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# Nonuniqueness in the Martingale Problem and the Dirichlet Problem for Uniformly Elliptic Operators

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*Dedicated to Professor Carlo Pucci on the occasion of his seventieth birthday*

## 1. – Introduction

Let $L$ be an elliptic operator defined on $\mathbb{R}^n$:

$$L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j}, \tag{1}$$

where $a_{ij} = a_{ji}$ are measurable functions such that

$$\lambda^{-1}|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \lambda|\xi|^2,$$

$\lambda \geq 1$ being an ellipticity constant.

There exists a diffusion $(\xi_t, P_x)$ related to the operator $L$, [K1], [S-V]. Such a diffusion can be defined as a solution of the martingale problem:

1. $P_x(\xi_0 = x) = 1$,
2. $\phi(\xi_t) - \phi(\xi_0) - \int_0^t L\phi(\xi_s)ds$ is a $P_x$-martingale for all $\phi \in C^2(\mathbb{R}^n)$,

which has always a solution (Stroock, Varadhan, [S-V]).

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary $\partial\Omega$ and let $g \in C^2(\partial\Omega)$. Consider the following Dirichlet problem:

$$\begin{aligned} Lu &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

The diffusion $(\xi_t, P_x)$ defines a solution to problem (2):

$$u(x) = E_x g(\xi_\tau),$$

where $\tau$ is the first time when the path leaves the domain $\Omega$.

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Below we also give an analytic definition of a solution to the Dirichlet problem (2). There is an important problem on the uniqueness of the solutions which we define. If the coefficients $a_{ij}$ are smooth functions then the uniqueness of the solution of the Dirichlet problem and of the martingale problem is well known. The uniqueness is also known for special types of elliptic equations, see the discussion below, but for the general elliptic operator (1) with measurable coefficients the problem of uniqueness is open. This problem was rised and discussed by various authors, see e.g. [K3], [L1], [P-T]. The aim of this paper is to show that there is *NO* uniqueness in general.

We study directly the Dirichlet problem (2) and prove a nonuniqueness result without using probability methods. We give now an analytic definition of a solution to the Dirichlet problem (2). Let $L^k, 1, 2, \dots$, be a sequence of elliptic operators in $\Omega$ :

$$L^k = \sum_{i,j=1}^{n} a_{ij}^k(x) \frac{\partial^2}{\partial x_i \partial x_j}$$

with a common ellipticity constant $\lambda$ and with smooth coefficients such that $a_{ij}^k \to a_{ij}$ a.e. in $\Omega$ as $k \to \infty$. Let $u_k$ be the smooth solution of the Dirichlet problem

$$\begin{aligned} L^k u_k &= 0 \quad \text{in } \Omega, \\ u_k &= g \quad \text{on } \partial\Omega. \end{aligned}$$

From Krylov-Safonov's theorem [K-S] it follows that there exists a convergent subsequence $u_{k_m}$. We define a solution $u$ of the problem (2) as a limit of $u_{k_m}$:

$$u := \lim_{m \to \infty} u_{k_m}$$

A so defined solution $u$ was called in [C-E-F2] "good solution" to the problem (2).

THEOREM. *There exists an elliptic operator $L_1$ of the form* (1) *defined in the unit ball $B_1 \subset \mathbb{R}^n$, $n \geq 3$, and there is a function $g \in C^2(\partial B_1)$, such that the Dirichlet problem* (2) *has at least two good solutions.*

REMARK. Nonuniqueness in the martingale problem simply follows from the Theorem, see [K-2], [K-3].

In view of the Theorem it is interesting to discuss when uniqueness holds. The uniqueness holds: if dimension $n = 2$ (Bers, Boyarskii, Morrey, Nirenberg, see [B-J-S]), if $\lambda - 1 \leq \epsilon(n)$ where $\epsilon(n) > 0$ is a sufficiently small constant (Cordes, [C], Talenti, [T]), if coefficients $a_{ij}$ are continuous functions (Strook, Varadhan, [S-V]). For other results, see [A-M], [A-T], [B-P], [M-P], [L2], [L].

Let $E \subset \Omega$ be the set of points of discontinuity for the coefficients $a_{ij}$. The uniqueness of good solution to the Dirichlet problem (2) holds if $E$ is sufficiently “small” (Caffarelli, Cerutti, Escauriaza, Fabes, Krylov, Safonov, [C-E-F1], [C-E-F2], [K3], [S2]). In this direction the best known result is the following: the uniqueness holds if the Hausdorff dimension of the set $E$ is less than $\epsilon = \epsilon(\lambda, n) > 0$, [S2].

There is also a uniqueness result via Alexandrov-Pucci maximum principle, [A], [P]: the Dirichlet problem (2) has no more than one good solution in the Sobolev space $W^{2,n}(\Omega)$. But generically the problem (2) is unsolvable in the space $W^{2,n}(\Omega)$, [S1].

Regularity and some other properties of good solutions were studied in [N].

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## 2. – Proof of the theorem

**(2.1)** Assumptions. We assume that dimension $n = 3$. For any $n > 3$ the construction of the required elliptic operator is analogous.

We assume by contradiction that the good solution of the Dirichlet problem (2) is always unique.

Thus to prove the Theorem it will be sufficient to show the existence of two sequences $L^k$, $L_0^k$, $k = 1, 2, \dots$, of uniformly elliptic operators of the form (1) with measurable coefficients defined in $B_1 \subset \mathbb{R}^3$ and a function $g \in C^2(\partial B_1)$ such that if $u^k, u_0^k$, $k = 1, 2, \dots$ are good solutions of the Dirichlet problems:

$$\begin{aligned} L^k u^k &= 0 \quad \text{in } B_1, \\ u^k &= g \quad \text{on } \partial B_1, \end{aligned}$$

$$\begin{aligned} L_0^k u_0^k &= 0 \quad \text{in } B_1, \\ u_0^k &= g \quad \text{on } \partial B_1, \end{aligned}$$

then

$$\lim_{k\to\infty} u^k \neq \lim_{k\to\infty} u_0^k$$

**(2.2)** We consider an elliptic operator $L$ of type (1) defined in $\mathbb{R}^n$ whith smooth coefficients $a_{ij}$ wich are periodic functions of each variable with period 1. Denote

$$L_t = \sum_{i,j=1}^{n} a_{ij}(tx) \frac{\partial^2}{\partial x_i \partial x_j}.$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a smooth boundary, $g \in C(\partial\Omega)$,

$$L_t u_t = 0 \quad \text{in } \Omega,$$
$$u_t = g \quad \text{on } \partial\Omega.$$

As the parameter $t$ goes to $\infty$ the functions $u_t$ tend uniformly in $\Omega$ to a solution $u$ of a problem

$$\hat{L}u = 0 \quad \text{in } \Omega,$$
$$u = g \quad \text{on } \partial\Omega.$$

where

$$\hat{L} = \sum_{i,j=1}^{n} \hat{a}_{ij} \frac{\partial^2}{\partial x_i \partial x_j},$$

an elliptic operator with constant coefficients, cf.[B-L-P]

If we consider $L_1$ as an elliptic operator defined on the $n$-dimensional torus $T^n$, then by standart arguments from the Fredholm alternative it follows that there exists a unique equilibrium probability measure $mdx$ on $T^n$ such that $L^*m = 0$ on $T^n$, $m > 0$ on $T^n$ and

$$\int_{T^n} mdx = 1,$$

where $L^*$ is an operator adjoint to the operator $L$. The following equalities hold (Freidlin [F], see also [B-L-P]:

(3) $$\hat{a}_{ij} = \int_{T^n} a_{ij} m dx$$

Let $p$ be a quadratic form on $\mathbb{R}^n$ and $\hat{L}p = 0$. Set $f = Lp$. Then $f$ is a periodic function of each variable with period 1, such that

$$\int_{T^n} mf dx = 0.$$

Therefore there exists a solution $v$ of the following problem

$$Lv = f \qquad \text{on } T^n.$$

REMARK. Let $f$ be a positive scalar function on $T^n$. Then evidently we have $(fL)\hat{} = C\hat{L}$ for some positive constant $C$. Thus we can consider the operator $\hat{}$ as a map from the field of positive defined quadrics on the cotangent bundle to $T^n$ into the set of positive defined quadrics.

**(2.3)** Now let us assume that the elliptic operators $L_t$ defined in (2.2) have measurable coefficients. Let $L^k$, $k = 1, 2, \dots$, be a sequence of uniformly elliptic operators on $T^n$ with smooth coefficients which tend a.e. to the coefficients of

$L_1 = L$ as $k \to \infty$. Let $m^k$ be an equilibrium probability measure of $L^k$. We assume that $m^k$ converge weakly in $L_{n/(n-1)}$ to a function $m$ as $k \to \infty$. Let $\hat{a}_{ij}$ be given by (3) and let $p$ be a quadratic form in $\mathbb{R}^n$ such that $\hat{L}p = 0$. From assertion (2.2) it follows that there exists a good solution $v$ of the problem

$$L_1 v = L_1 p \qquad \text{in } \mathbb{R}^n$$

such that $v$ is a periodic function of each variable with period 1. We have

$$L_1(p - v) = 0 \qquad \text{in } \mathbb{R}^n \tag{4}$$

We denote by $\Sigma_t(\Omega)$ the set of all solutions $u$ of the equation $L_t u = 0$ in $\Omega \subset \mathbb{R}^n$. Let $\Sigma(\Omega)$ be a limit of $\Sigma_t(\Omega)$ in the $C(\Omega)$-topology as $t \to \infty$. Let $B_r$ be the ball $|x| < r$. If $u \in \Sigma(B_2)$, $e \in B_1$ then $u(x - e)|_{B_1} \in \Sigma(B_1)$ hence if $\Psi \in C_0^\infty(B_1)$ then $(u * \Psi)|_{B_1} \in \Sigma(B_1)$. Denote $\Lambda = (\Sigma(B_2) * \Psi)|_{B_1}$. We have

$$\Lambda \subset \Sigma(B_1) \cap C^\infty(B_1)$$

Let $v \in C^\infty(B_1)$. Denote by $j(v)$ the two-jet of function $v$ at the point the 0 (Taylor expansions up to the oder 2), and let $J$ be the set of all two-jets at 0. By the maximum principle $j(|x|^2) \notin j(\Lambda)$. Thus $j(\Lambda)$ is a proper linear subspace of $j$. From (4) it follows that $p \in \Lambda$. Therefore for all $\hat{L}u(0) = 0$ for all $u \in \Lambda$ and hence $\hat{L}u = 0$ in $B_1$. Since we can assume supp$\Psi \in B_\epsilon$ may be chosen for arbitrary small $\epsilon > 0$ we obtain that any function $h \in \Sigma(B_1)$ satisfies the equation $\hat{L}h = 0$. Thus we proved that the result of Assertion (2.2) is valid for good solution of elliptic equations with measurable coefficients.

**(2.4)** Let $L_t$, $\Omega$ be defined as in (2.2). Set

$$\Omega_\epsilon = [x \in \Omega, \operatorname{dist}(x, \partial\Omega) > \epsilon]$$

Let $P$ be an elliptic operator of the type (1) such that $P \equiv L_t$ on $\Omega_\epsilon$ for some $t$, $\epsilon > 0$. Let $u_1$, $u_2$ be solutions of the two following problems:

$$\begin{aligned} \hat{L}u_1 &= 0 \quad \text{in } \Omega, \\ u_1 &= g \quad \text{on } \partial\Omega, \end{aligned}$$

and

$$\begin{aligned} Pu_2 &= 0 \quad \text{in } \Omega, \\ u_2 &= g \quad \text{on } \partial\Omega. \end{aligned}$$

PROPOSITION. *$\partial u_2/\partial n \to \partial u_1/\partial n$ on $\partial\Omega$ as $t \to \infty$, $\epsilon \to 0$.*

PROOF. Since $\|Pu_1\|_{L_n} < C_1\epsilon$ in $\Omega$ it follows that $|u_1 - u_2| < C_2\epsilon$ on $\Omega\setminus\Omega_\epsilon$, $C_1$, $C_2 > 0$. From the Assertion (2.3) and from the maximum principle it follows that $|u_1 - u_2| < 2C_1\epsilon$ in $\Omega$ for sufficiently large $t > 0$. Pick $\delta > \epsilon$ and set $w = u_1 - u_2$. We have $|Pw| < C_3$ in $\Omega\setminus\Omega_\delta$, $w = 0$ on $\partial\Omega$, $|w| < 2C_1\epsilon$ on $\partial\Omega_\delta$. For sufficiently small $\delta > 0$ we get $P((\mathrm{dist}(x, \partial\Omega_\delta))^2) > 1$ in $\Omega\setminus\Omega_\delta$ and hence by the maximum principle the following inequality holds for sufficiently small $\epsilon > 0$:

$$C(\delta^2 - (\mathrm{dist}(x, \partial\Omega_\delta))^2) > |w| \qquad \text{in } \Omega \setminus \Omega_\delta.$$

We conclude $|\partial w/\partial n| < 4C\delta$ for sufficiently small $\delta$, $\epsilon > 0$. Since the constants $\delta$, $\epsilon > 0$ can be chosen arbitrarily small the Proposition follows.

**(2.5)** PROPOSITION. *Let $L$ be an elliptic operator of the type* (1) *defined in a bounded domain $\Omega \subset \mathbb{R}^n$ with a smooth boundary. Let $\Omega_0 \subset\subset \Omega$ be a subdomain with a smooth boundary. Let $\sigma$ be the distribution in $\Omega$ given by the surface area of $\partial\Omega_0$. Let $u$ satisfies the following differential inequality in the sense of distributions:*

$$\begin{aligned} -Lu &\le \sigma && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

*Then $u < C$, where $C = C(\Omega, \Omega_0, \lambda)$.*

PROOF. Let $\Gamma$ be a component of $\partial\Omega_0$ and $\Omega_1$ be the part of $\Omega$ bounded by $\partial\Omega$ and $\Gamma$. Let $v$ be a solution of the problems:

$$\begin{aligned} Lv &= 0 && \text{in } \Omega_1, \\ v &= 0 && \text{on } \partial\Omega, \\ v &= 1 && \text{on } \Gamma. \end{aligned}$$

Then $-\partial v/\partial n > C_1$ where $C_1 = C_1(\Omega, \Gamma, \lambda)$, see [G-T]. We set $V := v$ in $\Omega_1$, $V := 1$ in $\Omega \setminus \Omega_1$. Then $-LV > \lambda C_1$ and hence the Proposition follows from the maximum principle.

**(2.6)** Let $\Omega_1 \subset\subset \Omega_2 \subset \Omega \subset \mathbb{R}^n$ be bounded domains and let $\partial\Omega_1$, $\partial\Omega$ be smooth surfaces. Let

$$Q = \sum_{i,j=1}^{n} q_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}$$

an elliptic operator of the type (2) defined in $\Omega$ and $q_{ij}|_{\Omega_1} \equiv \text{const}$. Let $L_t$ be the elliptic operator defined in Section (2.2). Assume that

$$\hat{a}_{ij} = q_{ij}(\Omega_1)$$

We define $P_t^1 := Q$ on $\Omega \setminus \Omega_2$, $P_t^1 := L_t$ on $\Omega_2$, $P_t^2 := Q$ on $(\Omega \setminus \Omega_2) \cup \Omega_1$, $P_t^2 := L_t$ on $\Omega_2 \setminus \Omega_1$. Let $u_t^1$, $u_t^2$ be solutions of the following problems

$$\begin{aligned} P_t^i u_t^i &= 0 && \text{in } \Omega, \\ u_t^i &= g && \text{on } \partial\Omega, \\ i &= 1, 2, && g \in C(\partial\Omega) \end{aligned}$$

PROPOSITION. *$|u_t^1 - u_t^2| \to 0$ as $t \to \infty$;*

PROOF. Assume by contradiction that there exists a sequence $t_k \to \infty$ such that the following limits

$$u^i := \lim_{k\to\infty} u^i_{t_k}(x), \quad x \in \Omega$$

exist and $u^1 \not\equiv u^2$.

We have $Qu^1 = 0$ on $\Omega_2$ and hence $u^1$ is a smooth function on $\Omega_2$. Let $v_t$ be a solution of the following problem

$$\begin{aligned} P^1_t v_t &= 0 \quad \text{in } \Omega \setminus \Omega_1, \\ v_t &= g \quad \text{on } \partial\Omega, \\ v_t &= u \quad \text{on } \partial\Omega_1. \end{aligned}$$

By the maximum principle $|v_{t_k} - u^1_{t_k}| \to \Omega$ on $\Omega \setminus \Omega_1$ as $k \to \infty$. From Assertion (2.4) it follows that

$$|u^1_{t_k} - u^2_{t_k}| \to 0$$

as $k \to 0$. The Proposition is proved.

LEMMA. *Let $w$ be a good solution of the problem*

$$\begin{aligned} Qw &= 0 \quad \textit{in } \Omega, \\ w &= g \quad \textit{on } \partial\Omega. \end{aligned}$$

*Then $|u^2_t - w| \to 0$ in $\Omega$ as $t \to \infty$.*

PROOF. Choose domains $G_i \subset\subset \Omega_2$, $i = 1, 2, \dots$, such that

$$\cup_{i=1}^{\infty} G_i = \Omega_2$$

Set $R^i_t := Q$ on $(\Omega \setminus \Omega_2) \cup G_i$, $R^i_t := L_t$ on $\Omega_2 \setminus G_i$. Let $h^i_t$ be a solution of the problem

$$\begin{aligned} R^i_t h^i_t &= 0 \quad \text{on } \Omega, \\ h^i_t &= g \quad \text{on } \partial\Omega. \end{aligned}$$

By the Proposition there exists $\alpha(i)$ such that for any $\tau(i) > \alpha(i)$ the following inequality holds:

$$|u^2_{\tau(i)} - h^i_{\tau(i)}| < 1/i \quad \text{in } \Omega$$

Since the coefficients of the operators $R^n_{\tau(i)}$ as converge a.e. in $\Omega$ as $i \to \infty$ to the coefficients of $Q$, then $h^i_{\tau(i)} \to w$ as $i \to \infty$ and thus the Lemma immediately follows.

**(2.7)** Let $L$ be an elliptic operator of the type (1) with continuous coefficients defined in a bounded domain $\Omega \subset \mathbb{R}^n$. Let

$$E = \sum_{i,j=1}^{n} e_{ij}(x) \frac{\partial^2}{\partial x_i; \partial x_j}$$

a differential operator of the second order with measurable coefficients $e_{ij}$, $|e_{ij}| < \epsilon$. It is well known, e.g. [G-T], that for $1 < p < \infty$ the map

$$L: \ W_0^{2,p}(\Omega) \to L_p(\Omega)$$

is an isomorphism. Hence for sufficiently small $\epsilon > 0$, $\epsilon = \epsilon(L, \Omega, p)$ the map

$$(L + E): \ W_0^{2,p}(\Omega) \to L_p(\Omega)$$

is also an isomorphism.

**(2.8)** Let $\|a_{ij}\|$ be a 3x3-matrix. We say that $\|a_{ij}\|$ is in the class $M$ if its eigenvalues are $1, 1, 3$.

Below in Section (2.13) we define an elliptic operator $H$ on $T^3$ of the type (1):

$$H = \sum_{i,j=1}^{3} h_{ij}(x) \frac{\partial^2}{\partial x_1 \partial x_j}$$

with continuous coefficients and with the following properties:

(a) $\hat{h}_{11} = 1$, $\hat{h}_{22} = 1$, $\hat{h}_{33} = 3$, $\hat{h}_{ij} = 0$ if $i \neq j$;

(b) there is an open non-empty set $G \subset T^3$ with smooth boundary such that $\|h_{ij}(z)\| \in M$ for any $z \in G$;

(c) set $H_0 := H$ on $T^3 \setminus G$, $H_0 = \Delta$ on $G$, then $\hat{H}_0 = \Delta$

**(2.9)** Let $L$ be an elliptic operator of the type (1) defined in $B_1 \subset \mathbb{R}^3$. Set

$$\phi = \left( x_1^2 + x_2^2 - \frac{2}{3} x_3^2 \right) \Big|_{\partial B_1}$$

Let $u$ be a good solution of the Dirichlet problem:

$$\begin{aligned} Lu &= 0 \quad \text{in } B_1, \\ u &= \phi \quad \text{on } \partial B_1. \end{aligned}$$

By Assumptions (2.1) $u$ is unique. We denote

$$\alpha(L) := u(0).$$

It is elementary to check that

$$\alpha \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 3 \frac{\partial^2}{\partial x_3^2} \right) = 0$$

and

$$\alpha(\Delta) = \frac{4}{9}.$$

**(2.10)** This is the main Section in the proof of the Theorem. We define by induction a sequence of open sets $G_k \subset B_1 \subset \mathbb{R}^3$ and a sequence of elliptic operators $L^k$, $k = 1, 2, \dots$, in $B_1$ with a common ellipticity constant $\lambda$,

$$L^k = \sum_{i,j=1}^{3} a_{ij}^k(x) \frac{\partial^2}{\partial x_i \partial x_j},$$

such that the following holds:

(a) $G_{k+1} \subset G_k$ and meas $G_{k+1} \le c$ meas $G_k$ for some constant $c < 1$;

(b) $\alpha(L^k) < \frac{1}{4}$;

(c) set $L_0^k := L^k$ on $B_1 \setminus G_k$, $L_0^k := \Delta$ on $G_k$, then $\alpha(L_0^k) > \frac{1}{3}$.

We will consider the elliptic operator $H_0$ of Section (2.8) as an elliptic operator defined in $\mathbb{R}^3$ with periodic coefficients $h_{ij}$ and the set $G$ as a periodic set in $\mathbb{R}^3$. We define $G_1 := G$

$$L^1 := \sum_{i,j=1}^{3} h_{ij}(t_1 x) \frac{\partial^2}{\partial x_i \partial x_j},$$

where the constant $t_1$ is chosen so large that

$$\alpha(L^1) < \frac{1}{27},$$
$$\alpha(L_0^1) > \frac{4}{9} - \frac{1}{27}.$$

Assume now that operator $L^k$ is already defined for some $k \le 1$.

Let $D$ be an elliptic operator on $B_1$,

$$D = \sum_{i,j=1}^{n} d_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j},$$

such that $D = L^k$ on $B_1 \setminus G_k$, $|a_{ij}^k - d_{ij}| < \epsilon$ on $G_k, \epsilon > 0$, $d_{ij}$ are piecewise constant on $G_k$, $\|d_{ij}(z)\| \in M$ for $x \in G_k$.

By Assertion (2.7) for a sufficiently small $\epsilon > 0$ the following inequality holds:

$$|\alpha(L^k) - \alpha(D)| < 1/3^{k+4}$$

Let $z \in G_k$ and $l_z \in \mathbb{R}^3$ be an eigenvector of the matrix $\|d_{ij}(z)\|$ corresponding to its largest eigenvalue. Let

$$U_z : \ x \to \sum_{i,j=1}^{3} s_{ij}^z x_i$$

be an unitary map of $\mathbb{R}^3$ which maps the vector $l_z$ into the axis $x_3$. We define an elliptic operator $P$ on $B_1$: let $T > 0$ be a sufficiently large constant, set

$$P := \sum_{i,j,m,n=1}^{3} (TU_z(z))s^z_{im}s^z_{jn}\frac{\partial^2}{\partial x_i \partial x_j} \quad \text{on } G_k,$$
$$P := L^k \qquad \text{on } B_1 \setminus G_k.$$

We set $L^{k+1} := P$,

$$G_{k+1} := \{z \in G_k, TU_z(z) \in G_1\}.$$

By Assertion (2.6), the following inequalities hold for a sufficiently large $T$:

$$|\alpha(D) - \alpha(P)| < 1/3^{k+4},$$
$$|\alpha(L_0^k) - \alpha(L_0^{k+1})| < 1/3^{k+4},$$

hence

$$|\alpha(L^k) - \alpha(L^{k+1})| < 1/3^{k+3},$$

and therefore $\alpha(L^k) < 1/4$, $\alpha(L^k) > 1/3$.

We have $L^k = L^{k+1} = L_0^k = L_0^{k+1}$ on $B_1 \setminus G_k$. Thus the coefficients of the sequences $L^k$ and $L_0^k$ are convergent on the set $B \setminus E$ where $E = \cap_{i=1}^{\infty} G_i$. By (a) $meas E = 0$. Thus from (a) and (b) it follows that the existence of the sequences $L^k$, $L_0^k$ proves the Theorem.

In order to complete the proof of the existence of $L^k$, $L_0^k$ we still need to show the existence of the operator $H$ of Section (2.8).

**(2.11)** Let $Q$ be a Gilbarg-Serrin operator on $\mathbb{R}^2$

$$Q = \frac{1}{2}\sum_{i,j=1}^{2} q_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j},$$

where $q_{ij} = \delta_{ij} + 2x_i x_j/|x|^2$. It is easy to check, e.g. [G-S], that

$$Q|x|^{2/3} = 0 \quad \text{on } \mathbb{R}^2 \setminus 0,$$
$$Q^*|x|^{-2/3} = 0 \quad \text{on } \mathbb{R}^2 \setminus 0.$$

**(2.12)** Let $c_\rho(t)$, $t \in \mathbb{R}$, be a smooth function such that $0 \le c_\rho \le 1$, $c_\rho(t) \equiv 1$ for $t > \rho$, $c_\rho(t) \equiv 0$ for $t < \rho/2$, $\rho > 0$. Set

$$S = c_\rho(|x|)Q + (1 - c_\rho(|x|))\Delta$$

Let $v_\rho$ be a solution of the Dirichlet problem:

$$S^* v_\rho = 0 \quad \text{in } B_1,$$
$$v_\rho = 1 \quad \text{on } \partial B_1.$$

From (2.11) it follows that $v_\rho \to \infty$ on $B_\rho$ as $\rho \to \infty$.

By the Alexandrov-Pucci theorem we have

$$\int_{B_1} v_\rho^2 < C$$

where the constant $C > 0$ may be chosen independently of $\rho > 0$.

**(2.13)** Let $0 < \rho < \tau < 1/4$. We define an elliptic operator $L$ on $T^2$:

$$L = c(\tau + \rho - |x|)S + (1 - c(\tau + \rho - |x|))\Delta$$

By symmetry $\hat{L} = \Delta$. Let $m$ be an equilibrium probability measure on $T^2$ for the operator $L$. Pick constants $\tau, \rho > 0$ so small that

$$\pi\tau^2 < 1/10,$$
$$\gamma = \int_{B_\tau \setminus B_\rho} m dx < 1/10,$$
$$m > 0 \quad \text{on } B_\rho.$$

By Assertion (2.12), the last inequalities hold for sufficiently small $\tau, \rho > 0$.

Set

$$a = \int_{B_\rho} m dx/\pi\rho^2,$$
$$b = \int_{T^2 \setminus B_\rho} m dx/(1 - \pi\rho^2).$$

We set $L_0 = L$ on $T^2 \setminus B_\tau$, $L_0 = \Delta$ on $B_\tau$. Let $m_0$ be a probability equilibrium measure for $L_0$ on $T^2$. Set

$$d = \int_{B_\rho} m_0 dx,$$
$$e = \int_{T^2 \setminus B_\rho} m_0 dx.$$

For sufficiently small $\rho > 0$ we obtain : $|d - \pi\rho^2| < 1/10$, $|e - 1 + \pi\tau^2| < 1/10$

Let $y$,$z$ satisfy the following linear system:

$$ay + bz + \frac{1}{2}(1 - a - b) = 3$$
$$dy + ez + 1 - d - e = 1$$

It is elementary to check that $y > 0$, $z > 0$. Let us define functions $\psi_1, \psi_2$ on $T^2$ by:

$$\psi_1 = \begin{cases} y & \text{on } B_\rho, \\ z & \text{on } T^2 \setminus B_r, \\ 1/2 & \text{on } B_r \setminus B_\rho, \end{cases}$$

$$\psi_2 = \begin{cases} \psi_1 & \text{on } (T^2 \setminus B_r) \cup B_\rho, \\ 1 & \text{on } B_r \setminus B_\rho. \end{cases}$$

Then

$$\int_{T^2} \psi_1 m dx = 3, \tag{5}$$

$$\int_{T^2} \psi_2 m_0 dx = 1. \tag{6}$$

We defined $\psi_1$ as a piecewise continuous function on $T^2$. Evidently $\psi_1$ may be defined as a continuous function such that equalities (5), (6) hold and $\psi_1 = 1/2$ on $B_\tau \setminus B_\rho$.

We set

$$E := \{x \in T^3, \ (x_1, x_2) \in B_r \setminus B_\rho\}.$$

We define elliptic operators $H, \ H_0$ on $T^3$:

$$\begin{aligned} &H(x_1, x_2, x_3) = L(x_1, x_2) + \psi_1 \partial^2 \setminus \partial x_3^2, \\ &H_0 = H \quad \text{on } T^3 \setminus E, \\ &H_0 = \Delta \quad \text{on } E. \end{aligned}$$

We denote by $m_1(m_2)$ the probability equilibrium measures associated to the operators $H(H_0)$. Since the operators $H_1, \ H_0$ are invariant with respect to the shifts along variable $x_3$ we obtain $m_1(x_1, x_2, x_3) \equiv m(x_1, x_2), \ m_2(x_1, x_2 x_3) \equiv m_0(x_1, x_2)$. Hence

$$\begin{aligned} &\hat{H} = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + 3\frac{\partial^2}{\partial x_3^2}, \\ &\hat{H}_0 = \Delta. \end{aligned}$$

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