It remains to prove that (ψ_n) converges to ψ strongly in $L^q(U, \mathbb{R}^N)$. Since (u_n) converges to u strongly in $H^{1,r}(U)$ for every $r < p$, passing to a subsequence, we may assume that (u_n) converges to u and (Du_n) converges to Du almost everywhere in U. Since $a(x, \cdot)$ is continuous, we have also that (ψ_n) converges to ψ almost everywhere in U. If $2 \leq p < +\infty$, by (1.5) there exists a constant C such that

$$|\psi_n|^q \leq C\left(|Du_n|^{q(p-2)} + |u_n|^{q(p-2)}\right)|u_n|^q = C\,|Du_n|^{q(p-2)}|u_n|^q + C\,|u_n|^p$$

for a.e. $x \in U$. As $\mathrm{supp}(\psi_n) \subseteq \mathrm{supp}(\varphi)$, by Hölder's inequality for every measurable set $E \subseteq U$ we have

$$\int_E |\psi_n|^q dx \le C\left(\int_U |Du_n|^p dx\right)^{(p-2)/(p-1)} \left(\int_{E\cap K} |u_n|^p dx\right)^{q/p} + C\int_{E\cap K} |u_n|^p dx,$$

where $K = \mathrm{supp}(\varphi)$. Since (u_n) converges strongly in $L^p(K)$, the sequence $(|\psi_n|^q)$ is equi-integrable on U. If $1 < p \le 2$, by (1.7) we have

$$|\psi_n|^q \le c_1^q |D\varphi|^p |u_n|^p \qquad \text{a.e. in } U,$$

and $(|\psi_n|^q)$ is equi-integrable on U in this case too. As (ψ_n) converges to ψ almost everywhere in U, by the dominated convergence theorem (ψ_n) converges to ψ strongly in $L^q(U, \mathbb{R}^N)$. Therefore (g_n) converges to g strongly in $H^{-1,q}(\Omega)$. Since (z_n) converges to z weakly in $H_0^{1,p}(\Omega)$, by (6.18) and by Theorem 6.8 the function z is the solution of the problem

$$\text{(6.19)} \qquad \begin{cases} z \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega), \\ \langle Az, v\rangle_\Omega + \displaystyle\int_\Omega |z|^{p-2} z v \, d\mu = \langle g, v\rangle_\Omega \quad \forall\, v \in H_0^{1,p}(\Omega) \cap L_\mu^p(\Omega). \end{cases}$$

As $\varphi = 1$ in U', we have $u = z$ q.e. in U', hence $u \in L_\mu^p(U')$. Moreover, if $v \in H_0^{1,p}(U) \cap L_\mu^p(U)$ with $\mathrm{supp}(v) \subset\subset U'$, then $\langle g, v\rangle = \langle f, v\rangle$, and so (6.19) implies (6.17). The convergence of the energy can be proved as in Theorem 6.8. □

THEOREM 6.12. *Let (μ_n) a sequence of measures of $\mathcal{M}_0^p(\Omega)$ which γ^A-converges in Ω to a measure $\mu \in \mathcal{M}_0^p(\Omega)$, and let U be an open subset of Ω. Then (μ_n) γ^A-converges to μ in U.*

PROOF. Let us fix $f \in H^{-1,q}(U)$. For every $n \in \mathbb{N}$ let u_n be the solution of problem (6.1), with Ω replaced by U. By estimate (2.3) we know that a subsequence, still denoted by (u_n), converges weakly in $H_0^{1,p}(U)$ to a function $u \in H_0^{1,p}(U)$. Then by Theorem 6.11 $u \in L_\mu^p(U')$ for every open set $U' \subset\subset U$ and u is a solution of problem (6.17).

It remains to prove that $u \in L_\mu^p(U)$. It is easy to construct a sequence (v_n) in $H_0^{1,p}(U)$, converging to u strongly in $H_0^{1,p}(U)$, such that $\mathrm{supp}(v_n) \subset\subset U$, $|v_n| \le |u|$ q.e. in U, and $uv_n \ge 0$ q.e. in U. Since $u \in L_\mu^p(U')$ for every open set $U' \subset\subset U$, each function v_n belongs to $L_\mu^p(U)$. We may also assume that (v_n) converges to u quasi everywhere in U and thus by Fatou's lemma

$$\int_U |u|^p d\mu \le \liminf_{n\to\infty} \int_U |u|^{p-2} u v_n \, d\mu.$$

Taking $v = v_n$ in (6.17) we get $\int_U |u|^{p-2} u v_n \, d\mu = \langle f, v_n\rangle - \langle Au, v_n\rangle$. As $n \to \infty$ we obtain $\int_U |u|^p d\mu \le \langle f, u\rangle - \langle Au, u\rangle < +\infty$, and thus $u \in L_\mu^p(U)$. By Proposition 5.5 the functions $v \in H_0^{1,p}(U) \cap L_\mu^p(U)$ with $\mathrm{supp}(v) \subset\subset U$ are dense in $H_0^{1,p}(U) \cap L_\mu^p(U)$. As $u \in L_\mu^p(U)$, (6.17) implies that u is the solution of problem (6.2) with Ω replaced by U. Since the limit does not depend on the subsequence, the proof is complete. □

COROLLARY 6.13. *Let μ_n, $\mu \in \mathcal{M}_0^p(\Omega)$. Let $(\Omega_i)_{i\in I}$ be a family of open subsets of Ω which covers Ω. Then (μ_n) γ^A-converges to μ in Ω if and only if (μ_n) γ^A-converges to μ in Ω_i for every $i \in I$.*

PROOF. The conclusion follows easily from the compactness of the γ^A-convergence (Theorem 6.5), from the localization property (Theorem 6.12), and from the uniqueness of the γ^A-limit (Remark 6.4). □

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