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Rotating Drops Trapped Between Parallel Planes

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Abstract. We derive the existence of local minimizers of the functional $\mathcal{F}_\Omega(E)$ describing the energy of a liquid drop $E \subset \mathbb{R}^n$, trapped between two parallel hyperplanes and rotating by a constant angular velocity $\sqrt{2\Omega}$, for small $\Omega > 0$.

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1. – Introduction

One of the interesting questions in the calculus of variations is the existence of rotating drops, especially the stability of connected drops.

The problem arises from the description of objects studied in astrophysics as rotating homogeneous masses.

In this connection many physicists and mathematicians can be named such as Newton,Mac Laurin [23], Jacobi [20], Plateau [25], Poincaré [26], Darwin [11], Lord Rayleigh [27], Hölder [19], Appell [2], Lichtenstein [21], Lyttleton [22], Chandrasekhar [7], [8], Auchmuty [4], Caffarelli and Friedman [6], Friedman and Turkington [16], [17], Brown and Scriven [5].

The methods used in the present paper will be those introduced by De Giorgi [13], [14] for the treatment of variational problems (compare also [15], [24]), related to the notion of sets of finite perimeter.

A Lebesgue measurable set $E \subset \mathbb{R}^n$, with characteristic function χ_E, is said to have *finite perimeter* in A, $A \subset \mathbb{R}^n$ open, if the total variation of the vector valued measure $D\chi_E$ satisfies

$$\int_A |D\chi_E| = \sup\left\{\int_A \chi_E \operatorname{div} g(x)dx : g \in C_0^1(A, \mathbb{R}^n),\ |g(x)| \leq 1 \text{ for } x \in E\right\} < +\infty .$$

Given two parallel hyperplanes $\Pi_1 = \{x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : z = 0\}$, $\Pi_2 = \{x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : z = d\} \subset \mathbb{R}^n$, $d > 0$, and the domain $G = \{x =$

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$(y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : 0 < z < d\}$ between Π_1 and Π_2, we want to minimize the functional

$$\mathcal{F}_\Omega(E) = \int_G |D\chi_E| + \nu \sum_{i=1}^{2} \int_{\Pi_i} \chi_E^+ d\mathcal{H}^{n-1} - \Omega \int_G |y|^2 \chi_E \, dz \, dy$$

where $\nu, \Omega \in \mathbb{R}$, $0 \leq \nu < 1$, $0 \leq \Omega$. By χ_E^+ we denote the trace of χ_E for $x \in \Pi_i$, $i = 1, 2$, (compare [14], [15], [24]). The class of admissible sets is chosen to be

$$\mathcal{C} = \{E \subset G \text{ Lebesgue measurable } : \int_G |D\chi_E| < \infty, \quad |E| = 1 \text{ and } \int_E y_i \, dx = 0, \ i = 1, \dots, n-1\},$$

that is, the sets E with prescribed volume and barycenter lying on the axis $(0, \dots, 0, z)$.

In $\mathbb{R}^3$ the functional $\mathcal{F}_\Omega$ describes the energy of a liquid drop, trapped between the parallel planes Π_1 and Π_2, the system rotating by a constant angular velocity $\sqrt{2\Omega}$ around its own barycenter.

The energy functional being unbounded from below, we shall here treat the question of the existence of local minimizers for $\mathcal{F}_\Omega$.

Let $G(R) = \{(y, z) \in G : |y| < R\}$. We call $E \in \mathcal{C}$ a *local minimizer* if there exists $R > 0$ such that

(i) $E \subset\subset G(R)$
(ii) $\mathcal{F}_\Omega(E) \leq \mathcal{F}_\Omega(F)$ for all $F \in \mathcal{C}$, $F \subset G(R)$.

We define $\mathcal{C}_R = \{E \in \mathcal{C} : E \subset G(R)\}$.

The techniques are the same as those used by Albano and Gonzalez in [1]. In our case, the special difficulty arises from the "free boundary" of E in Π_i, due to the additional capillarity term in the functional. In particular, we need to understand the behaviour of long, thin liquid bridges, i.e. drops of small enclosed volume related to the distance of the planes. In 2 we present a geometric pinching argument for such drops.

Related results for rotating drops with obstacles are also obtained by Congedo, Emmer and Gonzalez [10], and Congedo [9] — here the obstacle is assumed to be a graph with a certain growth at infinity. Sturzenhecker [28] treats the cases of pendent and rotating drops.

The main result we present is

THEOREM. *There exists* $\tilde{\Omega}_0 > 0$ *such that for* $0 < \Omega < \tilde{\Omega}_0$ *the energy functional* $\mathcal{F}_\Omega$ *has a (connected) local minimizer.*

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2. – A stability result

For the existence proof we need some information about the minimizer E_0 of the functional $\mathcal{F}_0$. In [3] the author proved that E_0 is rotationally symmetric, with analytic radius ρ_{E_0}. By the Euler equation ∂E_0 is shown to have constant mean curvature H, and prescribed contact angle γ, with $\cos\gamma = \nu$, at Π_i. For rotationally symmetric constant mean curvature surfaces Delaunay [12], in the case $n = 3$, and Hsiang and Yu [18], for general n, proved that ρ_{E_0} is periodic, and by [3] at most one period can occur. Furthermore, if (just for the present chapter) we denote by $v = |E_0|$ the volume of E_0, we have the following

LEMMA 2.0. *Given $\nu \geq 0$ and $d > 0$. The minimizer E_0 of $\mathcal{F}_0$ satisfies*

$$\int_G |D\chi_{E_0}| \geq (n-1)\omega_{n-1}\left(\frac{r}{2}\right)^{n-2}\frac{d}{8},$$

where $r = \max_{z\in(0,d)} \rho_{E_0}(z)$.

PROOF. As already observed, we have an exact description of the shape of E_0. E_0 being rotationally symmetric, the problem is one-dimensional and the constant mean curvature equation becomes

$$\frac{\ddot\rho_{E_0}(z)}{\left(1+\dot\rho^2_{E_0}(z)\right)^{3/2}} = H + (n-2)\frac{1}{\rho_{E_0}(z)\left(1+\dot\rho^2_{E_0}(z)\right)^{1/2}}. \tag{1}$$

The radius ρ_{E_0} is analytic ([3]), periodic ([12], [18]) with period P, monotonically increasing between $\rho_1 = \min_{z\in[0,P]} \rho_{E_0}(z)$ and $\rho_2 = \max_{z\in[0,P]} \rho_{E_0}(z)$ ([3]). At most one period is stable ([3]), i.e. $P \geq d$.
For convenience, we translate the origin in z-direction, to obtain $\rho_{E_0}(0) = \rho_2$. We observe that $\ddot\rho_{E_0}(z) \leq 0$ for z near 0.

The case $\ddot\rho_{E_0}(z) \leq 0$ throughout $[0, \frac{P}{2}]$ is handled in (i) below.

We now assume that, as ρ_{E_0} decreases from ρ_2 to ρ_1, there exists a first point $w \in [0, \frac{P}{2}]$, such that $\ddot\rho_{E_0}(w) = 0$; i.e. $\ddot\rho_{E_0}(z) < 0$ for $0 < z < w$, and $\ddot\rho_{E_0}(z) > 0$ in some interval (w, w_1), $w_1 \leq \frac{P}{2}$. We observe that $\dot\rho_{E_0}$ is negative and monotonically decreasing in $[0, w]$, strict monotonically increasing in (w, w_1). By the analyticity and monotonicity of the radius we conclude that w, with $\ddot\rho_{E_0}(w) = 0$, is unique in $[0, \frac{P}{2}]$. In particular the constant H is determined as

$$H = -(n-2)\frac{1}{\rho_{E_0}(w)\left(1+\dot\rho^2_{E_0}(w)\right)^{1/2}}.$$

The differential equation (1) is actually the curvature equation of the curve parametrized by ρ_{E_0}. One proves by contradiction

$$w > \frac{P}{4}.$$

In order to see this, we choose the points $0 < w_{+\epsilon} < w < w_{-\epsilon} < \frac{P}{2}$ with $\rho_{E_0}(w_{+\epsilon}) = \rho_{E_0}(w) + \epsilon$ and $\rho_{E_0}(w_{-\epsilon}) = \rho_{E_0}(w) - \epsilon$. We observe that $\dot\rho^2_{E_0}(w) > \dot\rho^2_{E_0}(w_{+\epsilon})$ and $\dot\rho^2_{E_0}(w) > \dot\rho^2_{E_0}(w_{-\epsilon})$. An easy computation leads to

$$\frac{\ddot\rho_{E_0}(w_{-\epsilon})}{\left(1+\dot\rho^2_{E_0}(w_{-\epsilon})\right)^{3/2}} + \frac{\ddot\rho_{E_0}(w_{+\epsilon})}{\left(1+\dot\rho^2_{E_0}(w_{+\epsilon})\right)^{3/2}} > 0.$$

We assume that $w \le \frac{P}{4}$. We reflect the part of the curve with $0 < z < w$ at the point $(w, \rho_{E_0}(w))$. By the above we have

$$\left|\frac{\ddot\rho_{E_0}(w_{+\epsilon})}{\left(1+\dot\rho^2_{E_0}(w_{+\epsilon})\right)^{3/2}}\right| < \left|\frac{\ddot\rho_{E_0}(w_{-\epsilon})}{\left(1+\dot\rho^2_{E_0}(w_{-\epsilon})\right)^{3/2}}\right|,$$

that means that the original curve is lying everywhere above the reflected one for $w < z < \frac{P}{2}$. We obtain $\rho_1 < \min_{z\in[0,P]} \rho_{E_0}(z)$, contradicting the assumption $w \le \frac{P}{4}$.

The plane curve parametrized by ρ_{E_0} is concave for $z \in [-w, w]$. Furthermore, there are two points $z_i \in [0, \frac{P}{2}]$, $0 < z_1 < w < z_2 < \frac{P}{2}$, for which E_0 and Π_i form the given contact angle γ.

We conclude that there are three possible cases for E_0:

(i) E_0 having a symmetry plane parallel to Π_i; $\frac{d}{2} = z_1$,
(ii) E_0 having a symmetry plane parallel to Π_i; $\frac{d}{2} = z_2$,
(iii) the asymmetric case with $d = z_1 + z_2$.

We observe that in all four cases a single cone over the (n-1)-dimensional ball of radius ρ_2 and distance $\frac{d}{4} \le \frac{P}{4}$ from the center of the ball to its vertex, is contained in E_0. Therefore we can estimate the perimeter of E_0 in G from below, by comparing to the cylinder of radius $\frac{\rho_2}{2}$, height $\frac{d}{8}$, which is contained in the cone, concluding the result of the lemma.

LEMMA 2.1. *Given $\nu \ge 0$ and $d > 0$. If*

$$v^{\frac{1}{n(n-1)}} < \frac{(n-1)}{2^{n+1}}(\omega_{n-1}d)^{\frac{1}{n-1}}\frac{1}{n\omega_n^{\frac{1}{n}}},$$

then the minimizer E_0 of $\mathcal{F}_0$ is the part $B^v \subset G$ cut from the ball B by the hyperplane Π_1, with $|B^v| = v$ and contact angle γ across $\Pi_1 \cap \partial B^v$, with $\cos\gamma = \nu$.

PROOF. Assume E_0 to satisfy

$$\int_{\Pi_i} \chi_E^+ d\mathcal{H}^{n-1} \neq 0 \qquad \text{for both } i = 1, 2 .$$

By [3], we have $\int_{\{z=t\}} \chi_{E_0} d\mathcal{H}^{n-1} \neq 0$ for all $t \in (0, d)$. We want to contradict this for small v.

By the above remarks ρ_{E_0} attains a maximum

$$r = \max_{z \in (0,d)} \rho_{E_0}(z) .$$

By the volume constraint it is $r \geq \left(\frac{v}{\omega_{n-1} d}\right)^{\frac{1}{n-1}}$. E_0 being a minimizer, and by Lemma 2.0, we obtain

$$n\omega_n^{\frac{1}{n}} v^{\frac{n-1}{n}} \geq \int_G |D\chi_{E_0}| \geq (n-1)\omega_{n-1} \left(\frac{r}{2}\right)^{n-2} \frac{d}{8} ,$$

i.e.

$$v^{\frac{1}{n(n-1)}} \geq \frac{n-1}{2^{n+1}} (\omega_{n-1} d)^{\frac{1}{n-1}} \frac{1}{n w_n^{\frac{1}{n}}} .$$

This is not possible for small v.

3. – Existence

We proceed as in [1] with some modifications due to the capillarity term in the functional. For the sake of completeness we repeat all results but the proofs are only given in case the difference is not obvious.

THEOREM 3.1. *Let $R \in \mathbb{R}$, $R > 0$, be such that $|G(R)| > 1$. Then, for each $\Omega \geq 0$ there exists $E_\Omega \in \mathcal{C}_R$ minimizing $\mathcal{F}_\Omega$*

$$\mathcal{F}_\Omega(E_\Omega) = \inf \left\{ \mathcal{F}_\Omega(F) : F \in \mathcal{C}_R \right\} .$$

The proof is based on the compactness of $BV(G(R))$-functions bounded in the $BV(G(R))$-norm (see [15]) and the lower semicontinuity of $\mathcal{F}_\Omega$ with respect to the L^1-convergence.

THEOREM 3.2. *For a sequence $\{\Omega_j\}_{j \in \mathbb{N}}$ with $\Omega_j \to 0$, as $j \to \infty$, we obtain*

$$E_{\Omega_j} \to E_0 \quad \text{in } L^1\big(G(R)\big) ,$$

where E_0 is $\mathcal{F}_0$-minimizing.

Theorem 3.2 allows the use of the author's results in [3] where a detailed discussion of the geometrical properties of E_0 is given.

We want to prove that for small Ω, we have $E_\Omega \subset\subset G(R)$, i.e. a local minimizer of $\mathcal{F}_\Omega$. We use the following notations: we write E for E_Ω and define

$$H(y,z) = -\Omega|y|^2$$

and, for $\frac{R}{2} \le t_1 < t_2 < t_3 \le \frac{3R}{4}$

$$v_1 = |E \cap G(t_1, t_2)|,$$

$$v_2 = |E \cap G(t_2, t_3)|,$$

$$v = v_1 + v_2,$$

where $G(t_i, t_{i+1}) = \{x = (y,z) \in G : t_i < |y| < t_{i+1}\}$, $i = 1, 2$. We assume the trace of E to be continuous on $G \cap \{x = (y,z) : |y| = t_i\}$, $i = 1, 2, 3$, and define

$$m = \max\left\{\int_{G\cap\{|y|=t_i\}} \chi_E d\mathcal{H}^{n-1}, \quad i = 1, 2, 3\right\}.$$

In the following lemma we prove an isoperimetric type inequality, satisfied by $\mathcal{F}_\Omega$-minimizers E.

LEMMA 3.1. *There are two constants c, K such that, if v satisfies the condition of Lemma 2.1 and*

$$\text{(1)} \qquad v \le \min\left\{\omega_n\left(\frac{d}{4}\right)^n, \omega_n\left(\frac{t_3 - t_1}{4}\right)^n\right\},$$

$$\text{(2)} \qquad 4vR \le \frac{t_3 - t_1}{2},$$

$$\text{(3)} \qquad \left|E \cap G\left(\frac{R}{2}\right)\right| \ge \frac{1}{2},$$

then

$$\text{(4)} \qquad \min_{i=1,2} v_i \le c\left(2m + \int_{G(t_1,t_3)} |H(x)|\,|\chi_{B_3^\nu}(x) - \chi_E(x)|dx + Kv\right)^N,$$

where c is a constant depending on n and ν, $K = 8(n-1)\Omega R^2$, $N = \frac{n}{n-1}$, and B_3^ν is defined as the part cut from an n-dimensional ball by Π_1, such that the outer normals of B_3^ν and G at $x \in \partial B_3^\nu \cap \Pi_1$ form an angle γ, with $\cos\gamma = \nu$ and $|B_3^\nu| = v$.

PROOF. Here we have to take care of the boundary terms, occuring by the capillarity forces. We take the unit ball $B \subset \mathbb{R}^n$ and cut by the hyperplane Π_1 the part B^ν of it, such that the angle γ formed by the outer normals has cosine equal to ν. For $\nu \geq 0$ we obtain at least the half ball. We observe that there are two constants $c_1(\nu)$, $c_2(\nu)$ such that the volume and the surface area of B^ν are given by $c_1(\nu)\omega_n$ and $c_2(\nu)n\omega_n$, respectively. Similarly, if B_1^ν, B_2^ν are the parts of the balls $B_1, B_2 \subset \mathbb{R}^n$, cut by Π_1 at an angle γ, with $\cos\gamma = \nu$, and such that $|B_1^\nu| = v_1$, $|B_2^\nu| = v_2$, the surface contained in G is expressed as

$$\int_G |D\chi_{B_i^\nu}| = c_2(\nu)n\omega_n \left(\frac{v_i}{c_1(\nu)\omega_n}\right)^{\frac{n-1}{n}} \qquad i = 1, 2. \tag{5}$$

The intersection of B_i^ν with Π_1 has the $(n-1)$-volume

$$\int_{\Pi_1} \chi_{B_i^\nu}^+ d\mathcal{H}^{n-1} = \omega_{n-1}(\sin\gamma)^{n-1} \left(\frac{v_i}{c_1(\nu)\omega_n}\right)^{\frac{n-1}{n}}, \quad \text{for } \nu > 0. \tag{6}$$

Let $E_i = E \cap G(t_i, t_{i+1})$, $i = 1, 2$, with $v_i = |E_i|$. By Lemma 2.1 it follows for small v_1, v_2,

$$\int_G |D\chi_{B_i^\nu}| + \nu \int_{\Pi_1} \chi_{B_i^\nu}^+ d\mathcal{H}^{n-1} \leq \int_G |D\chi_{E_i}| + \nu \sum_{j=1}^{2} \int_{\Pi_j} \chi_{E_i}^+ d\mathcal{H}^{n-1}, \quad i = 1, 2.$$

Using (5) and (6), and taking the sum over $i = 1, 2$, we conclude

$$\begin{aligned} &c_3\left(v_1^{\frac{n-1}{n}} + v_2^{\frac{n-1}{n}}\right) - \int_{G\cap\{|y|=t_1\}} \chi_E d\mathcal{H}^{n-1} - \int_{G\cap\{|y|=t_3\}} \chi_E d\mathcal{H}^{n-1} \\ &\qquad - 2\int_{G\cap\{|y|=t_2\}} \chi_E d\mathcal{H}^{n-1} \\ &\qquad \leq \int_{G(t_1,t_3)} |D\chi_E| + \nu \sum_{j=1}^{2} \int_{\Pi_j\cap\{t_1<|y|<t_3\}} \chi_E^+ d\mathcal{H}^{n-1}, \end{aligned} \tag{7}$$

where $c_3 = (c_2(\nu)n\omega_n + \nu\omega_{n-1}(\sin\gamma)^{n-1})\left(\frac{1}{c_1(\nu)\omega_n}\right)^{\frac{n-1}{n}}$.
We construct the following set F

$$F = \begin{cases} E & \text{in } G(R) \setminus G(t_3) \\ B_3^\nu & \text{in } G(t_3) \setminus G(\frac{t_1+t_3}{2}) \\ (E \cap G(t_1))_T & \text{in } G(\frac{t_1+t_3}{2}), \end{cases}$$

with B_3^ν as above and $|B_3^\nu| = v$, T a translation to obtain the condition on the barycenter. By assumption (1) we have $F \subset G(R)$. One checks $|T_i| < \frac{t_3-t_1}{2}$ exactly as in ([1], Lemma 2.1). The set E minimizing $\mathcal{F}_\Omega$ we have

$$\mathcal{F}_\Omega(E) \leq \mathcal{F}_\Omega(F).$$

A computation similar to the respective one in ([1], Lemma 2.1), but with the additional boundary terms, leads to

$$
\begin{aligned}
&\int_{G(t_1,t_3)} |D\chi_E| + \sum_{j=1}^{2} \nu \int_{\Pi_j \cap \{t_1<|y|<t_3\}} \chi_E^+ d\mathcal{H}^{n-1} \\
&\quad - \Omega \int_{G(t_1,t_3)} |y|^2 (\chi_E - \chi_{B_3^\nu}) dz\, dy \qquad (8)\\
&\quad \le c_3 v^{\frac{n-1}{n}} + \int_{G\cap\{|y|=t_1\}} \chi_E d\mathcal{H}^{n-1} + \int_{G\cap\{|y|=t_3\}} \chi_E d\mathcal{H}^{n-1} + Kv\,.
\end{aligned}
$$

The factor in front of $v^{\frac{n-1}{n}}$ being c_3, we finally derive

$$
\min_{i=1,2} v_i \le c\left\{2m + \int_{G(t_1,t_3)} |H(x)|\, |\chi_{B_3^\nu}(x) - \chi_E(x)| dx + Kv\right\}^N ,
$$

with

$$
c = \left(2(1 - 2^{-\frac{1}{n}})c_3\right)^{-1},
$$

$K = 8(n-1)\Omega R^2$, and $N = \frac{n}{n-1}$.

The result of Lemma 3.1 has the same form as in ([1], Lemma 2.1). This enables us to carry over the proof of the following

THEOREM 3.3. *Choose R so large that $\omega_n(\frac{R}{2})^n \ge 1$ or $(\frac{R}{2})^{n-1} \ge \frac{n}{\omega_{n-1}}\frac{4}{d}$, depending on whether the set minimizing $\mathcal{F}_0$ is part of a ball, or $\int_{\Pi_i} \chi_{E_0}^+ d\mathcal{H}^{n-1} \neq 0$ for both $i = 1, 2$ (compare Lemma 2.1). Then there exists $\Omega_0 > 0$ such that, for $0 < \Omega < \Omega_0$, there exists t, $\frac{R}{2} \le t \le \frac{3R}{4}$, with*

$$
\int_{G\cap\{|y|=t\}} \chi_E d\mathcal{H}^{n-1} = 0\,.
$$

REMARK 3.1. The condition on R guarantees, in view of Theorem 3.2, $|E \cap G(\frac{R}{2})| \ge \frac{1}{2}$ and that $|E \cap G(\frac{R}{2}, \frac{3R}{4})|$ is small enough, as needed in the proof of Theorem 3.3. To see this we remark that by the observations of Lemma 2.1 E_0 is either part of a ball or $\int_{\Pi_i} \chi_{E_0}^+ dx \neq 0$, $i = 1, 2$. We can guarantee that $r = \max_{z\in(0,d)} \rho_{E_0}(z)$ satisfies $r < \frac{R}{2}$ by comparing to the cone over the ball of radius ρ_{max}, with height $\frac{d}{4}$ and volume 1. We obtain

$$
\rho_{max}^{n-1} = \frac{n}{\omega_{n-1}}\frac{4}{d}
$$

and hence the second condition of the above theorem.

The main result leading to the existence of local minimizers of $\mathcal{F}_\Omega$ will be

THEOREM 3.4. *Choose R as large as in Theorem 3.3. There exists $\Omega_1 > 0$ such that, for $0 < \Omega < \Omega_1$ there exists t, $\frac{R}{2} \le t \le \frac{3R}{4}$, with*

$$\int_{G(R)\setminus G(t)} \chi_E(x)dx = 0.$$

PROOF. We choose Ω_1 so small that, by Theorem 3.3, we have the existence of t, $\frac{R}{2} \le t \le \frac{3R}{4}$, with

$$\int_{G\cap\{|y|=t\}} \chi_E d\mathcal{H}^{n-1} = 0.$$

Let $E(t) = E \cap G(t)$ and $v = |E \setminus G(t)|$. Define the set

$$F = \left\{x = (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R} : y = \mu\tilde{y} + T,\ (\tilde{y}, z) \in E(t),\ T \in \mathbb{R}^{n-1}\right\},$$

where $\mu = (\frac{1}{1-v})^{\frac{1}{n-1}}$ and T is a translation, such that

$$\int_G y_i \chi_F(x)dx = 0.$$

F is the set $E(t)$ blown up horizontally (radially in the y-directions, normal to the axis $(0, \dots, 0, z)$, where the barycenter of $E(t)$ lies).

We prove that F is an admissible set, i.e. $F \subset G(R)$. We have

$$\begin{aligned} 0 = \int_F y_i dx &= \int_{E(t)} (\mu y_i + T_i)\mu^{n-1} dx \\ &= \int_{E(t)} \mu^n y_i dx + T_i \mu^{n-1}|1 - v| + \int_{E\setminus E(t)} \mu^n y_i dx - \int_{E\setminus E(t)} \mu^n y_i dx \\ &= \mu^{n-1} T_i (1 - v) - \mu^n \int_{E\setminus E(t)} y_i dx \end{aligned}$$

and, because $\mu = \frac{1}{(1-v)^{\frac{1}{n-1}}}$, and $\sup_{E\setminus E(t)} |y_i| = R$, it follows

$$|T_i| \le \frac{v}{(1 - v)^{\frac{n}{n-1}}} R. \tag{1}$$

To show $F \subset G(R)$ we need $|\mu y_i + T_i| < R$ for all $y \in \mathbb{R}$, such that $(y, z) \in E(t)$.

As $|y_i| \le t \le \frac{3R}{4}$ and $|T_i| \le \frac{v}{(1-v)^{\frac{n}{n-1}}} R$ by (1), it suffices to show

$$\frac{1}{(1 - v)^{\frac{1}{n-1}}} \frac{3R}{4} + \frac{v}{(1 - v)^{\frac{n}{n-1}}} R < R.$$

Furthermore, with $0 \leq 1 - v < 1$, and $\frac{n}{n-1} < 2$ for $n \geq 3$, we finally need

$$\frac{3}{4} + v < (1-v)^2 \quad \text{or} \quad \left(\frac{3}{4} + 1\right) + (v-1) < (1-v)^2$$

to guarantee $F \subset G(R)$.

We define $f(\eta) = \eta^2 + \eta - \frac{7}{4}$, and obtain $f(\eta) > 0$ for $\eta > -\frac{1}{2} + \sqrt{2}$, and $\eta < -\frac{1}{2} - \sqrt{2}$, i.e. for

$$0 \leq v < \frac{3}{2} - \sqrt{2}$$

$F \subset G(R)$ is an admissible set, and by the minimality of E

$$\mathcal{F}_\Omega(E) \leq \mathcal{F}_\Omega(F) . \tag{2}$$

We shall contradict (2) for small v.

In order to do so, we first compare the perimeters of E and F in G. We need the notion of partial perimeter, as defined in ([24], 2.2.1). We observe, for fixed $z \in \mathbb{R}$, that

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} |D_y \chi_{F_z}| &= \mu^{n-2} \int_{\mathbb{R}^{n-1}} |D_y \chi_{(E(t))_z}| \\ &< \mu^{n-1} \int_{\mathbb{R}^{n-1}} |D_y \chi_{(E(t))_z}| , \end{aligned} \tag{3}$$

where F_z denotes the horizontal slice of F at height z. On the other hand, for fixed $y \in \mathbb{R}^{n-1}$, and the one-dimensional "slice" F_y, we have

$$\int_{\mathbb{R}^{n-1}} \left(\int_\Omega |D_n \chi_{F_y}| \right) dy = \int_{\mathbb{R}^{n-1}} \left(\int_\Omega |D_n \chi_{(E(t))_y}| \right) \mu^{n-1} dy \tag{4}$$

for all open $\Omega \subset \mathbb{R}$.

We define the vector-valued Radon measures

$$\alpha(\Omega) = \left(\int_\Omega \left(\int_{\mathbb{R}^{n-1}} |D_y \chi_{(E(t))_z}| \right) dz, \int_{\mathbb{R}^{n-1}} \left(\int_\Omega |D_n \chi_{(E(t))_y}| \right) dy \right) ,$$

$$\beta(\Omega) = \left(\int_\Omega \left(\int_{\mathbb{R}^{n-1}} |D_y \chi_{F_z}| \right) dz, \int_{\mathbb{R}^{n-1}} \left(\int_\Omega |D_n \chi_{F_y}| \right) dy \right) ,$$

for which by (3) and (4) holds the estimate

$$\beta(\Omega) \leq \mu^{n-1} \alpha(\Omega)$$

for all open $\Omega \subset \mathbb{R}$.

A lemma by De Giorgi, used to prove the isoperimetric property of spheres (see [13], [24]) gives

$$\int_G |D\chi_F| \le \mu^{n-1} \int_G |D\chi_{E(t)}| . \tag{5}$$

For the perimeters of E and F in G, we obtain by (5)

$$\int_G |D\chi_E| - \int_G |D\chi_F| \ge \int_{G\setminus G(t)} |D\chi_E| + \left(1-\mu^{n-1}\right) \int_G |D\chi_{E(t)}| , \tag{6}$$

and for the difference of the boundary terms

$$\begin{aligned} &\int_{\Pi_i} \chi_E^+ d\mathcal{H}^{n-1} - \int_{\Pi_i} \chi_F^+ d\mathcal{H}^{n-1} \\ &= \int_{\Pi_i \cap \{|y|>t\}} \chi_E^+ d\mathcal{H}^{n-1} + \left(1-\mu^{n-1}\right) \int_{\Pi_i} \chi_{E(t)}^+ d\mathcal{H}^{n-1} . \end{aligned} \tag{7}$$

Finally, we compare the rotational energies:

$$\begin{aligned} -\Omega &\int_G |y|^2 \chi_E(x)dx + \Omega \int_G |y|^2 \chi_F(x)dx \\ &= -\Omega \int_{G\setminus G(t)} |y|^2 \chi_E(x)dx - \Omega \int_G |y|^2 \chi_{E(t)}(x)dx + \Omega \int_G |y|^2 \chi_F(x)dx \\ &\ge -\Omega R^2 v - \Omega \int_G \left(|y|^2 - |\mu y + T|^2 \mu^{n-1}\right) \chi_{E(t)}(x)dx \\ &= -\Omega R^2 v - \Omega \int_G \left||y|^2(1-\mu^{n+1}) - 2\mu^n (y, T) - \mu^{n-1}|T|^2\right| \chi_{E(t)}(x)dx . \end{aligned}$$

We have $|T_i| \le \frac{v}{(1-v)^{\frac{n}{n-1}}} R$ by (1), $|y| \le R$, and $\mu = \frac{1}{(1-v)^{\frac{1}{n-1}}}$,

$$\begin{aligned} &- \Omega \int_G |y|^2 (\chi_E(x) - \chi_F(x))dx \\ &\ge -\Omega R^2 v - \Omega\left(|1-\mu^{n+1}|R^2 + 2\sqrt{n-1}\mu^{2n} R^2 v + (n-1)\mu^{3n-1} R^2 v^2\right)(1-v) \\ &= c_1 v + \left(c_2 |1-\mu^{n+1}| + c_4 \mu^{2n} v + c_5 \mu^{3n-1} v^2\right)(1-v) . \end{aligned} \tag{8}$$

Later on we shall let $v \to 0$, so we can already assume v to be so small, that by Lemma 2.1, and the observations at the beginning of the proof of Lemma 3.1, we have

$$\int_{G\setminus G(t)} |D\chi_E| + \nu \sum_{i=1}^{2} \int_{\Pi_i \cap \{|y|>t\}} \chi_E^+ d\mathcal{H}^{n-1} \ge c_3 v^{\frac{n-1}{n}} . \tag{9}$$

By (6), (7), (8), and (9), we conclude

$$
\begin{aligned}
\mathcal{F}_\Omega(E) - \mathcal{F}_\Omega(F) \geq &\int_{G\setminus G(t)} |D\chi_E| + (1-\mu^{n-1}) \int_G |D\chi_{E(t)}| \\
&+ \nu \sum_{i=1}^{2} \left(\int_{\prod_i \cap\{|y|>t\}} \chi_E^{+} d\mathcal{H}^{n-1} + \left(1-\mu^{n-1}\right) \int_{\prod_i} \chi_{E(t)}^{+} d\mathcal{H}^{n-1} \right) \\
&+ c_1 v + \left(c_2|1-\mu^{n+1}| + c_4\mu^{2n} v + c_5 \mu^{3n-1} v^2\right)(1-v) \\
= &\left(1-\mu^{n-1}\right) \left(\int_G |D\chi_{E(t)}| + \nu \sum_{i=1}^{2} \int_{\prod_i} \chi_{E(t)}^{+} d\mathcal{H}^{n-1} \right) \\
&+ \int_{G\setminus G(t)} |D\chi_E| + \nu \sum_{i=1}^{2} \int_{\prod_i \cap\{|y|>t\}} \chi_E^{+} d\mathcal{H}^{n-1} \\
&+ c_1 v + \left(c_2|1-\mu^{n+1}| + c_4\mu^{2n} v + c_5 \mu^{3n-1} v^2\right)(1-v) \\
\geq &\left(1-\mu^{n-1}\right) \left(\int_G |D\chi_{E(t)}| + \nu \sum_{i=1}^{2} \int_{\prod_i} \chi_{E(t)}^{+} d\mathcal{H}^{n-1} \right) \\
&+ c_3 v^{\frac{n-1}{n}} + c_1 v + \left(c_2|1-\mu^{n+1}| + c_4\mu^{2n} v + c_5 \mu^{3n-1} v^2\right)(1-v),
\end{aligned}
$$

with constants $c_3 > 0$, and $c_i < 0$ for $i = 1, 2, 4, 5$.

The final step is to expand $(1-\mu^{n-1})$, $|1-\mu^{n+1}|$, μ^{2n} and μ^{3n-1}, in Taylor's series in the above inequality. The dominating term here being $c_3 v^{\frac{n-1}{n}}$ with $c_3 > 0$ we conclude

$$\mathcal{F}_\Omega(E) - \mathcal{F}_\Omega(F) > 0$$

for v small but strictly positive, contradicting (2). we already had that $v \to 0$ as $\Omega \to 0$, so that it must be $v = 0$ for small Ω.

By the above we derive the main result

THEOREM 3.5. *There exists* $\tilde{\Omega}_0 > 0$ *such that for* $0 < \Omega < \tilde{\Omega}_0$ *the energy functional* $\mathcal{F}_\Omega$ *has a (connected) local minimizer.*

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