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Exact Boundary Controllability of Galerkin's Approximations of Navier-Stokes Equations

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Abstract. We consider the 2-d and 3-d Navier - Stokes equations in a bounded smooth domain with a boundary control acting on the system through the Navier slip boundary conditions. We introduce a finite-dimensional Galerkin approximation of this system. Under suitable assumptions on the Galerkin basis we prove that this Galerkin approximation is exactly controllable. Moreover we prove that the cost of controlling is independent of the presence of the nonlinearity on the system. Our assumptions on the Galerkin basis are related to the linear independence of suitable traces of its elements over the boundary. At this respect, the one-dimensional Burgers equation provides a particularly degenerate example that we study in detail. In this case we prove local controllability results.

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1. – Introduction

In a bounded domain Ω of $\mathbb{R}^3$ (we can consider the 2-dimensional case as well) we consider a flow governed by the Navier-Stokes equations. If $y = y(x, t)$ denotes the velocity of the flow and $p = p(x, t)$ the pressure (defined up to a function of time), they satisfy the Navier-Stokes equations

$$(1.1) \qquad \begin{cases} y_t + y \cdot \nabla y - \mu \Delta y = -\nabla p & \text{in } \Omega \times (0, T) \\ \operatorname{div} y = 0 & \text{in } \Omega \times (0, T) \end{cases}$$

where $\mu > 0$ denotes the viscosity and $T > 0$ a given value of time.

We assume that we act on the flow through a *boundary control.* Let $\tau^j = \left(\tau_1^j, \tau_2^j, \tau_3^j\right) : \Gamma = \partial\Omega \to \mathbb{R}^3$, $j = 1, 2$ be two smooth vector fields constituting an orthonormal basis of the tangent plane to Ω at each $x \in \Gamma$. Let

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us denote by $\gamma(x)$ another tangent vector field, which is the direction in which the scalar controls we shall use will be applied in the system. Let us denote by n the unit outward normal to Ω, and by $\partial/\partial n$ the normal derivative. On the other hand, let us denote by $D(y)$ the symmetric part of the gradient of y so that

$$D_{ij}(y) = \frac{1}{2}(\partial y_i/\partial x_j + \partial y_j/\partial x_i).$$

Let Γ_0 be an open non-empty subset of Γ.

We can now write the boundary conditions we consider:

$$(1.2) \qquad y \cdot n = 0 \text{ on } \Gamma \times (0,T)$$

$$(1.3) \qquad y = 0 \text{ on } (\Gamma \backslash \Gamma_0) \times (0,T)$$

$$(1.4) \qquad (2\mu D_{ij}(y) n_j + \lambda y_i - v\gamma_i)\tau_i^k = 0 \text{ on } \Gamma_0 \times (0,T), \; k = 1,2$$

with $\lambda \geq 0$ and where $v \in L^2(\Gamma_0 \times (0,T))$ is a scalar function that plays the role of the *control*.

In case $\Gamma_0 = \Gamma$ we shall assume that $\lambda > 0$.

Observe that the control function v is *scalar* and that the control it produces $v\gamma$ is oriented in the direction of the given tangent vector field γ. Observe also that we act on the system only on *the subset* Γ_0 *of the boundary* and that the control enters in the system as a tangential friction.

Condition (1.2) guarantees that $\int_\Gamma y \cdot n d\Gamma = 0$, which is necessary for the compatibility with $\operatorname{div} y = 0$.

We assume, to fix ideas, that

$$(1.5) \qquad y(x,0) = 0 \quad \text{in} \quad \Omega$$

although all our results are valid as well if the initial data is not zero.

Then, given v *smoooth enough* it is known (cf. for instance J.-L. Lions [L1]) that there exists at least one weak solution of (1.1)-(1.5). More precisely, consider the Hilbert spaces

$$(1.6) \quad V = \left\{\varphi \in \left(H^1(\Omega)\right)^3 : \operatorname{div}\varphi = 0 \text{ in } \Omega, \varphi \cdot n = 0 \text{ on } \Gamma, \varphi = 0 \text{ on } \Gamma \backslash \Gamma_0\right\}$$

and

$$(1.7) \qquad H = \left\{\varphi \in \left(L^2(\Omega)\right)^3 : \operatorname{div}\varphi = 0 \text{ in } \Omega, \varphi \cdot n = 0 \text{ on } \Gamma\right\}.$$

Multiply (1.1) by φ, integrating formally by parts and using the boundary conditions we obtain:

$$(1.8) \quad (y_t, \varphi) + 2\mu(D_{ij}y, D_{ij}\varphi) + (y \cdot \nabla y, \varphi) + \lambda(y,\varphi)_{\Gamma_0} = (v\gamma, \varphi)_{\Gamma_0}, \; \forall \varphi \in V$$

$$(1.9) \qquad y(0) = 0.$$

Here and in what follows $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\Gamma_0}$ denote the scalar product in $(L^2(\Omega))^3$ and $(L^2(\Gamma_0))^3$. It is easy to see by classical methods that this system admits a solution $y \in L^\infty(0,T;H) \cap L^2(0,T;V)$. However, uniqueness is an open problem. We denote by $y(x,t;v)$ the set of all possible solutions.

We want to address the problem of (*approximate*) *controllability*. Namely, let y^T be given, satisfying at least

(1.10) $y^T \in \left(L^2(\Omega)\right)^3, \quad \operatorname{div} y^T = 0 \text{ in } \Omega, y^T \cdot n = 0 \text{ on } \Gamma, \quad y^T = 0 \quad \text{on } \Gamma\backslash\Gamma_0.$

The problem may be formulated as follows: To find a (tangential) control v such that there exists, among the set (possibly reduced to one element) of all $y(x,t;v)$, one of them which satisfies

(1.11) $y(\cdot,T;v)$ is a close as we wish (in the L^2 sense) to y^T. □

Some years ago it was *conjectured* by the first author (cf. J.-L. Lions [L1]) that this is indeed the case for *distributed control*, i.e. when the control *vector* is distributed over an open subset of Ω (which can be arbitrarily small).

Many important steps in the direction of a positive answer to the conjecture have been made.

Actually, for the case $\mu = 0$ (Euler's equations), it has been proven by J. M. Coron [C1] that one has *exact controllability* (in suitable spaces) in the 2-dimensional case. Later on, in [C2] it was proved that 2-dimensional Navier-Stokes equations are approximately controllable for some choices of the boundary conditions. On the other hand, A. Fursikov and O. Yu. Imanuvilov [FI], by different techniques that rely on Carleman's estimates, have proved the null controllability of small solutions. More recently, J. M. Coron and A. Fursikov [CF] have proved the null controllability of the two-dimensional Navier-Stokes equations in a manifold without boundary.

We conjecture that similar results hold true for the tangential boundary control considered here.

REMARK 1.1. In the problem (1.1)-(1.5) under consideration we have "very few" control functions since we have only *one scalar* function as control for a 3D problem. It could be that, in this situation, approximate controllability is only "*generically*" true, i.e. that, if not true for a given set $\Gamma_0 \subset \Gamma$, then it will be true after an arbitrarily small modification of Ω and Γ_0. Such a result has been proven for the *linear Stokes system* with Dirichlet boundary conditions and for distributed control by the authors in [LZ1]. □

In a recent paper (cf. [LZ2]) we have proven the conjecture for *Galerkin's approximations* of the system for *distributed control*. We have also proven that one can achieve the result (actually, in the finite dimensional approximation, we have *exact* controllability) at a "cost" (i.e. an estimate of the L^2–norm of the control function in terms of y^T) *which can be bounded independently of the non linearity* (i.e. with the term $y \cdot \nabla y$ present in the system or not) and of $0 \le \mu_0 \le \mu \le \mu_1 < \infty$.

We are going to show that similar results hold true in the present situation where boundary controls are used.

In order to show and to make explicit that the "cost of control" can be bounded "independently of the non linearity", we introduce the (non physical) family of state equations

$$y_t + \alpha y \cdot \nabla y - \mu \Delta y = -\nabla p \tag{1.12}$$

all other conditions being unchanged, with $\alpha \in \mathbb{R}$. We will prove that the estimates we obtain *do not depend on* α.

In the following section we introduce the Galerkin's approximations and we state the main result.

Section 3 is devoted to the proof of this result. In Section 4 we analyze similar questions for the 1D Burgers equations which, in some sense that we will explain below, is a degenerate situation for what concerns the boundary controllability.

2. – Galerkin's approximations and main result

Let us consider a basis $\{e_i\}_{i\geq 1}$ of V such that $\{e_i \cdot \gamma\}_{i\geq 1}$ are linearly independent in $L^2(\Gamma_0)$. The existence of this basis is guaranteed by the following abstract result proved by the authors in [LZ3].

PROPOSITION 2.1. *Let H_1 and H_2 be two separable Hilbert spaces. Let $L : H_1 \to H_2$ be a bounded linear operator with an infinite dimensional range. Then, there exists a Riesz basis $\{e_i\}_{i\geq 1}$ of H_1 such that $\{Le_i\}_{i\geq 1}$ are linearly independent in H_2.*

We consider a finite dimensional space E generated by

$$E = \operatorname{span}[e_1, \cdots, e_N]. \tag{2.1}$$

Of course $E \subset V$.

We now introduce the *Galerkin approximation of the variational formulation* (1.8)-(1.9) *of the state equation*:

$$y \in C([0,T]; E);\ y(0) = 0 \tag{2.2}$$

$$(y_t, e) + 2\mu(D_{ij}y, D_{ij}e) + \alpha(y \cdot \nabla y, e) + \lambda(y, e)_{\Gamma_0} = (v\gamma, e)_{\Gamma_0},\ \forall e \in E. \tag{2.3}$$

REMARK 2.1. System (2.3) is a set of N ordinary differential equations which are non linear. Global existence in time of *a unique solution* is insured by the fact that $(e \cdot \nabla e, e) = 0$ for all $e \in E$ which is a consequence of the fact that $e \in V$. On the other hand, it is easy to see that for v smooth fixed, solutions of (2.2)-(2.3) as $N \to \infty$ converge to a solution of the variational form of the state equation. □

REMARK 2.2. A simple inspection of (2.3) shows that in order to obtain results of controllability, the following condition on the basis of E is rather natural

$$\text{(2.4)} \qquad \textit{the dimension of} \operatorname{span} \left[\gamma \cdot e_j\right]_{1 \le j \le N} \quad \textit{on } \Gamma_0 \quad \textit{equals } N.$$

In other words, it is natural to assume that the functions

$$\text{(2.5)} \qquad \gamma \cdot e_j,\ j = 1, \cdots, N \textit{ are linearly independent on } \Gamma_0.$$

As we have seen above the choice of the functions $\{e_j\}_{1 \le j \le N}$ satisfying this condition *is always possible.* □

REMARK 2.3. The analysis of the controllability of (2.2)-(2.3) when the dimension of the span $\left[\gamma \cdot e_j\right]_{1 \le j \le N}$ is less than N is an interesting open problem.

In this sense, 1D problems are always degenerate cases in which the dimension of this span is at most 2. To illustrate the type of results one may expect when (2.4) fails, in Section 4 we address the controllability of the Galerkin's approximations of the 1D Burgers equation. □

We now consider

$$\text{(2.6)} \qquad y^T \text{ given in } E.$$

In the next section we prove the following results:

THEOREM 2.1. *We assume that* (2.5) *holds true. Then, one can find* $v \in L^2(\Gamma_0 \times (0, T))$ *such that the solution* $y(t; v)$ *of* (2.2)-(2.3) *satisfies*

$$\text{(2.7)} \qquad y(T; v) = y^T.$$

We can then introduce *the cost to achieve* (2.7). Namely,

$$\text{(2.8)} \qquad C(v) = \frac{1}{2} \int_{\Gamma_0 \times (0,T)} v^2 d\Gamma dt.$$

One has

THEOREM 2.2. *For any* $\alpha \in \mathbb{R}$ *and any* μ *such that* $0 \le \mu \le \mu_1 < \infty$ *one can achieve* (2.7) *by a control* $v = v(\alpha, \mu)$ *such that*

$$\text{(2.9)} \qquad C\left(v(\alpha, \mu)\right) \le \textit{ constant independent of } \alpha \textit{ and } \mu.$$

REMARK 2.4. Note that the statement of Theorem 2.2 remains valid as $\mu \to 0$. More precisely the cost of controlling remains bounded as $\mu \to 0$ as well. It can then be proved that as $\mu \to 0$ the controls v_μ of minimal norm converge weakly in $L^2(\Gamma_0 \times (0,T)$ to a control v such that the solution of

$$(y_t, e) + \alpha(y \cdot \nabla y, e) + \lambda(y, e)_{\Gamma_0} = (v\gamma, e)_{\Gamma_0}, \ \forall e \in E \tag{2.10}$$

$$y(0) = 0 \tag{2.11}$$

satisfies

$$y(T) = y^T. \tag{2.12}$$

Note however that (2.10) is not a realistic approximation of Euler's equations with boundary control. □

REMARK 2.5. The results of Theorems 2.1 and 2.2 are also valid if the control is taken a priori in a finite-dimensional subspace of $L^2(\Gamma_0 \times (0,T)$ of dimension N. Indeed, let us assume that the control function has the following structure

$$v(x,t) = \sum_{j=1}^{N} v_j(t) m_j(x)$$

where the control "actions" are $v_j(t) \in L^2(0,T)$ and where the m_j's are N linearly independent smooth scalar functions, with support in Γ_0.

Physically the m_j's correspond to "actuators" and the v_j's describe "how we use" these actuators. (Of course in practice there are often constraints on the v_j's which are not taken into account here).

If one chooses the finite-dimensional subspace E (of dimension N), where $e_1, \dots, e_N$ are such that

$$\det \int_{\Gamma_0} m_i(x) \gamma \cdot e_j(x) d\Gamma_0 \neq 0,$$

then all results apply to the present situation. □

REMARK 2.6. The results of Theorems 2.1 and 2.2 are also valid if the initial condition $y(0) = 0$ is replaced by $y(0) = y^0$ for any $y^0 \in E$. □

REMARK 2.7. In the begining of this section we have introduced a special basis of V such that (2.5) holds for all N. However it is also natural to consider the problem of whether a given particular basis satisfies (2.5) or not. In this sense one can consider an orthogonal basis of the subspace of V, constituted by the eigenfunctions of the Stokes operator with appropriate boundary conditions

$$\begin{cases} -\Delta e = \lambda e - \nabla p & \text{in } \Omega \\ e \cdot n = 0 & \text{on } \Gamma \\ e = 0 & \text{on } \Gamma \backslash \Gamma_0 \\ (2\mu D_{ij}(e) n_j + \lambda e_i)\tau_i^k = 0, k = 1,2 & \text{on } \Gamma_0 \\ \operatorname{div} e = 0 & \text{in } \Omega. \end{cases} \tag{2.13}$$

Given a finite N and a tangent vector field γ, the question of whether $\gamma \cdot e_j, j = 1, \cdots, N$ are linearly independent on Γ_0 or not is an interesting open problem.

We conjecture however that (2.4) holds at least *generically* for the eigenfunctions of (2.13). More precisely, if $\Omega, \Gamma_0 \subset \Gamma$ and γ are such that (2.4) does not hold for a finite number of eigenfunctions of the Stokes operator, very probably (2.4) will hold after an arbitrarily small perturbation of Ω, Γ_0 and γ. □

REMARK 2.8. It is obvious that an assumption of the form (2.4) does not make sense for 1D problems. We will return to this question in Section 4. □

The two Theorems stated in this section will be proved simultaneously in the following section.

3. – Proof of the main results

Theorems 2.1 and 2.2 are going to be proven simultaneously according to the following steps.

STEP 1. LINEARIZATION.

STEP 2. ESTIMATES USING DUALITY ARGUMENTS.

STEP 3. CONCLUSIONS.

STEP 1. LINEARIZATION. We consider a function h such that

$$h \in L^2(0, T; E), \tag{3.1}$$

and we consider the *linear* state equation

$$\left\{ \begin{array}{l} (y_t + \alpha h \cdot \nabla y, e) + 2\mu \left(D_{ij}(y), D_{ij}(e)\right) + \lambda (y, e)_{\Gamma_0} = (v\gamma, e)_{\Gamma_0} \quad \forall e \in E \\ y \in C\left([0, T]; E\right), y(0) = 0. \end{array} \right. \tag{3.2}$$

Let us check that system (3.2) is *exactly controllable*, i.e. the existence of $v \in L^2\left(\Gamma_0 \times (0, T)\right)$ such that the solution of (3.2) satisfies

$$y(T; v) = y^T. \tag{3.3}$$

Clearly, it suffices to prove that if $f \in E$ satisfies

$$(y(T; v), f) = 0, \quad \forall v \in L^2\left(\Gamma_0 \times (0, T)\right), \tag{3.4}$$

then $f \equiv 0$.

Let us introduce φ defined by

$$\left\{ \begin{array}{l} (-\varphi_t - \alpha h \cdot \nabla \varphi, e) + 2\mu \left(D_{ij}(\varphi), D_{ij}(e)\right) + \lambda (\varphi, e)_{\Gamma_0} = 0, \ \forall e \in E \\ \varphi \in C\left([0, T]; E\right), \varphi(T) = f. \end{array} \right. \tag{3.5}$$

Clearly (3.5) has a unique solution. Thus, φ is well defined.

Let us take $e = y(t)$ in (3.5). We observe that

$$-(h(t)\cdot\nabla\varphi(t), y(t)) = (h(t)\cdot\nabla y(t), \varphi(t)), \ \forall t\in[0,T],$$

so that, after integration by parts in t, we obtain

$$-(\varphi(T), y(T)) + \int_0^T [(\varphi, y_t + \alpha h\cdot\nabla y) + 2\mu\left(D_{ij}(\varphi), D_{ij}(y)\right) + \lambda(\varphi, y)_{\Gamma_0}]dt = 0.$$

Hence, using (3.2),

$$(y(T), f) = \int_{\Gamma_0\times(0,T)} v\gamma\cdot\varphi d\Gamma dt. \tag{3.6}$$

If (3.4) holds true, then (3.6) implies that

$$\gamma\cdot\varphi = 0 \quad \text{on} \quad \Gamma_0\times(0,T). \tag{3.7}$$

But $\varphi(t) = \sum_{i=1}^N \varphi_i(t)e_i$ and thanks to (2.5), it follows from (3.7) that $\varphi_i(t) = 0$ for $i = i, \cdots, N$ so that $\varphi\equiv 0$ and $f\equiv 0$.

STEP 2. ESTIMATES USING DUALITY. Thanks to the results of Step 1 one can define

$$M(h) = \inf_{v\in\mathcal{U}_{ad}} \frac{1}{2}\int_{\Gamma_0\times(0,T)} v^2 d\Gamma dt \tag{3.8}$$

where $\mathcal{U}_{ad}$ is the set of admissible controls

$$\mathcal{U}_{ad} = \left\{v\in L^2\left(\Gamma_0\times(0,T)\right) : y \text{ solution of (3.2) satisfies (3.3)}\right\}.$$

We define in this way a function of $h\in L^2(0,T;E)$.

We are going to prove that

(3.9) $M(h) \leq$ *constant independent of* $h, \alpha\in\mathbb{R}$ *and* $\mu\in[0,\mu_1]$.

We use for that a duality argument.

We define the continuous linear operator $L : L^2(\Gamma_0\times(0,T)) \to E$ defined by

$$Lv = y(T; v) \tag{3.10}$$

and we introduce

$$F_1(v) = \frac{1}{2}\int_{\Gamma_0\times(0,T)} v^2 d\Gamma dt \tag{3.11}$$

$$F_2(f) = \begin{cases} 0 \quad \text{if} & f = y^T \\ \infty \ \text{otherwise on} & E. \end{cases} \tag{3.12}$$

Then

$$M(h) = \inf_{v \in L^2(\Gamma_0 \times (0,T))} \left[F_1(v) + F_2(Lv) \right], \tag{3.13}$$

and by the duality theorem of Fenchel and Rockafellar [FR] we have

$$-M(h) = \inf_{f \in E} \left[F_1^* \left(L^* f \right) + F_2^*(-f) \right], \tag{3.14}$$

where L^* is the adjoint of L.

Using (3.6) one sees that

$$L^* f = \gamma \cdot \varphi \text{ on } \Gamma_0 \times (0,T) \tag{3.15}$$

so that (3.14) gives

$$M(h) = \inf_{f \in E} \left[\frac{1}{2} \int_{\Gamma_0 \times (0,T)} (\gamma \cdot \varphi)^2 \, d\Gamma_0 dt - \left(f, y^T \right) \right]. \tag{3.16}$$

But, in view (2.4) and (2.5), $(\int_{\Gamma_0} |\gamma \cdot e|^2 d\Gamma)^{1/2}$ is a norm on E, so that

$$C(E) \parallel e \parallel^2 \geq \int_{\Gamma_0} (\gamma \cdot e)^2 \, d\Gamma \geq c(E) \parallel e \parallel^2, \forall e \in E \tag{3.17}$$

with $c(E), C(E) > 0$ positive constants that only depend on E.

Thus, (3.16) gives

$$-M(h) \geq \inf_{f \in E} \left[\frac{c(E)}{2} \int_0^T \parallel \varphi(t) \parallel^2 dt - \left(f, y^T \right) \right]. \tag{3.18}$$

We now take $e = \varphi(t)$ in (3.5) and we integrate from t to T. Then the term containing h drops out. We obtain

$$\frac{1}{2} \parallel \varphi(t) \parallel^2 + 2\mu \int_t^T \parallel D(\varphi(t)) \parallel^2 dt + \lambda \int_t^T \int_{\Gamma_0} |\varphi(t)|^2 d\Gamma dt = \frac{1}{2} \parallel f \parallel^2, \tag{3.19}$$

where $||D(\varphi)||^2 = \sum (D_{ij}(\varphi), D_{ij}(\varphi))$.

We integrate (3.19) in $(0,T)$. We obtain

$$\begin{aligned} &\frac{1}{2} \int_0^T \parallel \varphi(t) \parallel^2 dt + 2\mu \int_0^T t \parallel D(\varphi(t)) \parallel^2 dt \\ &\qquad + \lambda \int_t^T \int_{\Gamma_0} t |\varphi(t)|^2 d\Gamma dt = \frac{T}{2} \parallel f \parallel^2 . \end{aligned} \tag{3.20}$$

Since E is finite-dimensional

$$\lambda \parallel e \parallel^2_{\Gamma_0} + 2\mu \parallel D(e) \parallel^2 \leq C(\lambda, \mu) \parallel e \parallel^2, \forall e \in E \tag{3.21}$$

for a suitable $C > 0$ that only depends on E. Hence (3.20) gives

$$\frac{T}{2} \parallel f \parallel^2 \leq \left(\frac{1}{2} + C(\lambda, \mu)T\right) \int_0^T \parallel \varphi \parallel^2 dt. \tag{3.22}$$

In view of (3.18) we get

$$-M(h) \geq \inf_{f \in E} \left[\frac{c(E)T}{2(1 + 2C(\lambda, \mu)T)} \parallel f \parallel^2 - (f, y^T) \right].$$

Hence

$$M(h) \leq \frac{1 + 2C(\lambda, \mu)T}{c(E)T} \parallel y^T \parallel^2 . \tag{3.23}$$

If $0 \leq \mu \leq \mu_1$ (and even if $|\mu| \leq \mu_1$) we have, of course,

$$M(h) \leq \frac{1 + 2C(\lambda, \mu_1)T}{c(E)T} \parallel y^T \parallel^2 . \tag{3.24}$$

Hence (3.9) follows.

REMARK 3.1. We do not know if (3.22) provides "the best" estimate but it is probably not far of being a good indication of what is going on. It gives a number of important indications:

(i) Increasing μ *does not help*, and probably, *on the contrary*. In fact from (3.18) it is clear that estimate (3.22) is sharp. Thus, when the flow becomes more viscous, it becomes more difficult to control

(ii) The cost remains bounded as $T \to \infty$. But the cost could become infinite as $T \to 0$, according to common sense. (A precise result along these lines has been given, for other purposes, and for a much simpler system, in J.-L. Lions [L3]). □

STEP 3. CONCLUSIONS. Let h be given in $L^2(0, T; E)$. We choose for v the unique element such that

$$\frac{1}{2} \int_{\Gamma_0 \times (0,T)} v^2 d\Gamma dt = M(h). \tag{3.25}$$

We define in this way a continuous mapping $h \to v = v(h)$ from $L^2(0, T; E)$ into $L^2(\Gamma_0 \times (0, T))$. We denote by $y = y(h)$ the solution of (3.2) with the control $v = v(h)$.

Taking $e = y(t)$ in (3.2) gives

$$\frac{1}{2}\frac{d}{dt} \| y(t) \|^2 + 2\mu \| D(y) \|^2 + \lambda \| y \|_{\Gamma_0}^2 = \int_{\Gamma_0} v\gamma \cdot y d\Gamma$$

so that

$$\begin{aligned} &\frac{1}{2} \| y(t) \|^2 + 2\mu \int_0^t \| D(y(s)) \|^2 ds + \lambda \int_{\Gamma_0} v(s)\gamma \cdot y(s) d\Gamma \\ &= \int_{\Gamma_0} \int_0^t v\gamma \cdot y d\Gamma ds \\ &\leq \| v \|_{L^2(\Gamma_0 \times (0,T))} \| \gamma \cdot y \|_{L^2(\Gamma_0 \times (0,t))} . \end{aligned} \tag{3.26}$$

Combining (3.17) and (3.26) with inequality

$$\| y \|^2 \leq C[2\mu \| D(y) \|^2 + \lambda \| y \|_{\Gamma_0}^2], \ \forall y \in V \tag{3.27}$$

we deduce that

$$\frac{1}{2} \| y(t) \|^2 + C \int_0^t \| y(s) \|^2 ds \leq \| v \|_{L^2(\Gamma_0 \times (0,T))} \left(\int_0^t \| y(s) \|^2 ds \right)^{1/2} . \tag{3.28}$$

In view of (3.25) it follows that

(3.29) y remains a bounded subset K of $L^2(0, T; E)$ when h varies in $L^2(0, T; E)$.

We claim that

(3.30) the map $h \to y(h)$ admits a fixed point in K.

Assuming for a moment that (3.30) holds let us conclude the proofs of Theorem 2.1 and 2.2.

If h is a fixed point, then, in view of (3.2)-(3.3), Theorem 2.1 holds. Moreover, for any h we have the uniform estimate (3.9), so that the control $v(h)$ satisfies the conditions of Theorem 2.2.

It only remains to prove (3.30).

According to Schauder's fixed point theorem, it suffices to show that the range of $y(h)$ when h spans K is *compact* in K, which follows from the following estimate

(3.31) y_t remains in a bounded set of $L^1(0, T; E)$ when h describes K.

To prove (3.31), we observe that, from (3.2), the following holds

$$
\begin{aligned}
|(y_t, e)| &\leq \alpha \parallel h(t) \parallel \parallel \nabla y(t) \parallel \parallel e \parallel + 2\mu \parallel D_{ij}(y(t)) \parallel \parallel D_{ij}(e) \parallel \\
&\quad + \lambda \parallel y(t) \parallel_{\Gamma_0} \parallel e \parallel_{\Gamma_0} \\
&\quad + \parallel v(t) \parallel_{L^2(\Gamma_0)} \parallel \gamma \cdot e \parallel_{L^2(\Gamma_0)} \\
&\leq C \left[\alpha \parallel h(t) \parallel \parallel y(t) \parallel + (\lambda + \mu) \parallel y(t) \parallel + \parallel v(t) \parallel_{L^2(\Gamma_0)} \right] \parallel e \parallel
\end{aligned}
$$

for all $e \in E$, since on the finite-dimensional space E all the norms are equivalent

Therefore

$$
\parallel y_t(t) \parallel \leq C \left[\alpha \parallel h(t) \parallel \parallel y(t) \parallel + (\lambda + \mu) \parallel y(t) \parallel + \parallel v(t) \parallel_{L^2(\Gamma_0)} \right]
$$

which implies (3.31). □

REMARK 3.2. During the proof of Theorems 2.1 and 2.2 the finite-dimensionality of E has been used several times:

(i) In Step 1, when deducing that $\varphi \equiv 0$ from the fact that $\gamma \cdot \varphi = 0$ on $\Gamma_0 \times (0, T)$. For the continuous model this would constitute a difficult unique continuation problem that, according to Remark 2.7, one can only expect to hold in some generic sense.

(ii) In Step 2 inequality (3.17) relies in an essential way on the finite-dimensional character of E. This allows to prove immediatelly the coercivity of the quadratic functional appearing in (3.16) in the characterization of the cost. The situation is analogous to the case in which the continuous model is controlled by means of distributed controls everywhere in the domain Ω.

(iii) In deriving the estimate (3.22) by using (3.21).

Obviously this argument fails in the continuous model and this fact is related to the irreversibility (in time) of the system. Indeed, notice that, at the level of the continuous model, this difficulty disappears if, instead of the problem of driving $y(0) = 0$ into $y(T) = y^T$, we consider the problem of driving $y(0) = y^0$ into $y(T) = 0$. In that case the functional appearing in the characterization of M in (3.16) takes the form

$$
\frac{1}{2} \int_{\Gamma_0 \times (0,T)} (\gamma \cdot \varphi)^2 d\Gamma dt + \left(\varphi(0), y^0 \right).
$$

Thus, this time, (3.5) has to be considered backwards in time and therefore (3.21) is useless.

(iv) The finite dimensional character of E is used again in Step 3 to derive the uniform bounds in the state y. □

REMARK 3.3. Inequality (3.27) in infinite space dimensions corresponds to the classical inequality of Korn. However, since we are in finite space dimensions, in (3.27) we just claim that $[2\mu||D(y)||^2 + \lambda||y||^2_{\Gamma_0}]^{1/2}$ defines a norm in E. As we said in the introduction, when $\mu > 0$ this only fails when $\Gamma_0 = \Gamma$ and $\lambda = 0$ due to the rigid motions. In view of the assumption (2.4) the same is true over E when $\mu = 0$. □

4. – The 1-D Burgers equation

As we have seen in Section 3 above, the assumption guaranteeing that the dimension of the span$[\gamma \cdot e_i]_{1\leq i\leq N}$ on Γ_0 coincides with N plays a crucial role in the proof of our main results. It is evident that this kind of condition is never fulfilled for 1-D problems. However, we of course do not exclude a priori the controllability of the Galerkin approximation when this condition is not fulfilled.

As a first attempt to address the situations in which condition (2.4) is not fulfilled we study here the 1-D Burgers equation. For the sake of completeness we first consider the distributed control problem. As we shall see, in this case the system is exactly controllable. We then address the boundary control problem and prove local controllability results.

4.1. – Distributed control

We consider the 1-D Burgers equation in the interval $\Omega = (0,1)$ with control in an open subset $\omega = (a,b) \subset (0,1)$ with $0 \leq a < b \leq 1$:

$$
\left\{\begin{array}{ll} y_t - y_{xx} + yy_x = v\chi_\omega & \text{for } 0<x<1, 0<t<T \\ y(0,t) = y(1,t) = 0 & \text{for } 0<t<T \\ y(x,0) = 0 & \text{for } 0<x<1. \end{array}\right. \tag{4.1}
$$

Following [LZ2], we introduce a basis $\{e_i\}_{i\geq 1}$ of $H_0^1(0,1)$ such that $\{e_i|_\omega\}_{i\geq 1}$ are linearly independent in $L^2(\omega)$. Given a finite-dimensional subspace $E =$ span $[e_1, ..., e_N]$ we introduce the Galerkin approximation of (4.1):

$$
\left\{\begin{array}{l} y \in C([0,T];E)\,;\, y(0) = 0 \\ (y_t, e) + (y_x, e_x) + (yy_x, e) = (v\chi_\omega, e)\,, \forall e \in E. \end{array}\right. \tag{4.2}
$$

Taking into account that $(yy_x, y) = 0$ for all $y \in H_0^1(0,1)$, it is easy to see that for any $v \in L^2(\omega \times (0,T))$ system (4.2) has a unique (global in time) solution.

Our main result is as follows:

THEOREM 4.1. *System (4.2) is exactly controllable. More precisely for any* $y^T \in E$ *there exists* $v \in L^2(\omega \times (0,T))$ *such that the solution of* (4.2) *satisfies*

$$
y(T) = y^T. \tag{4.3}
$$

REMARK 4.1. The same result holds when the initial condition $y(0) = 0$ is replaced by $y(0) = y^0$ for any $y^0 \in E$. Thus, system (4.2) is exactly controllable in the classical sense. □

REMARK 4.2. As we shall see in the proof of the theorem, the cost of controlling is independent of the presence of the non-linearity in the system. □

REMARK 4.3. When proving Theorem 4.1 the fact that

$$\dim E = \dim \ \text{span} \ [e \mid_{\omega}]_{e \in E} \tag{4.4}$$

plays a crucial role. This is the analogous of condition (2.4) for the boundary controllability of Navier-Stokes equations. Obviously, (4.4) is fulfilled for many choices of the basis $\{e_i\}_{i \geq 1}$, in particular when $e_j = \sin(j\pi x)$. □

REMARK 4.4. It is by now well known that the continuous Burgers equation (4.1) is not approximately controllable (see [FI]). Thus, the exact controllability of the Galerkin approximation is in contrast with the behaviour of the continuous model. This implies that the cost of controlling tends to infinity when E increases to cover the whole space. □

PROOF OF THEOREM 4.1. We proceed as in [LZ2]. We just give a sketch of the proof.

STEP 1. LINEARIZATION. This has to be done carefully in order to preserve the analogous of the properties that the non-linearity satisfies. Given $h \in L^2(0, T; E)$ we consider the system

$$\begin{cases} (y_t, e) + (y_x, e_x) + b(h, y, e) = (v\chi_\omega, e) \, , \ \forall e \in E \\ y \in C([0, T]; E) \, ; \, y(0) = 0 \end{cases} \tag{4.5}$$

with

$$b(h, y, e) = \int_0^1 \left(\frac{2}{3} h y_x + \frac{1}{3} h_x y \right) e dx \tag{4.6}$$

The 3-linear form b has the following properties:

(a) $b(h, y, y) = 0$.

(b) $b(y, y, e) = (y y_x, e)$.

This shows, in particular, that when $h = y, y$ solves the non-linear system (4.2). Moreover, (a) indicates that the cancellation property of the nonlinearity of Burgers equation is preserved by the linearization.

In view of condition (4.4) and proceeding as in [LZ2] or Section 3 above we deduce easily that system (4.5) is exactly controllable for any $h \in L^2(0, T; E)$. Moreover, given $y^T \in E$, the cost of controlling system (4.5) to achieve the final condition (4.3) is independent of $h \in L^2(0, T; E)$.

STEP 2 FIXED POINT. Taking into account that the cost of controlling is independent of $h \in L^2(0, T; E)$, the exact controllability of (4.2) follows by Schauder's fixed point Theorem as in [LZ2] or Section 3 above. □

4.2. – Boundary control

We now consider the case in which the control acts on the extreme $x = 1$:

$$(4.7)\qquad \begin{cases} y_t - y_{xx} + yy_x = 0 & \text{for } 0 < x < 1, 0 < t < T \\ y(0,t) = 0;\ y_x(1,t) = v(t) & \text{for } 0 < t < T \\ y(0) = 0. \end{cases}$$

We introduce the Hilbert space $V = \{u \in H^1(0,1) : u(0) = 0\}$. We also introduce a basis $\{e_i\}_{i\geq 1}$ of V. We then define the finite-dimensional subspace $E = \text{span}\,[e_1, ..., e_N]$ of V and introduce the Galerkin approximation:

$$(4.8)\qquad \begin{cases} (y_t, e) + (y_x, e_x) + (yy_x, e) = v(t)e(1),\ \forall e \in E \\ y \in C[(0,T]; E);\ y(0) = 0. \end{cases}$$

We analyze the exact controllability of (4.8), i.e. given $y^T \in E$ we look for $v \in L^2(0,T)$ such that the solution of (4.8) satisfies (4.3).

Clearly this is a completely degenerate situation since, whatever the space E is, the subspace generated by $\{e(1)\}_{e\in E}$ is of dimension 1. Thus, the analogous of (2.4) is never satisfied. In fact system (4.8) is constituted by $N = \dim E$ equations and we only dispose of a control $v \in L^2(0,T)$ for all of them. This is in contrast with the situation we encountered in Section 3 under assumption (2.4): That time we had N equations but also N controls.

To analyze the controllability of (4.8) we consider the particular case in which

$$(4.9)\qquad E = \text{span}\,[\sin(\pi x/2), \cdots, \sin((2N+1)\pi x/2)].$$

In this case $e_j = \sin((2j+1)\pi x/2)$ are the eigenfunctions of the Laplacian with boundary conditions $u(0) = u_x(1) = 0$.

We analize first the linearization of (4.8) around $y = 0$, i.e.

$$(4.10)\qquad \begin{cases} (y_t, e) + (y_x, e_x) = v(t)e(1),\ \forall e \in E \\ y \in C([0,T]; E);\ y(0) = 0. \end{cases}$$

Clearly, (4.10) is the Galerkin approximation of the heat equation:

$$(4.11)\qquad \begin{cases} y_t - y_{xx} = 0,\ 0 < x < 1,\ 0 < t < T \\ y(0,t) = 0,\ y_x(1,t) = v(t),\ 0 < t < T \\ y(x,0) = 0, 0 < x < 1. \end{cases}$$

The following holds:

THEOREM 4.2. *The Galerkin approximation* (4.10) *of the heat equation with E as in* (4.9) *is exactly controllable.*

REMARK 4.5. Note that, although the analogous of condition (2.4) is not verified, system (4.10) is controllable. □

PROOF OF THEOREM 4.2. Proceeding as in Section 3 we introduce the adjoint system:

$$\left\{\begin{array}{l} -(\varphi_t, e) + (\varphi_x, e_x) = 0,\ \forall e \in E \\ \varphi \in C([0,T]; E),\ \varphi(T) = f \in E. \end{array}\right. \tag{4.12}$$

The problem is then reduced to prove the existence of a constant $C > 0$ such that

$$\int_0^T |\varphi(1,t)|^2\, dt \geq c \parallel f \parallel_E^2,\ \forall f \in E. \tag{4.13}$$

Writing

$$f = \sum_{j=1}^N \alpha_j e_j$$

the solution φ of (4.12) can be computed explicitely:

$$\varphi(t) = \sum_{j=1}^N \alpha_j e^{-((2j+1)/2)^2 \pi^2 (T-t)} e_j.$$

Then

$$\int_0^T |\varphi(1,t)|^2\, dt = \int_0^T \left| \sum_{j=1}^N \alpha_j (-1)^j e^{-((2j+1)/2)^2 \pi^2 (T-t)} \right|^2 dt. \tag{4.14}$$

In view of (4.14) and taking into account that the functions

$$\left\{ e^{-\pi^2 (T-t)/4}, \dots, e^{-(2N+1)^2 \pi^2 (T-t)/4} \right\}$$

are linearly independent in $L^2(0,T)$ we deduce that (4.13) holds. □

Combining the exact controllability of the linear system (4.10) and the Inverse Function Theorem (IFT) it is easy to deduce the following local controllability result for (4.8).

THEOREM 4.3. *The Galerkin approximation* (4.8) *of the Burgers equation with E as in* (4.9) *is locally controllable in the following sense: For all* $T > 0$ *there exists* $\varepsilon > 0$ *such that for every* $y^T \in E$ *with* $\parallel y^T \parallel \leq \varepsilon$ *there exists* $v \in L^2(0,T)$ *such that the solution of* (4.8) *satisfies* (4.3).

REMARK 4.6. Note that Theorem 4.3 does not provide any information about the reachability of large final states y^T. In this sense the result is much weaker than those of Section 2. Whether system (4.8) is globally exactly controllable or not is an open problem whose solution very possibly requires the use of deep tools from the theory of controllability of non-linear finite-dimensional systems. □

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