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On Homogenization of Solutions of Boundary Value Problems in Domains, Perforated Along Manifolds

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This paper is dedicated to the memory of E. De Giorgi,
a great mathematician of the XX century

1. In this paper the problem of homogenization of solutions to the Poisson equation in domains, perforated along a manifold, is considered with the Neumann boundary condition, with the Dirichlet boundary condition or the mixed condition on cavities. Some particular problems of this kind were considered in [1], [2]. The same method can be applied also to boundary value problems in domains, perforated in some subdomains. The short note on these results is published in [3]. We study here also the corresponding spectral problems. E.De Giorgi was one of the first mathematicians, who considered homogenization problems [4].

Let Ω be a bounded domain in R^n with a smooth boundary $\partial\Omega$ and γ be a manifold in $\overline{\Omega}$. Let P_j ($j = 1, \dots, N(\varepsilon)$ with $N(\varepsilon) \leq d_0\varepsilon^{1-n}$, $d_0 = \text{const}$) be a point such that $P_j \in \gamma$; ε is a small parameter. We denote by $G^j(a_\varepsilon^j)$ a domain which belongs to Ω, has a smooth boundary $\partial G^j(a_\varepsilon^j)$, $P_j \in \overline{G^j(a_\varepsilon^j)}$, the diameter of $G^j(a_\varepsilon^j)$ is a_ε^j and $a_\varepsilon^j \leq C_0\varepsilon$, $C_0 = \text{const} > 0$, $G^j(a_\varepsilon^j) \cap G^i(a_\varepsilon^i) = \emptyset$ for $i \neq j$. We consider all possible behavior of a_ε^j as $\varepsilon \to 0$.

We set

$$G_\varepsilon = \bigcup_{j=1}^{N(\varepsilon)} G^j(a_\varepsilon^j),\ \Omega_\varepsilon = \Omega \setminus \overline{G_\varepsilon},\ S'_\varepsilon = \bigcup_{j=1}^{N(\varepsilon)} \partial G^j(a_\varepsilon^j),$$

$$S_\varepsilon = \bigcup_{j=1}^{N(\varepsilon)} \partial G^j(a_\varepsilon^j) \cap \Omega,\ \Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus S_\varepsilon.$$

We assume that $G^j(a_\varepsilon^j)$ are such that any function $u \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ can be extended on Ω as a function $\tilde{u} \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ in such a way that

$$\|\tilde{u}\|_{H_1(\Omega,\Gamma_\varepsilon)} \leq K_1 \|u\|_{H_1(\Omega_\varepsilon,\Gamma_\varepsilon)}, \tag{1}$$

$$\|\nabla\tilde{u}\|_{L_2(\Omega)} \leq K_2 \|\nabla u\|_{L_2(\Omega_\varepsilon)}, \tag{2}$$

where K_j here and in what follows are constants which do not depend on ε.

The space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ is defined as a closure of $C^\infty(\overline{\Omega_\varepsilon})$-functions, which are equal to zero in a neighbourhood of Γ_ε, in the norm

$$\|u\|_{H_1(\Omega_\varepsilon,\Gamma_\varepsilon)} \equiv \left(\int_{\Omega_\varepsilon} (u^2 + |\nabla u|^2)dx\right)^{1/2}.$$

The cases, when it is possible, are considered in [5], [6].

In this domain Ω_ε we consider the boundary value problems for the equation

$$-\Delta u_\varepsilon = f \tag{3}$$

with the boundary conditions

$$\frac{\partial u_\varepsilon}{\partial \nu} = 0 \quad \text{on} \quad S_\varepsilon,\ u_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon, \tag{4}$$

or

$$\frac{\partial u_\varepsilon}{\partial \nu} + \beta u_\varepsilon = 0 \quad \text{on} \quad S_\varepsilon,\ u_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon,\ \beta(x) \geq \beta_0 = \text{const} > 0, \tag{5}$$

or

$$u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\varepsilon, \tag{6}$$

where ν is an outward unit normal vector to S_ε.

2. Let us consider the problem (3), (4) (the Neumann condition on S_ε).

We define a weak solution of the problem (3), (4) as a function $u_\varepsilon \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ which satisfies the integral identity

$$A_\varepsilon(u_\varepsilon, \varphi) \equiv \int_{\Omega_\varepsilon} (\nabla u_\varepsilon, \nabla\varphi)dx = \int_{\Omega_\varepsilon} f\varphi dx \tag{7}$$

for any function $\varphi \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$.

We also assume that for functions u from the space $H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ the Friedrichs inequality is valid:

$$\|u\|_{L_2(\Omega_\varepsilon)} \leq K_0\|\nabla u\|_{L_2(\Omega_\varepsilon)} \tag{8}$$

with the constant K_0, independent on ε. This inequality is satisfied if, for example, $\Gamma \subset \Gamma_\varepsilon$ and Γ is a smooth piece of $\partial\Omega$ with a positive measure on $\partial\Omega$, $\Gamma \cap \overline{G_\varepsilon} = \emptyset$ (see [5]).

From (7) and the Friedrichs inequality it follows that $\|u_\varepsilon\|_{H_1(\Omega_\varepsilon)} \leq K_3$.

Using the Riesz theorem, it is easy to prove that the problem (3), (4) has a unique weak solution in Ω_ε.

Let the function v_0 be a solution of the problem

$$-\Delta v = f \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega, \tag{9}$$

and f is a smooth function in $\overline{\Omega}$.

First we consider the case when $S'_\varepsilon \cap \partial\Omega = \emptyset$. Using the integral identity (7) for problem (3),(4) and the integral identity for the problem (9), we get

$$\int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - v_0)|^2 dx = \int_{G_\varepsilon} f(\tilde{u_\varepsilon} - v_0)dx + \int_{G_\varepsilon} (\nabla v_0, \nabla(\tilde{u_\varepsilon} - v_0))dx, \tag{10}$$

where $\tilde{u_\varepsilon}$ is an extention of u_ε in Ω, such that $\tilde{u_\varepsilon} - v_0$ satisfies (1),(2). From (10) we have

$$\begin{aligned} \|\nabla(u_\varepsilon - v_0)\|^2_{L_2(\Omega_\varepsilon)} \le {} & \max_{\overline{\Omega}} |f||G_\varepsilon|^{1/2}\|\tilde{u}_\varepsilon - v_0\|_{L_2(G_\varepsilon)} \\ & + \max_{\overline{\Omega}} |\nabla v_0||G_\varepsilon|^{1/2}\|\nabla(\tilde{u}_\varepsilon - v_0)\|_{L_2(G_\varepsilon)}. \end{aligned} \tag{11}$$

Using inequalities (1), (2) for $\tilde{u_\varepsilon} - v_0$, and the Friedrichs inequality, we obtain from (11) that

$$\begin{aligned} K_4\|\nabla(\tilde{u}_\varepsilon - v_0)\|_{L_2(\Omega)} \le \|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)} & \le K_5\big(\max_{\overline{\Omega}} |f| + \max_{\overline{\Omega}} |\nabla v_0|\big)|G_\varepsilon|^{1/2} \\ & \le K_6|G_\varepsilon|^{1/2} \le K_7(\max_j a_\varepsilon^j)^{n/2}\varepsilon^{(1-n)/2}, \end{aligned}$$

where $|G|$ is the measure of the set G.

Therefore,

$$\|u_\varepsilon - v_0\|^2_{H^1(\Omega_\varepsilon)} \le K_8(\max_j a_\varepsilon^j)^n \varepsilon^{1-n}. \tag{12}$$

Let us consider the case, when $S'_\varepsilon \cap \partial\Omega \neq \emptyset$, and let $G^j(a_\varepsilon^j)$, $j = 1, \dots, M(\varepsilon)$, be such that $\overline{G^j}(a_\varepsilon^j) \cap \partial\Omega \neq \emptyset$, $M(\varepsilon) \le d_1\varepsilon^{-n+2}$, $d_1 = \text{const} > 0$, and $|\partial G^j(a_\varepsilon^j)| \le d_2(a_\varepsilon^j)^{n-1}$. We set

$$l_\varepsilon = \partial\Omega \cap \bigcup_{j=1}^{M(\varepsilon)} \overline{G^j}(a_\varepsilon^j).$$

Since v_0 is a smooth function, we derive from the Green formula that

$$\int_{\Omega} (\nabla v_0, \nabla(\tilde{u}_\varepsilon - v_0))dx = \int_{\Omega} f(\tilde{u}_\varepsilon - v_0)dx + \int_{l_\varepsilon} \frac{\partial v_0}{\partial \nu}(\tilde{u}_\varepsilon - v_0)ds. \tag{13}$$

From integral identity (7) with $\varphi = (u_\varepsilon - v_0)$ and (13), we obtain

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - v_0)|^2 dx = {} & \int_{G_\varepsilon} (\nabla v_0, \nabla(\tilde{u}_\varepsilon - v_0))dx - \int_{G_\varepsilon} f(\tilde{u}_\varepsilon - v_0)dx \\ & - \int_{l_\varepsilon} \frac{\partial v_0}{\partial \nu}(\tilde{u}_\varepsilon - v_0)ds \equiv J^\varepsilon. \end{aligned} \tag{14}$$

Let us estimate J^ε. Using (1), (2), the Friedrichs inequality and the imbedding theorem, we get

$$\begin{aligned}\|\nabla(u_\varepsilon - v_0)\|^2_{L_2(\Omega_\varepsilon)} &\le K_9\{|G_\varepsilon|^{1/2}(\|\nabla(\tilde{u}_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)} \\ &\quad + \|\tilde{u}_\varepsilon - v_0\|_{L_2(\Omega_\varepsilon)}) + \|\tilde{u}_\varepsilon - v_0\|_{L_2(\partial\Omega)}|l_\varepsilon|^{1/2}\} \\ &\le K_{10}\{|G_\varepsilon|^{1/2} + |l_\varepsilon|^{1/2}\}\|\tilde{u}_\varepsilon - v\|_{H_1(\Omega_\varepsilon)} \\ &\le K_{11}(|G_\varepsilon|^{1/2} + |l_\varepsilon|^{1/2})\|\nabla(u_\varepsilon - v_0)\|_{L_2(\Omega_\varepsilon)}.\end{aligned} \tag{15}$$

Due to assumptions on $M(\varepsilon)$ and $\partial G^j(a^j_\varepsilon)$ we have

$$|l_\varepsilon| \le K_{12}(\max_j a^j_\varepsilon)^{n-1}\varepsilon^{2-n}. \tag{16}$$

From estimates (15), (16) it follows that

$$\|u_\varepsilon - v_0\|^2_{H_1(\Omega_\varepsilon)} \le K_{13}\max_j (a^j_\varepsilon)^{n-1}\varepsilon^{2-n}. \tag{17}$$

Hence, we proved the following theorem.

THEOREM 1. *Let u_ε be a weak solution of the problem* (3), (4), *v_0 be a solution of the problem* (9), *$S'_\varepsilon \cap \partial\Omega = \emptyset$. Then*

$$\|u_\varepsilon - v_0\|^2_{H_1(\Omega_\varepsilon)} \le K_{14}(\max_j a^\varepsilon_j)^n \varepsilon^{1-n}.$$

If $S'_\varepsilon \cap \partial\Omega \ne \emptyset$ and $M(\varepsilon) \le d_1\varepsilon^{2-n}$, $|\partial G^j(a^j_\varepsilon)| \le d_2(a^j_\varepsilon)^{n-1}$, then

$$\|u_\varepsilon - v_0\|^2_{H_1(\Omega_\varepsilon)} \le K_{15}\max_j (a^j_\varepsilon)^{n-1}\varepsilon^{2-n}.$$

We note that in the proof of Theorem 1 the assumption that $P_j \in \gamma$ is not used.

3. Let us consider the problem (3), (5) (the mixed boundary condition). Let $\beta(x) \ge \beta_0 = \text{const} > 0$ and v_0 be a solution of problem (9). Then the function $w_\varepsilon = u_\varepsilon - v_0$ is a weak solution of the problem

$$\begin{aligned}&-\Delta w_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon,\ w_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon, \\ &\frac{\partial w_\varepsilon}{\partial \nu} + \beta(x)w_\varepsilon = -(\frac{\partial v_0}{\partial \nu} + \beta(x)v_0) \quad \text{on} \quad S_\varepsilon.\end{aligned} \tag{18}$$

From the integral identity for the problem (18) we have

$$\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 dx + \int_{S_\varepsilon} \beta(x) w^2_\varepsilon ds = -\int_{S_\varepsilon} (\frac{\partial v_0}{\partial \nu} + \beta(x)v_0)w_\varepsilon ds. \tag{19}$$

For the right-hand side of (19) we get

$$\begin{aligned}
&\left|\int_{S_\varepsilon}\left(\frac{\partial v_0}{\partial \nu}+\beta(x)v_0\right)w_\varepsilon ds\right| \\
&\le \frac{1}{2}\beta_0\int_{S_\varepsilon} w_\varepsilon^2 ds + K_{16}\int_{S_\varepsilon}\left(\frac{\partial v_0}{\partial \nu}+\beta(x)v_0\right)^2 ds \\
&\le \frac{1}{2}\int_{S_\varepsilon}\beta(x) w_\varepsilon^2 ds + K_{16}\int_{S_\varepsilon}\left(\frac{\partial v_0}{\partial \nu}+\beta(x)v_0\right)^2 ds.
\end{aligned} \tag{20}$$

From inequalities (20) and (19) it follows that

$$\|\nabla(u_\varepsilon - v_0)\|^2_{L_2(\Omega_\varepsilon)} \le K_{17}\|\frac{\partial v_0}{\partial \nu}+\beta(x)v_0\|^2_{L_2(S_\varepsilon)}. \tag{21}$$

Taking into account that $v_0(x)$ is a smooth function, we obtain

$$\|u_\varepsilon - v\|^2_{H_1(\Omega_\varepsilon)} \le K_{18}|S_\varepsilon| \le K_{19}(\max_j a_\varepsilon^j)^{n-1}\varepsilon^{1-n}.$$

From this estimate we derive the following theorem.

THEOREM 2. *Let* $\beta(x)\ge\beta_0=\text{const}>0$, u_ε *be a solution of the problem* (3), (5), v_0 *be a solution of the problem* (9). *Assume that* $|\partial G^j(a_\varepsilon^j)|\le K_{20}(a_\varepsilon^j)^{n-1}$, $\eta_\varepsilon \equiv (\max_j a_\varepsilon^j)\varepsilon^{-1}\to 0$ *as* $\varepsilon\to 0$. *Then* $\|u_\varepsilon - v_0\|_{H_1(\Omega_\varepsilon)}\to 0$ *as* $\varepsilon\to 0$ *and*

$$\|u_\varepsilon - v_0\|^2_{H_1(\Omega_\varepsilon)} \le K_{21}\eta_\varepsilon^{n-1}.$$

In the proof of Theorem 2 we do not use that $P_j\in\gamma$.

4. Assume that γ is a domain in the hyperplane $\{x : x_1 = 0\}$ and $\gamma = \Omega\cap\{x : x_1 = 0\}$, $Q=\{x : -1/2 < x_j < 1/2,\ j=1,\dots,n\}$, $G'_\varepsilon = \bigcup_{z\in Z}(a_\varepsilon G_0 + \varepsilon z)$, where G_0 is a smooth domain and $a_\varepsilon \le C\varepsilon$, $C=\text{const}$, $\overline{G_0}\subset Q$, Z is the set of vectors with integer components. Let $a_\varepsilon\varepsilon^{-1}\to C_0=\text{const}>0$ as $\varepsilon\to 0$ and $C_0\overline{G_0}\subset Q$.

We set

$$\Pi_\varepsilon = \Omega\cap\{x : |x_1| < \varepsilon/2\},\ \Pi_\varepsilon^* = \Pi_\varepsilon\setminus\overline{G'_\varepsilon},\ \gamma_\varepsilon^\pm = \Omega\cap\{x : x_1 = \pm\varepsilon/2\},$$

$$\Omega_\varepsilon^+ = \Omega\cap\{x : x_1 > \varepsilon/2\},\ \Omega_\varepsilon^- = \Omega\cap\{x : x_1 < -\varepsilon/2\},\ G_\varepsilon = G'_\varepsilon\cap\Omega,$$

$$\Omega_\varepsilon = \Omega_\varepsilon^+\cup\gamma_\varepsilon^+\cup\Pi_\varepsilon^*\cup\gamma_\varepsilon^-\cup\Omega_\varepsilon^-, \tag{22}$$

$$S_\varepsilon = \partial G_\varepsilon\cap\Omega,\ \Gamma_\varepsilon = \partial\Omega_\varepsilon\setminus S_\varepsilon,\ l_\varepsilon = \overline{G}_\varepsilon\cap\partial\Omega.$$

We assume that $\beta(x) = \beta_0 = \text{const} > 0$. Let v be a weak solution of the problem

$$
\begin{aligned}
&-\Delta v = f \quad \text{in} \quad \Omega \setminus \gamma, \quad v = 0 \quad \text{on} \quad \partial\Omega, \\
&[v]\big|_\gamma = 0, \ \left[\frac{\partial v}{\partial x_1}\right]\big|_\gamma = \mu v\big|_\gamma,
\end{aligned} \tag{23}
$$

where $\mu = \beta_0 C_0^{n-1}|\partial G_0|$, $[\varphi]\Big|_\gamma = \varphi(x_1+0, x') - \varphi(x_1-0, x')$, $(x_1, x') \in \gamma$, $x' = (x_2, \dots, x_n)$.

We assume that $|v(x)| < K_{22}$ for $x \in \overline{\Omega}$, $|\nabla v(x)| \le K_{23}$ in $\Omega^+ = \Omega \cap \{x : x_1 > 0\}$ and $|\nabla v(x)| \le K_{24}$ in $\Omega^- = \Omega \cap \{x : x_1 < 0\}$. From the integral identity for the problem (3), (5) we obtain

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} (\nabla u_\varepsilon, \nabla(u_\varepsilon - v))dx + \beta_0 \int_{S_\varepsilon} (u_\varepsilon - v)^2 ds + \beta_0 \int_{S_\varepsilon} v(u_\varepsilon - v)ds \\
&= \int_{\Omega_\varepsilon} f(u_\varepsilon - v)dx.
\end{aligned} \tag{24}
$$

Using the assumptions on $v(x)$ and the Green formula we get

$$
\begin{aligned}
&\int_\Omega (\nabla v, \nabla(\tilde{u}_\varepsilon - v))dx + \beta_0 C_0^{n-1}|\partial G_0| \int_\gamma v(\tilde{u}_\varepsilon - v)dx' \\
&= \int_\Omega f(\tilde{u}_\varepsilon - v)dx + \int_{l_\varepsilon} \frac{\partial v}{\partial \nu}(\tilde{u}_\varepsilon - v)ds,
\end{aligned} \tag{25}
$$

where $l_\varepsilon = \partial\Omega \cap \overline{G}_\varepsilon$, $\tilde{u}_\varepsilon - v$ is an extention of $u_\varepsilon - v$ in Ω such that (1), (2) are satisfied.

From (24), (25) we derive

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} |\nabla(u_\varepsilon - v)|^2 dx + \beta_0 \int_{S_\varepsilon} (u_\varepsilon - v)^2 ds \\
&= \beta_0 \{C_0^{n-1}|\partial G_0| \int_\gamma v(\tilde{u}_\varepsilon - v)dx' - \int_{S_\varepsilon} v(u_\varepsilon - v)ds\} + P_\varepsilon,
\end{aligned} \tag{26}
$$

where

$$
P_\varepsilon \equiv -\int_{G_\varepsilon} f(\tilde{u}_\varepsilon - v)dx - \int_{l_\varepsilon} \frac{\partial v}{\partial \nu}(\tilde{u}_\varepsilon - v)ds. \tag{27}
$$

LEMMA 1. *Assume that* $v \in H_1(\Omega_\varepsilon, \Gamma_\varepsilon)$ *and* $(a_\varepsilon \varepsilon^{-1} - C_0) \to 0$ *as* $\varepsilon \to 0$, Ω_ε *is defined by* (22). *Then*

$$
\left| \int_{S_\varepsilon} v ds - C_0^{n-1}|\partial G_0| \int_\gamma v dx' \right| \le K_{25}\{\sqrt{\varepsilon} + |a_\varepsilon \varepsilon^{-1} - C_0|\}\|v\|_{H_1(\Omega)}. \tag{28}
$$

PROOF. Consider the function $\theta_\varepsilon(y)$, $y = \varepsilon^{-1}x$, as a solution of the problem

$$(29)\quad \begin{cases} \Delta_y \theta_\varepsilon = 0 & y \in Q \setminus \overline{a_\varepsilon \varepsilon^{-1} G_0} = Y_\varepsilon, \\ \dfrac{\partial \theta_\varepsilon}{\partial \nu} = 1 & \text{on} \quad a_\varepsilon \varepsilon^{-1} \partial G_0, \\ \dfrac{\partial \theta_\varepsilon}{\partial y_1} = (a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| & \text{on} \quad \Sigma_0 = \partial Q \cap \{y : y_1 = -1/2\}, \\ \dfrac{\partial \theta_\varepsilon}{\partial y_1} = 0 & \text{on} \quad \Sigma_1 = \partial Q \cap \{y : y_1 = 1/2\}. \\ \theta_\varepsilon(y) & \text{is 1 -periodic in} \quad y' = (y_2, \dots, y_n). \end{cases}$$

Since $|a_\varepsilon \varepsilon^{-1} \partial G_0| = (a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0|$, the problem (29) has a unique weak solution to within a constant. We define the constant in such a way that

$$\int_{Y_\varepsilon} \theta_\varepsilon(y) dy = 0.$$

From the integral identity for problem (29) it follows that

$$(30)\quad \|\nabla \theta_\varepsilon\|_{L_2(Y_\varepsilon)} \le K_{26}.$$

Indeed,

$$(31)\quad \int_{Y_\varepsilon} |\nabla_y \theta_\varepsilon|^2 dy = -(a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| \int_{\Sigma_0} \theta_\varepsilon dy' + \int_{a_\varepsilon \varepsilon^{-1} \partial G_0} \theta_\varepsilon ds_y.$$

From the imbedding theorem and the Poincaré inequality it follows that

$$(32)\quad \|\theta_\varepsilon\|_{L_2(\Sigma_0)} \le K_{27} \|\nabla_y \theta_\varepsilon\|_{L_2(Y_\varepsilon)}, \ \|\theta_\varepsilon\|_{L_2(a_\varepsilon \varepsilon^{-1} \partial G_0)} \le K_{28} \|\nabla_y \theta_\varepsilon\|_{L_2(Y_\varepsilon)}.$$

The inequalities (31) and (32) imply (30). We set

$$P_j^\varepsilon = \frac{\partial \theta_\varepsilon}{\partial y_j}, \ j = 1, \dots, n.$$

For the vector-function $P^\varepsilon(y) = \left(P_1^\varepsilon(y), \dots, P_n^\varepsilon(y)\right)$ we have

$$\operatorname{div}_y P^\varepsilon = 0, \quad \text{if} \quad y \in Y_\varepsilon, \ (P^\varepsilon, \nu) = 1 \quad \text{on} \quad a_\varepsilon \varepsilon^{-1} \partial G_0,$$
$$(P^\varepsilon, \nu) = -(a_\varepsilon \varepsilon^{-1})^{n-1} |\partial G_0| \quad \text{on} \quad \Sigma_0 \quad \text{and} \quad (P^\varepsilon, \nu) = 0 \quad \text{on} \quad \Sigma_1,$$

where $\nu = (\nu_1, \dots, \nu_n)$ is a unit outward normal vector to ∂Y_ε. We denote by T_ε the set of cells of the form $(\varepsilon Q + \varepsilon z) \setminus (a_\varepsilon G_0 + \varepsilon z)$ which have a nonempty intersection with Π_ε^* and $\partial\Omega$. Let $\Pi_\varepsilon^1 = \Pi_\varepsilon^* \cup T_\varepsilon$. We extend the function v on T_ε by setting $v = 0$ on $T_\varepsilon \setminus \Omega$. It is easy to see that

$$\begin{aligned}\int_{\Pi_\varepsilon^1} \operatorname{div}_x(P^\varepsilon(y)v)dx &= \int_{S_\varepsilon}(P^\varepsilon, \nu)vds + \int_{\gamma_\varepsilon^-}(P^\varepsilon, \nu)vdx' \\ &= \int_{S_\varepsilon} vds - (a_\varepsilon\varepsilon^{-1})^{(n-1)}|\partial G_0| \int_{\gamma_\varepsilon^-} vdx'.\end{aligned} \tag{33}$$

From (33) it follows that

$$\begin{aligned}\left|\int_{S_\varepsilon} vds - C_0^{n-1}|\partial G_0| \int_\gamma vdx'\right| &\le \left|\int_{\Pi_\varepsilon^1} \operatorname{div}_x(P^\varepsilon(y)v)dx\right| \\ &+ \left|(a_\varepsilon\varepsilon^{-1})^{n-1}|\partial G_0| \int_{\gamma_\varepsilon^-} vdx' - C_0^{n-1}|\partial G_0| \int_\gamma vdx'\right|,\end{aligned} \tag{34}$$

Let us estimate the right-hand side of (34). We have

$$I_1^\varepsilon \equiv \left|\int_{\Pi_\varepsilon^1} \operatorname{div}_x(P^\varepsilon(y)v)dx\right| \le \int_{\Pi_\varepsilon^1} |\nabla v||P^\varepsilon(y)|dx. \tag{35}$$

Therefore,

$$I_1^\varepsilon \le \Big(\int_{\Pi_\varepsilon^1} |\nabla_y\theta|^2 dx\Big)^{1/2} \|v\|_{H_1(\Omega)}. \tag{36}$$

It is easy to see that

$$\|\nabla_y\theta\|^2_{L_2(\varepsilon Y_\varepsilon)} \le K_{29}\varepsilon^n. \tag{37}$$

Since Π_ε^1 can contain sets of the form $\varepsilon Y_\varepsilon + \varepsilon z$ no more than $a_1\varepsilon^{1-n}$, $a_1 =$ const > 0, from (36) and (37) we derive that

$$I_1^\varepsilon \le K_{30}\sqrt{\varepsilon}\|v\|_{H_1(\Omega)}. \tag{38}$$

In order to estimate the second term in the right-hand side of (34) we use the continuity of functions from $H_1(\Omega)$ on hyperplanes in L_2 - norm. We have

$$\begin{aligned}I_2^\varepsilon &\equiv |(a_\varepsilon\varepsilon^{-1})^{n-1}|\partial G_0| \int_{\gamma_\varepsilon^-} vdx' - C_0^{n-1}|\partial G_0| \int_\gamma vdx'| \\ &\le K_{31}\left\{(a_\varepsilon\varepsilon^{-1})^{n-1}\left|\int_{\gamma_\varepsilon^-} vdx' - \int_\gamma vdx'\right| + |(a_\varepsilon\varepsilon^{-1})^{n-1} - C_0^{n-1}| \int_\gamma |v|dx'\right\} \\ &\le K_{32}\{\sqrt{\varepsilon}\|v\|_{H_1(\Omega)} + |a_\varepsilon\varepsilon^{-1} - C_0|\|v\|_{H_1(\Omega)}\} \\ &\le K_{33}\{\sqrt{\varepsilon} + |a_\varepsilon\varepsilon^{-1} - C_0|\}\|v\|_{H_1(\Omega)}.\end{aligned} \tag{39}$$

From (38) and (39) it follows that (28) is valid. Lemma 1 is proved.

In order to prove Theorem 3 we note that from (26), (27) it follows that

$$\begin{aligned}\|\nabla(u_\varepsilon - v)\|^2_{L_2(\Omega_\varepsilon)} &\le K_{34}(\sqrt{\varepsilon} + |a_\varepsilon\varepsilon^{-1} - C_0| \\ &+ |G_\varepsilon|^{1/2} + |l_\varepsilon|^{1/2})\|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)}.\end{aligned} \tag{40}$$

We assume that $|l_\varepsilon| \le d_3\varepsilon$, $|G_\varepsilon| \le d_4\varepsilon$. Then we have from (40) and the Friedrichs inequality that

$$\|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)} \le K_{35}\{\sqrt{\varepsilon} + |a_\varepsilon\varepsilon^{-1} - C_0|\}.$$

Hence, we have the following theorem

THEOREM 3. *Let* u_ε *be a solution of the problem* (3), (5), *the domain* Ω_ε *be defined by* (22), v *be a solution of problem* (23), $a_\varepsilon\varepsilon^{-1} \to C_0 = \text{const} > 0$ *as* $\varepsilon \to 0$, $|l_\varepsilon| = |\overline{G}_\varepsilon \cap \partial\Omega| \le d_3\varepsilon$. *Then*

$$\|u_\varepsilon - v\|_{H_1(\Omega_\varepsilon)} \le K_{36}(\sqrt{\varepsilon} + |a_\varepsilon\varepsilon^{-1} - C_0|).$$

5. Consider now the problem (3), (6) (the Dirichlet boundary condition on cavities). We study the behavior of solutions of the problem

$$-\Delta u_\varepsilon = f \quad \text{in} \quad \Omega_\varepsilon,\ u_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\varepsilon$$

as $\varepsilon \to 0$.

Let us define the function φ_ε^j $(j = 1, \dots, N(\varepsilon))$ for $n \ge 3$, setting $\varphi_\varepsilon^j \equiv 0$ for $|x - P_j| \le a_\varepsilon^j$, $\varphi_\varepsilon^j \equiv \frac{1}{\ln c_0}\ln\left(\frac{|x-P_j|}{a_\varepsilon^j}\right)$ for $a_\varepsilon^j \le |x - P_j| \le c_0 a_\varepsilon^j$, $\varphi_\varepsilon^j \equiv 1$ for $|x - P_j| > c_0 a_\varepsilon^j$. For $n = 2$ we set $\varphi_\varepsilon^j = \varphi\left(\frac{|\ln|x-P_j||}{|\ln c_0 a_\varepsilon^j|}\right)$, where $\varphi(\xi) = 1$ for $\xi \le 1/2$, $\varphi = 0$ for $\xi \ge 1$, $0 \le \varphi \le 1$, $\varphi \in C^\infty(R^n)$, c_0 is a constant, $c_0 > 0$, $c_0 a_\varepsilon^j \le 1$.

We pose

$$\psi_\varepsilon(x) = \sum_{j=1}^{N(\varepsilon)} \psi_\varepsilon^j(x),$$

where $\psi_\varepsilon^j(x) = 1 - \varphi_\varepsilon^j(x)$ and $w_\varepsilon = v\psi_\varepsilon - v_\varepsilon$, v_ε is a solution of the problem

$$\Delta v_\varepsilon = 0 \quad \text{in} \quad \Omega_\varepsilon,\ v_\varepsilon = v \quad \text{on} \quad S_\varepsilon,\ v_\varepsilon = 0 \quad \text{on} \quad \Gamma_\varepsilon,$$

v is a solution of the problem (9).

It is evident that w_ε is a weak solution of the problem

$$\Delta w_\varepsilon = \Delta(v\psi_\varepsilon) \quad \text{in} \quad \Omega_\varepsilon,\ w_\varepsilon = 0 \quad \text{on} \quad \partial\Omega_\varepsilon. \tag{41}$$

Let us estimate w_ε and its derivatives. From the integral identity for the problem (41) it follows that

$$\begin{aligned}\int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 dx &\le K_{37}\int_{\Omega_\varepsilon} |\nabla(v\psi_\varepsilon)|^2 dx \le K_{38}\int_{\Omega_\varepsilon} (\psi_\varepsilon^2 + |\nabla\psi_\varepsilon|^2)dx \\ &\le K_{39}\left\{\max_j (a_\varepsilon^j)^n \varepsilon^{-n+1} + \sum_{j=1}^{N(\varepsilon)} \int_{a_\varepsilon^j}^{c_0 a_\varepsilon^j} r^{-3+n} dr\right\} \\ &\le K_{40}\max_j (a_\varepsilon^j)^{n-2}\varepsilon^{1-n},\end{aligned} \tag{42}$$

for $n \geq 3$. For $n = 2$ we have

$$\begin{aligned} \int_{\Omega_\varepsilon} |\nabla w_\varepsilon|^2 dx &\leq K_{41} \int_{\Omega_\varepsilon} (\psi_\varepsilon^2 + |\nabla \psi_\varepsilon|^2) dx \\ &\leq K_{42} \left(\max_j a_\varepsilon^j \varepsilon^{-1} + \sum_{j=1}^{N(\varepsilon)} |\ln c_0 a_\varepsilon^j|^{-2} \int_{c_0 a_\varepsilon^j}^{(c_0 a_\varepsilon^j)^{1/2}} r^{-1} dr \right) \\ &\leq K_{43} \max_j |\ln c_0 a_\varepsilon^j|^{-1} \varepsilon^{-1}. \end{aligned} \tag{43}$$

From (42), (43), the relation $v_\varepsilon = v - u_\varepsilon$, estimates for $|\nabla(v\psi_\varepsilon)|$ and the Friedrichs inequality it follows that

$$\|u_\varepsilon - v\|^2_{H_1(\Omega_\varepsilon)} \leq K_{44} (\max_j a_\varepsilon^j)^{n-2} \varepsilon^{1-n} \quad \text{for} \quad n \geq 3, \tag{44}$$

$$\|u_\varepsilon - v\|^2_{H_1(\Omega_\varepsilon)} \leq K_{45} (\max_j |\ln a_\varepsilon^j|)^{-1} \varepsilon^{-1} \quad \text{for} \quad n = 2. \tag{45}$$

The inequalities (44), (45) imply the following theorem.

THEOREM 4. *Assume that*

$$\lim_{\varepsilon \to 0} \max_j (a_\varepsilon^j)^{n-2} \varepsilon^{1-n} = 0, \quad \textit{if} \quad n \geq 3,$$

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} \max_j |\ln a_\varepsilon^j|^{-1} = 0, \quad \textit{if} \quad n = 2,$$

v is a solution of the problem (9), *u_ε is a solution of problem* (3), (6). *Then the estimates* (44), (45) *are valid.*

Here as in Theorems 1 and 2 we do not use that $P_j \in \gamma$.

6. Let us consider the problem (3), (6), when

$$\begin{cases} \lim_{\varepsilon \to 0} a_\varepsilon^{n-2} \varepsilon^{1-n} = C_1 = \text{const} > 0, & \text{if} \quad n \geq 3, \\ \lim_{\varepsilon \to 0} (\varepsilon |\ln a_\varepsilon|)^{-1} = C_2 = \text{const} > 0, & \text{if} \quad n = 2. \end{cases} \tag{46}$$

We assume that $G_0 = \{x : |x| < a\}$, $Q = \{x : -1/2 < x_j < 1/2, j = 1, \ldots, n\}$, $a_\varepsilon^j = a_\varepsilon$, Ω_ε has the form given by (22). In this case the limit problem for (3), (6) is

$$\begin{gathered} -\Delta v = f \quad \text{in} \quad \Omega^- \cup \Omega^+, \\ [v]\Big|_\gamma = 0, \ \left[\frac{\partial v}{\partial x_1}\right]\Big|_\gamma = \mu_1 v\Big|_\gamma, \ v = 0 \quad \text{on} \quad \partial\Omega, \end{gathered} \tag{47}$$

where $\Omega^- = \Omega \cap \{x : x_1 < 0\}, \Omega^+ = \Omega \cap \{x : x_1 > 0\}, \mu_1 = (n-2)a^{n-2}\omega(n)C_1$, if $n \geq 3$, $\mu_1 = 2\pi C_2$, if $n = 2$, $\omega(n)$ is the area of the unit sphere in R^n.

We assume that for a solution of the problem (47) the following estimates are valid

$$|v(x)| \le A_1 \quad \text{for} \quad x \in \overline{\Omega}, \ |\nabla v(x)| \le A_2 \quad \text{for} \quad x \in \overline{\Omega^-}$$
$$\text{and} \quad |\nabla v(x)| \le A_3 \quad \text{for} \quad x \in \overline{\Omega^+}.$$

Consider w_ε^j as a solution of the problem

$$\Delta w_\varepsilon^j = 0 \quad \text{in} \quad T_{b\varepsilon}^j \setminus T_{aa_\varepsilon}^j, \ w_\varepsilon^j = 0 \quad \text{on} \quad \partial T_{aa_\varepsilon}^j,$$
$$w_\varepsilon^j = 1 \quad \text{on} \quad \partial T_{b\varepsilon}^j, \ \varepsilon/2 > b\varepsilon > aa_\varepsilon, \ b = \text{const} > 0, \tag{48}$$

where T_s^j is the ball with the center P_j and with the radius s. It is easy to see that the solution of the problem (48) has the form:

$$w_\varepsilon^j = \frac{r^{2-n} - (a_\varepsilon a)^{2-n}}{(b\varepsilon)^{2-n} - (a_\varepsilon a)^{2-n}}, \quad \text{if} \quad n \ge 3,$$
$$w_\varepsilon^j = \left(\ln \frac{r}{a_\varepsilon a}\right)\left(\ln \frac{b\varepsilon}{a_\varepsilon a}\right)^{-1}, \ \text{if} \quad n = 2,$$

where $r = |x - P_j|$. We introduce the function w_ε such that $w_\varepsilon = w_\varepsilon^j$ for $T_{b\varepsilon}^j \setminus T_{a_\varepsilon a}^j$, $w_\varepsilon = 0$ for $x \in T_{a_\varepsilon a}^j$, $j = 1, \dots, N(\varepsilon)$, $w_\varepsilon = 1$ for $x \in R^n \setminus \sum_{j=1}^{N(\varepsilon)} T_{b\varepsilon}^j$.

Using the Green formula for functions w_ε and $\varphi \in H_1(\Omega, G_\varepsilon \cup \partial\Omega)$ we obtain

$$\begin{aligned} \sum_{j=1}^{n} \int_\Omega \frac{\partial w_\varepsilon}{\partial x_j}\frac{\partial \varphi}{\partial x_j} dx &= \sum_{i=1}^{N(\varepsilon)} \int_{T_{b\varepsilon}^i \setminus T_{a_\varepsilon a}^i} (\nabla w_\varepsilon^i, \nabla\varphi) dx \\ &= -\sum_{i=1}^{N(\varepsilon)} \int_{T_{b\varepsilon}^i \setminus T_{a_\varepsilon a}^i} (\Delta w_\varepsilon^i)\varphi dx + \sum_{i=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^i} \frac{\partial w_\varepsilon^i}{\partial \nu}\varphi ds \\ &= \sum_{i=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^i} \frac{\partial w_\varepsilon^i}{\partial \nu}\varphi ds. \end{aligned} \tag{49}$$

Since

$$\frac{\partial w_\varepsilon^i}{\partial \nu} = \frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1 - (a^{-1}b)^{2-n}(a_\varepsilon \varepsilon^{-1})^{n-2}} \quad \text{for} \quad x \in \partial T_{b\varepsilon}^j, \ n \ge 3,$$
$$\frac{\partial w_\varepsilon^j}{\partial \nu} = -\frac{1}{b\varepsilon \ln a_\varepsilon} \frac{1}{\left(1 - \frac{\ln b\varepsilon}{\ln a_\varepsilon} + \frac{\ln a}{\ln a_\varepsilon}\right)} \quad \text{for} \quad x \in \partial T_{b\varepsilon}^j, \ n = 2,$$

we derive from (49) that

$$\int_\Omega (\nabla w_\varepsilon, \nabla\varphi) dx = \frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1 - (a^{-1}b)^{2-n}(a_\varepsilon \varepsilon^{-1})^{n-2}} \sum_{j=1}^{N(\varepsilon)} \int_{\partial T_{b\varepsilon}^j} \varphi ds, \tag{50}$$

if $n \geq 3$, and

$$\int_{\Omega}(\nabla w_\varepsilon, \nabla\varphi)dx = \frac{1}{b\varepsilon|\ln a_\varepsilon|}\frac{1}{\left(1-\frac{\ln b\varepsilon}{\ln a_\varepsilon}+\frac{\ln a}{\ln a_\varepsilon}\right)}\sum_{j=1}^{N(\varepsilon)}\int_{\partial T_{b\varepsilon}^j}\varphi ds, \tag{51}$$

if $n = 2$.

Applying Lemma 1 proved in Section 4 we get

$$\left|\sum_{j=1}^{N(\varepsilon)}\int_{\partial T_{b\varepsilon}^j}\varphi ds - b^{n-1}\omega(n)\int_\gamma \varphi dx'\right| \leq K_{46}\sqrt{\varepsilon}\|\varphi\|_{H_1(\Omega)}. \tag{52}$$

Let us note that for a solution of problem (47) we have the integral identity

$$\int_{\Omega}(\nabla v, \nabla\varphi)dx + (n-2)a^{n-2}\omega(n)C_1\int_\gamma v\varphi dx' = \int_\Omega f\varphi dx, \tag{53}$$

if $n \geq 3$, and

$$\int_{\Omega}(\nabla v, \nabla\varphi)dx + 2\pi C_2\int_\gamma v\varphi dx_2 = \int_\Omega f\varphi dx, \tag{54}$$

in $n = 2$, $\varphi \in H_1(\Omega, \partial\Omega)$. From (50), (53) for $n \geq 3$ it follows that

$$\begin{aligned}
\int_{\Omega}(\nabla(\tilde{u}_\varepsilon - w_\varepsilon v), \nabla\tilde{\varphi})dx &= \int_\Omega(\nabla\tilde{u}_\varepsilon, \nabla\tilde{\varphi})dx - \int_\Omega(\nabla(w_\varepsilon v), \nabla\tilde{\varphi})dx \\
&= \int_{\Omega_\varepsilon} f\tilde{\varphi}dx - \int_\Omega(\nabla w_\varepsilon, v\nabla\tilde{\varphi})dx - \int_\Omega w_\varepsilon(\nabla v, \nabla\tilde{\varphi})dx \\
&= \int_\Omega f\tilde{\varphi}dx - \int_\Omega(\nabla w_\varepsilon, \nabla(v\tilde{\varphi}))dx + \int_\Omega(\nabla w_\varepsilon, \nabla v)\tilde{\varphi}dx \\
&\quad - \int_\Omega(\nabla v, \nabla\tilde{\varphi})dx + \int_\Omega(1-w_\varepsilon)(\nabla v, \nabla\tilde{\varphi})dx \\
&= \int_\Omega f\tilde{\varphi}dx - \frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1-(a^{-1}b)^{2-n}(a_\varepsilon\varepsilon^{-1})^{n-2}}\sum_{j=1}^{N(\varepsilon)}\int_{\partial T_{b\varepsilon}^j} v\tilde{\varphi}ds \\
&\quad - \int_\Omega f\tilde{\varphi}dx + (n-2)a^{n-2}\omega(n)C_1\int_\gamma v\tilde{\varphi}dx' \\
&\quad + \int_\Omega(1-w_\varepsilon)(\nabla v, \nabla\tilde{\varphi})dx + \int_\Omega(\nabla w_\varepsilon, \nabla v)\tilde{\varphi}dx \\
&= \Bigg\{(n-2)a^{n-2}\omega(n)C_1\int_\gamma v\tilde{\varphi}dx' \\
&\quad - \frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1-(a^{-1}b)^{2-n}(a_\varepsilon\varepsilon^{-1})^{n-2}}\sum_{j=1}^{N(\varepsilon)}\int_{\partial T_{b\varepsilon}^j} v\tilde{\varphi}ds\Bigg\} \\
&\quad + \int_\Omega(1-w_\varepsilon)(\nabla v, \nabla\tilde{\varphi})dx + \int_\Omega(\nabla w_\varepsilon, \nabla v)\tilde{\varphi}dx,
\end{aligned} \tag{55}$$

where $\tilde{\varphi} = \varphi$ for $x \in \Omega_\varepsilon$ and $\tilde{\varphi} = 0$ for $x \in \Omega \setminus \Omega_\varepsilon$, $\varphi \in H_1(\Omega_\varepsilon, \partial\Omega_\varepsilon)$.

In a similar way we get for $n = 2$

$$\begin{aligned}\int_\Omega(\nabla(\tilde{u_\varepsilon} - w_\varepsilon v), \nabla\tilde{\varphi})dx = &\left\{2\pi C_2 \int_\gamma v\tilde{\varphi}dx_2\right.\\ &\left.- \frac{1}{b\varepsilon|\ln a_\varepsilon|}\frac{1}{\left(1 - \frac{\ln b\varepsilon}{\ln a_\varepsilon} + \frac{\ln a}{\ln a_\varepsilon}\right)}\sum_{j=1}^{N(\varepsilon)}\int_{\partial T^j_{b\varepsilon}} v\tilde{\varphi}ds\right\}\\ &+ \int_\Omega(1 - w_\varepsilon)(\nabla v, \nabla\tilde{\varphi})dx + \int_\Omega(\nabla w_\varepsilon, \nabla v)\tilde{\varphi}dx.\end{aligned} \tag{56}$$

Let us estimate two last integrals in the right-hand side of (55) and (56) . We denote them by $J_\varepsilon(\varphi)$. Using Lemma 1, we get

$$\begin{aligned}&\left|\frac{(n-2)(b\varepsilon)^{1-n}(a_\varepsilon a)^{n-2}}{1-(a^{-1}b)^{2-n}(a_\varepsilon\varepsilon^{-1})^{n-2}}\sum_{j=1}^{N(\varepsilon)}\int_{\partial T^j_{b\varepsilon}} v\tilde{\varphi}ds - (n-2)a^{n-2}\omega(n)C_1\int_\gamma v\tilde{\varphi}dx'\right|\\ &\le K_{47}\left\{\left|\sum_{j=1}^{N(\varepsilon)}\int_{\partial T^j_{b\varepsilon}} v\tilde{\varphi}ds - b^{n-1}\omega(n)\int_\gamma \tilde{\varphi}vdx'\right|\right.\\ &\left.+ \int_\gamma|v\tilde{\varphi}|dx'|a_\varepsilon^{n-2}\varepsilon^{1-n} - C_1| + \alpha(\varepsilon)|\int_\gamma v\tilde{\varphi}dx'|\right\}\\ &\le K_{48}\{\sqrt{\varepsilon} + |a_\varepsilon^{n-2}\varepsilon^{1-n} - C_1|\}\|\tilde{\varphi}\|_{H_1(\Omega)},\end{aligned} \tag{57}$$

for $n \ge 3$ and $\alpha(\varepsilon) \le c_1\varepsilon$,and

$$\begin{aligned}&\left|2\pi C_2\int_\gamma v\tilde{\varphi}dx_2 - \frac{1}{b\varepsilon|\ln a_\varepsilon|}\frac{1}{(1 - \alpha(\varepsilon))}\sum_{j=1}^{N(\varepsilon)}\int_{\partial T^j_{b\varepsilon}} v\tilde{\varphi}ds\right|\\ &\le K_{49}\{\sqrt{\varepsilon} + |(\varepsilon|\ln a_\varepsilon|)^{-1} - C_2|\}\|\tilde{\varphi}\|_{H_1(\Omega)},\end{aligned} \tag{58}$$

for $n = 2$, $\alpha(\varepsilon) = \frac{\ln b\varepsilon}{\ln a_\varepsilon} - \frac{\ln a}{\ln a_\varepsilon} \to 0$ as $\varepsilon \to 0$.

For $J_\varepsilon(\tilde{\varphi})$ we have estimates:

$$\begin{aligned}\left|J_\varepsilon(\tilde{\varphi})\right| &\le K_{50}\{\|w_\varepsilon - 1\|_{L_2(\Omega)}\|\tilde{\varphi}\|_{H_1(\Omega)} + \sqrt{\varepsilon}\|\nabla w_\varepsilon\|_{L_2(\Omega)}\|\tilde{\varphi}\|_{L_2(\gamma)}\}\\ &\le K_{51}\sqrt{\varepsilon}\|\tilde{\varphi}\|_{H_1(\Omega)}.\end{aligned} \tag{59}$$

Taking $\tilde{\varphi} = \tilde{u}_\varepsilon - w_\varepsilon v$ in (55) and (56), we obtain

$$\|u_\varepsilon - w_\varepsilon v\|_{H_1(\Omega_\varepsilon)} \le K_{52}\{\sqrt{\varepsilon} + |C_1 - a_\varepsilon^{n-2}\varepsilon^{1-n}|\}, \tag{60}$$

if $n \geq 3$, and

$$\|u_\varepsilon - w_\varepsilon v\|_{H_1(\Omega_\varepsilon)} \leq K_{53}\{\sqrt{\varepsilon} + |C_2 - (\varepsilon|\ln a_\varepsilon|)^{-1}|\}, \tag{61}$$

if $n = 2$.

From (60) and (61) we derive

$$\|u_\varepsilon - v\|_{L_2(\Omega_\varepsilon)} \leq K_{54}\{\sqrt{\varepsilon} + |C_1 - a_\varepsilon^{n-2}\varepsilon^{1-n}|\}, \tag{62}$$

if $n \geq 3$, and

$$\|u_\varepsilon - v\|_{L_2(\Omega_\varepsilon)} \leq K_{55}\{\sqrt{\varepsilon} + |C_2 - (\varepsilon|\ln a_\varepsilon|)^{-1}|\}, \tag{63}$$

if $n = 2$.

Thus we have proved the following theorem.

THEOREM 5. *Let conditions* (46) *be satisfied. Assume that* Ω_ε *has the form* (22), u_ε *is a solution of the problem* (3), (6), v *is a solution of the problem* (47). *Then estimates* (62), (63) *hold.*

7. Consider the problem (3), (6) under conditions:

$$a_\varepsilon^{2-n}\varepsilon^{n-1} \to 0 \text{ as } \varepsilon \to 0,\ n \geq 3, \tag{64}$$

$$\varepsilon|\ln a_\varepsilon| \to 0 \text{ as } \varepsilon \to 0,\ n = 2. \tag{65}$$

We assume that $a_\varepsilon^j = a_\varepsilon$, the domain Ω_ε has the form (22). In this case the limit problem for the problem (3), (6) is:

$$-\Delta v^- = f \text{ in } \Omega^-,\ v^- = 0 \text{ on } \partial\Omega^-, \tag{66}$$

$$-\Delta v^+ = f \text{ in } \Omega^+,\ v^+ = 0 \text{ on } \partial\Omega^+. \tag{67}$$

We will use the following Lemma 2, proved in [6].

LEMMA 2. *Let* $u \in H_1(\Omega_\varepsilon, \partial\Omega_\varepsilon)$. *Then*

$$\|u\|_{L_2(\Pi_\varepsilon)} \leq K_{56}a_\varepsilon^{(2-n)/2}\varepsilon^{n/2}\|\nabla u\|_{L_2(\Pi_\varepsilon)}, \tag{68}$$

if $n \geq 3$, *and*

$$\|u\|_{L_2(\Pi_\varepsilon)} \leq K_{57}\varepsilon\sqrt{|\ln a_\varepsilon|}\|\nabla u\|_{L_2(\Pi_\varepsilon)}, \tag{69}$$

if $n = 2$.

From the integral identity for the problem (3), (6) we derive

$$\|u_\varepsilon\|_{H_1(\Omega_\varepsilon)} \le K_{58}. \tag{70}$$

Therefore, from (68)-(70) we get

$$\frac{1}{|\Pi_\varepsilon|}\int_{\Pi_\varepsilon} \tilde{u}_\varepsilon^2 dx \le K_{59} a_\varepsilon^{2-n}\varepsilon^{n-1}, \tag{71}$$

if $n \ge 3$, and

$$\frac{1}{|\Pi_\varepsilon|}\int_{\Pi_\varepsilon} \tilde{u}_\varepsilon^2 dx \le K_{60}\varepsilon|\ln a_\varepsilon|, \tag{72}$$

if $n = 2$. Here $\tilde{u}_\varepsilon = u_\varepsilon$ for $x \in \Omega_\varepsilon$ and $\tilde{u}_\varepsilon = 0$ for $x \in \Omega \setminus \Omega_\varepsilon$.
It is easy to prove that

$$\int_{\gamma_\varepsilon^\pm} u^2(x_1, x')dx' \le K_{61}\left(\frac{1}{|\Pi_\varepsilon|}\int_{\Pi_\varepsilon} u^2(x)dx + \varepsilon\|\nabla u\|^2_{L_2(\Pi_\varepsilon)}\right), \tag{73}$$

From this inequality and (71), (72), (73) it follows that

$$\|\tilde{u}_\varepsilon\|^2_{L_2(\gamma_\varepsilon^\pm)} \le K_{62}\{\varepsilon + a_\varepsilon^{2-n}\varepsilon^{n-1}\} \tag{74}$$

for $n \ge 3$, and

$$\|\tilde{u}_\varepsilon\|^2_{L_2(\gamma_\varepsilon^\pm)} \le K_{63}\{\varepsilon + \varepsilon|\ln a_\varepsilon|\} \tag{75}$$

for $n = 2$.

We set $w_\varepsilon^- = u_\varepsilon - v^-$, $w_\varepsilon^+ = u_\varepsilon - v^+$, $\Omega_\varepsilon^+ = \Omega \cap \{x : x_1 > \varepsilon/2\}$, $\Omega_\varepsilon^- = \Omega_\varepsilon \cap \{x : x_1 < -\varepsilon/2\}$. The functions $w_\varepsilon^\pm$ are weak solutions of the problems:

$$\Delta w_\varepsilon^\pm = 0 \quad \text{in} \quad \Omega_\varepsilon^\pm,\ w_\varepsilon^\pm = u_\varepsilon - v^\pm \quad \text{on} \quad \gamma_\varepsilon^\pm,\ w_\varepsilon^\pm = 0 \quad \text{on} \quad \partial\Omega_\varepsilon^\pm \setminus \gamma_\varepsilon^\pm.$$

From the inequality (see [7], ch 4, sec. 1)

$$\|w_\varepsilon^\pm\|_{L_2(\Omega_\varepsilon^\pm)} \le K_{64}\{\|u_\varepsilon\|_{L_2(\gamma_\varepsilon^\pm)} + \|v^\pm\|_{L_2(\gamma_\varepsilon^\pm)}\},$$

and (71), (72) we have estimates

$$\|\tilde{u}_\varepsilon - v^\pm\|^2_{L_2(\Omega^\pm)} \le K_{65} a_\varepsilon^{2-n}\varepsilon^{n-1}, \tag{76}$$

for $n \ge 3$, and

$$\|\tilde{u}_\varepsilon - v^\pm\|^2_{L_2(\Omega^\pm)} \le K_{66}\varepsilon|\ln a_\varepsilon|, \tag{77}$$

for $n = 2$.

From the estimates (76), (77) we get the following theorem.

THEOREM 6. *Let Ω_ε be a domain of the form* (22), *$a_\varepsilon^{2-n}\varepsilon^{n-1} \to 0$ as $\varepsilon \to 0$, $n \ge 3$, and $\varepsilon|\ln a_\varepsilon| \to 0$ as $\varepsilon \to 0$, $n = 2$, $a_\varepsilon^j = a_\varepsilon$. Let u_ε be a solution of the Dirichlet problem* (3), (6), *$v^\pm$ be solutions of problems* (66), (67). *Then estimates* (76), (77) *are valid.*

The Dirichlet problems in domains with holes were considered in [8].

8. Using Theorems 1-6, proved above, we can get theorems about the spectrum of corresponding eigenvalue problems. We apply here the theorem about the spectrum of a sequence of singularly perturbed operators proved in [5].

THEOREM 7. *Consider the eigenvalue problem*

$$\Delta u_\varepsilon^k + \lambda_\varepsilon^k u_\varepsilon^k = 0 \quad in \quad \Omega_\varepsilon, \tag{78}$$

$$\frac{\partial u_\varepsilon^k}{\partial \nu} = 0 \quad on \quad S_\varepsilon, \quad u_\varepsilon^k = 0 \quad on \quad \Gamma_\varepsilon, \tag{79}$$

and the eigenvalue problem

$$\Delta v^k + \lambda^k v^k = 0 \quad in \quad \Omega, \; v^k = 0 \quad on \quad \partial\Omega. \tag{80}$$

Assume that $S'_\varepsilon \cap \partial\Omega = \emptyset$. *Then*

$$|\lambda_\varepsilon^k - \lambda^k|^2 \le C_1 \left(\max_j a_\varepsilon^j\right)^n \varepsilon^{1-n}.$$

If $S'_\varepsilon \cap \partial\Omega \neq \emptyset$ *and* $M(\varepsilon) \le d_1 \varepsilon^{2-n}$, *then*

$$|\lambda_\varepsilon^k - \lambda^k|^2 \le C_2 \left(\max_j a_\varepsilon^j\right)^{n-1} \varepsilon^{2-n}.$$

where $\lambda_\varepsilon^1 \le \lambda_\varepsilon^2 \le \dots$ *is a nondecreasing sequence of eigenvalues to problem* (78), (79) *and* $\lambda^1 \le \lambda^2 \le \dots$ *is a nondecreasing sequence of eigenvalues to problem* (80) *and every eigenvalue is counted as many times as its multiplicity.*

Here and in what follows constants C_j do not depend on ε. This Theorem is a consequence of Theorem 1.

We note that in Theorem 1, from which Theorem 7 follows, it is not necessary to assume that P_j belongs to γ. We use only the fact, that the number $N(\varepsilon)$ of P_j is such that $N(\varepsilon) \le d_0 \varepsilon^{1-n}$ and $a_\varepsilon^j \le C_0 \varepsilon$.

THEOREM 8. *Consider the eigenvalue problem*

$$\Delta u_\varepsilon^k + \lambda_\varepsilon^k u_\varepsilon^k = 0 \quad in \quad \Omega_\varepsilon, \quad \frac{\partial u_\varepsilon^k}{\partial \nu} + \beta(x) u_\varepsilon^k = 0 \quad on \quad S_\varepsilon, \; u_\varepsilon^k = 0 \quad on \quad \Gamma_\varepsilon, \tag{81}$$

and the eigenvalue problem

$$\Delta v^k + \lambda^k v^k = 0 \quad in \quad \Omega, \; v^k = 0 \quad on \quad \partial\Omega. \tag{82}$$

Assume that $|\partial G^j(a_\varepsilon^j)| \le C_3 (a_\varepsilon^j)^{n-1}$, $\max_j a_\varepsilon^j \varepsilon^{-1} \to 0$ *as* $\varepsilon \to 0$, $\beta(x) \ge \beta_0 =$ const > 0. *Then*

$$|\lambda_\varepsilon^k - \lambda^k|^2 \le C_4 \left((\max_j a_\varepsilon^j)\varepsilon^{-1}\right)^{n-1}.$$

Here λ_ε^k *and* λ^k *are ordering in the same way as in Theorem 7.*

Now let us consider the case when Ω_ε has the structure, given by (22), and the eigenvalues of the problem (81).

THEOREM 9 (Critical case). *Let u_ε^k be an eigenfunction of the problem* (81) *and u^k be an eigenfunction of the problem*

$$\Delta v^k + \lambda^k v^k = 0 \quad in \quad \Omega \setminus \gamma,\ v^k = 0 \quad on \quad \partial\Omega,$$

$$[v^k]\Big|_\gamma = 0, \quad \left[\frac{\partial v^k}{\partial x_1}\right]\Big|_\gamma = \mu v^k,$$

where $\beta(x) = \beta_0 = \text{const} > 0$, $\mu = \beta_0 c_0^{n-1}|\partial \overline{G}_0|$, $a_\varepsilon \varepsilon^{-1} \to c_0 = \text{const} > 0$ *as* $\varepsilon \to 0$, $|\overline{G}_\varepsilon \cap \partial\Omega| \leq C_4\varepsilon$. *Let Ω_ε have the structure, defined by* (22). *Then*

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_5(\varepsilon + |a_\varepsilon \varepsilon^{-1} - c_0|^2).$$

This Theorem is a consequence of Theorem 3.

THEOREM 10. *Consider the Dirichlet eigenvalue problem*

$$\Delta u_\varepsilon^k + \lambda_\varepsilon^k u_\varepsilon^k = 0 \quad in \quad \Omega_\varepsilon,\ u_\varepsilon^k = 0 \quad on \quad \partial\Omega_\varepsilon, \tag{83}$$

and the Dirichlet eigenvalue problem

$$\Delta v^k + \lambda^k v^k = 0 \quad in \quad \Omega,\ v^k = 0 \quad on \quad \partial\Omega.$$

Assume that

$$\max_j (a_\varepsilon^j)^{n-2}\varepsilon^{1-n} \to 0 \quad as \quad \varepsilon \to 0 \quad for \quad n \geq 3,$$

$$\max_j (|\ln a_\varepsilon|\varepsilon)^{-1} \to 0 \quad as \quad \varepsilon \to 0 \quad for \quad n = 2.$$

Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_8(\max_j a_\varepsilon)^{n-2}\varepsilon^{1-n} \quad for \quad n \geq 3,$$

$$|\lambda_\varepsilon^k - \lambda^k|^2 \leq C_9(\max_j |\ln a_\varepsilon|\varepsilon)^{-1} \quad for \quad n = 2.$$

This Theorem is a consequence of Theorem 4.

THEOREM 11 (Critical case). *Let Ω_ε be given by* (22), $G_0 = \{x : |x| < a,\ 0 < a < 1/2\}$, $a_\varepsilon^j = a_\varepsilon$, $Q = \{x : -1/2 < x_i < 1/2,\ i = 1, \dots, n\}$. *Assume that* $a_\varepsilon^{n-2}\varepsilon^{1-n} \to c_0 = \text{const} > 0$ *as* $\varepsilon \to 0$ *for* $n \geq 3$ *and* $(\varepsilon|\ln a_\varepsilon|)^{-1} \to c_1 = \text{const} > 0$ *as* $\varepsilon \to 0$ *for* $n = 2$. *Let u_ε^k be a solution of the eigenvalue problem* (83) *and v^k be a solution of the eigenvalue problem*

$$\Delta v^k + \lambda^k v^k = 0 \quad in \quad \Omega,$$

$$[v^k]\Big|_\gamma = 0, \quad \left[\frac{\partial v^k}{\partial x_1}\right]\Big|_\gamma = \mu_1 v^k\Big|_\gamma, \quad v^k = 0 \quad on \quad \partial\Omega,$$

where $\mu_1 = (n-2)a^{n-2}\omega_n c_0$ *for* $n \geq 3$ *and* $\mu_1 = 2\pi c_1$ *for* $n = 2$.

Then

$$|\lambda_\varepsilon^k - \lambda^k|^2 \le C_{10}(\varepsilon + |c_0 - a_\varepsilon^{n-2}\varepsilon^{1-n}|^2) \quad \textit{for} \quad n \ge 3,$$

$$|\lambda_\varepsilon^k - \lambda^k|^2 \le C_{11}\{\varepsilon + |c_1 - (\varepsilon|\ln a_\varepsilon|)^{-1}|^2\} \quad \textit{for} \quad n = 2.$$

This theorem is a consequence of Theorem 5.

THEOREM 12. *Let* u_ε^k *be a solution of the eigenvalue problem* (83) *and let us consider two eigenvalue problems:*

$$\Delta v_+^k + \lambda_+^k v_+^k = 0 \quad \textit{in} \quad \Omega^+ = \Omega \cap \{x : x_1 > 0\},\ v_+^k = 0 \quad \textit{on} \quad \partial\Omega^+,$$

$$\Delta v_-^k + \lambda_-^k v_-^k = 0 \quad \textit{in} \quad \Omega^- = \Omega \cap \{x : x_1 < 0\},\ v_-^k = 0 \quad \textit{on} \quad \partial\Omega^-.$$

Let $\{\lambda^k\}$ *be a sequence, which is the set* $\{\lambda_-^k\} \cup \{\lambda_+^k\}$, *odered as a nondecreasing sequence and every eigenvalue is counted as many time as its multiplicity. Assume that* $a_\varepsilon^{2-n}\varepsilon^{n-1} \to 0$ *as* $\varepsilon \to 0$ *for* $n \ge 3$ *and* $\varepsilon|\ln a_\varepsilon| \to 0$ *as* $\varepsilon \to 0$ *for* $n = 2$. *Then*

$$|\lambda_\varepsilon^k - \lambda^k|^2 \le C_{12} a_\varepsilon^{2-n}\varepsilon^{n-1} \quad \textit{for} \quad n \ge 3,$$

$$|\lambda_\varepsilon^k - \lambda^k|^2 \le C_{13}\varepsilon|\ln a_\varepsilon| \quad \textit{for} \quad n = 2.$$

We note that in Theorems 1-12 the Laplace operator can be substituted by any elliptic second order selfajoint operator.

REFERENCES

[1] V. A. MARCHENKO – E. YA. KHRUSLOV, "Boundary problems with a finely grenulated boundaries", Naukova Dumka, Kiev, 1974.

[2] O. A. OLEINIK – T. A. SHAPOSHNIKOVA, *On homogenization of the biharmonic equation in a domain, perforated along manifolds of a small dimension*, Differential equations **6** (1996), 830-842.

[3] M. LOBO – O. A. OLEINIK – M. E. PEREZ – T. A. SHAPOSHNIKOVA, *On boundary value problems in a domain, perforated along manifolds*, Russian Math.Survey **52** (4) 1997.

[4] E. DE GIORGI – S. SPAGNOLO, *Sulla convergenza degli integrali dell'energia per operatori elliptici del secondo ordine*, Boll. Unione Mat. Ital. **8** (1973), 391-411.

[5] O. A. OLEINIK – A. S. SHAMAEV – G. A. YOSIFIAN, "Mathematical Problems in Elasticity and Homogenization", North-Holland, Amsterdam, 1992.

[6] O. A. OLEINIK – T. A. SHAPOSHNIKOVA, *On the homogenization of the Poisson equation in partially perforated domains with arbitrary density of cavities and mixed type conditions on their boundary*, Rendiconti Lincei, Matematica e Applicazioni **7** 1996, 129-146.

[7] O. A. OLEINIK, "Some Asymptotic problems in the Theory of Partial Differential Equations", Cambridge, University Press., 1996.

[8] D. Cioranescu – F. Murat, "Un terme etranger venu d'ailleurs. Nonlinear partial differential equations and their applications", College de France Seminar, v. II, III, edited by H. Brezis, J. L. Lions., Research Notes in Mathematics 60, 70. Pitman, London, 1982, p. 98-138, 154-178.

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